Contextuality in Bosonic Bunching

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In the classical probability theory a sum of probabilities of three pairwise exclusive events is always bounded by one. This is also true in quantum mechanics if these events are represented by pairwise orthogonal projectors. However, this might not be true if the three events refer to a system of indistinguishable particles. We show that one can find three pairwise exclusive events for a system of three bosonic particles whose corresponding probabilities sum to $3/2$. This can be done under assumptions of realism and noncontextuality, i.e., that it is possible to assign outcomes to events before measurements are performed and in a way that does not depend on a particular measurement setup. The root of this phenomenon comes from the fact that for indistinguishable particles there are events that can be deduced to be exclusive under the aforementioned assumptions, but at the same time are complementary because the corresponding projectors are not orthogonal.

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Introduction.—Three events α, β, and γ are pairwise exclusive if any two of them never occur together. Any experiment capable of testing all three events at once will show that at most one of the three events can occur in each run of the experiment. As a consequence, the sum of probabilities corresponding to these events cannot exceed one,

$$p(\alpha) + p(\beta) + p(\gamma) \leq 1. \tag{1}$$

Despite the fact that quantum mechanics is associated with many counterintuitive phenomena, it was noted by Specker [1] that even quantum events described by projectors cannot violate inequality (1), which is often referred to as the Specker's inequality. The violation of (1) has been assumed to exist only in theories that are more contextual than quantum mechanics [2–5]; i.e., in theories in which assumptions of measurement outcome preassignment (realism) and outcome independence of the measurement scenario (noncontextuality) cannot be simultaneously met [6].

Here, we show that a Specker-like inequality can be derived for a system of three bosonic particles; however, this time the fundamental properties of indistinguishable particles allow us to violate it. This result establishes a fundamental link between two seemingly distinct fields—quantum contextuality and quantum indistinguishability.

We present a physical system where, under certain assumptions, the sum of probabilities for three pairwise exclusive events exceeds the bound of 1 and reaches the value of $3/2$. Note, that $3/2$ is the maximal bound allowed by the no-disturbance principle [5,7,8] which is a generalization of the no-signaling. This principle states that probabilities do not depend on the measurement context. More precisely, apart from exclusivity, the maximal bound of $3/2$ results from the following assumptions:

1. *Complementarity*: it is not possible to directly test all three events; it is only possible to test pairs of exclusive events like α and β.

2. *No-disturbance*: Probability of any event does not depend on with which other event it is tested.

Complementarity does not allow us to test exclusivity of all three events. Pairwise exclusivity implies that

$$p(\alpha) + p(\beta) \leq 1,$$
$$p(\beta) + p(\gamma) \leq 1,$$
$$p(\alpha) + p(\gamma) \leq 1.$$

Moreover, no-disturbance guarantees that $p(\alpha)$ in the first inequality is the same as $p(\alpha)$ in the last inequality (same for β and γ). Therefore, summing all three inequalities and dividing them by 2 gives

$$p(\alpha) + p(\beta) + p(\gamma) \leq \frac{3}{2}.$$

System.—We focus on a particular realization of our bosonic system with three photons. Moreover, we utilize the bunching phenomenon which was demonstrated experimentally by Hong-Ou-Mandel (HOM) [9]. Let us consider three photons in three optical fibers A, B, and C (one photon per fiber). Next, let us consider three possible measurement scenarios M_i, $i = 1, 2, 3$ that utilize a single beam splitter (BS) and three detectors that are placed at the end of each fiber. M_1 uses BS's inputs and outputs to mix modes corresponding to fibers A and B, M_2 mixes B and C, and M_3 mixes A and C (see Fig. 1).

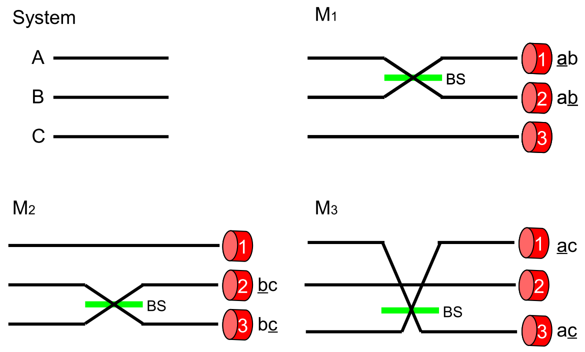

FIG. 1 (color online). Schematic picture representing the setup consisting of three photons in modes A, B, and C (one photon per mode) and three different measurements M_1, M_2, and M_3. The measurements use a single 50/50 beam splitter (BS) to mix two modes and three detectors 1, 2, and 3.

It is clear that every photon incident upon the BS can be either reflected or transmitted. Furthermore, according to HOM [9] photons will bunch; i.e., both photons from two inputs will exit together from a single output port (probability of 1/2 for each output port).

Assumptions and violation.—Let us introduce our assumptions upon which we will derive a variant of Specker's inequality (1) for three bosonic particles:

(i) *Mode distinguishability*: While bosons are still in fibers A, B, and C, it is possible to refer to the boson in fiber A as boson A, etc.

(ii) *Noncontextuality*: The scattering properties of each boson on the BS do not depend on which other fiber is connected to the other BS's input port and on the choice of the BS's input port.

(iii) *Realism*: It is possible to assign to each boson a binary variable x ($x = a$, b, c for bosons A, B, and C, respectively) describing the scattering properties upon the BS, i.e., whether it is transmitted (x) or reflected ($\underline{x}$).

The above assumptions are based on classical intuition. In addition, the assumption (ii) is also supported by the fact that bosons do not interact while they are scattered by the BS. The BS's Hamiltonian includes only single-particle terms. Moreover, this assumption also allows us to treat the BS as a deterministic memoryless device. If the scattering events were not due to the variables assigned to particles, the BS would have to be intrinsically random or would need to store a single bit of information for each scattering event.

Next, consider the following three events: $\mathbf{\underline{a}b}$—the photon A is reflected from BS AND B is transmitted through BS; $\mathbf{\underline{b}c}$—B is reflected AND C is transmitted; $\mathbf{a\underline{c}}$—C is reflected AND A is transmitted. These events are composed of two elementary events referring to the behavior of a single photon. Moreover, they are pairwise exclusive. The events $\underline{a}b$ and $\underline{b}c$ are exclusive because b is exclusive to $\underline{b}$. Exclusivity of $\underline{b}c$ and $a\underline{c}$ and exclusivity of $\underline{a}b$ and $a\underline{c}$ follows from similar arguments.

Note, that events like $\underline{a}bc$ (A is reflected and B is transmitted and C is transmitted), that include scattering of three photons, cannot be tested using our setup due to the fact that BS has only two input and two output ports. Interestingly, this fact is compliant with the complementarity assumption that was used to derive the bound of 3/2 for inequality (1). Nevertheless, the assumptions (i), (ii), and (iii) imply that one can construct a joint probability distribution over the space of eight events $\{abc, \underline{a}bc, a\underline{b}c, ab\underline{c}, \underline{a}\underline{b}c, \underline{a}b\underline{c}, a\underline{b}\underline{c}, \underline{a}\underline{b}\underline{c}\}$. The existence of such a joint probability distribution implies that the following variant of inequality (1) is obeyed:

$$p(\underline{a}b) + p(\underline{b}c) + p(a\underline{c}) \leq 1. \qquad (2)$$

This is because one can write $p(\underline{a}b) = p(\underline{a}bc) + p(\underline{a}b\underline{c})$, $p(\underline{b}c) = p(a\underline{b}c) + p(\underline{a}\underline{b}c)$, $p(a\underline{c}) = p(ab\underline{c}) + p(a\underline{b}\underline{c})$ and since the sum of all probabilities in the joint probability distribution is equal to 1 it is clear that

$$p(\underline{a}b) + p(\underline{b}c) + p(a\underline{c}) = 1 - p(abc) - p(\underline{a}\underline{b}\underline{c}) \leq 1.$$

However, inequality (2) can be violated by the setup presented in Fig. 1. It is well known that scattering of two bosons on a 50/50 BS leads to the bunching phenomenon in which both bosons exit always together through one of the BS's output ports with probability 1/2 for each output [9]. This implies the following bunching events each occur with probability $p(\underline{a}b) = p(\underline{b}c) = p(a\underline{c}) = 1/2$, which leads to violation of inequality (2) since these probabilities sum to 3/2. This in turn implies that at least one of the assumptions (i), (ii), and (iii) does not hold.

Source of violation.—Inequality (2) would not be violated if the exclusivity of events $\underline{a}b$, $\underline{b}c$, and $a\underline{c}$ were implemented by pairwise orthogonality of three von Neumann projectors. However, pairwise orthogonality of three projectors would imply their joint measurability which does not occur in our case. We are going to show that in the case of indistinguishable particles there are events that are complementary but which at the same time can be considered as exclusive if one takes into account assumptions (i), (ii), and (iii). This complementarity is the source of the violation of inequality (2).

The measurements M_i ($i = 1, 2, 3$) use three detectors and a single BS. Despite the fact that the measurement is active in the sense that it contains a BS transformation, the measurement setup can be considered as a black box. Note, that in an idealized scenario the detector that is coupled to a fiber which is not connected to the BS will always click (for example, in M_1 detector 3 always clicks). Therefore, in each measurement only three detection events contain useful information; i.e., both particles scattered by BS were detected by the upper detector (the particle from the upper mode was reflected and from the lower mode was transmitted), both particles were detected by the lower detector (particle from the upper mode was transmitted and from the lower mode was reflected), one particle was detected by the upper detector and one by the lower detector (however, we cannot say anything about which was reflected and which was transmitted). As a result, the black box implementing the measurement M_i contains three outputs corresponding to these three events.

Now, let us look at the problem from a different perspective using the Fock space approach. The initial state of the system can be expressed as $|1,1,1\rangle$, where the modes denote fibers A, B, and C, respectively. On the other hand, the event $\underline{a}b$ corresponds to a projection onto a state $U_{\rm BS}^{(1)\dagger}|2,0,1\rangle$, where $U_{\rm BS}^{(1)\dagger}$ is the reversed BS transformation for the measurement M_1. Analogically, $\underline{b}c$ corresponds to projection onto $U_{\rm BS}^{(2)\dagger}|1,2,0\rangle$ and $a\underline{c}$ to $U_{\rm BS}^{(3)\dagger}|0,1,2\rangle$. We refer to these states as to $|\underline{a}b\rangle$, $|\underline{b}c\rangle$, and $|a\underline{c}\rangle$, respectively. The BS transforms the creation operators of the upper (u) and lower (l) modes in the following way:

$$a_u^\dagger \to \frac{a_u^\dagger + i a_l^\dagger}{\sqrt{2}}, \qquad a_l^\dagger \to \frac{i a_u^\dagger + a_l^\dagger}{\sqrt{2}}.$$

It is therefore straightforward to show that

$$|\underline{a}b\rangle = \frac{1}{2}(-i\sqrt{2}|1,1,1\rangle + |2,0,1\rangle - |0,2,1\rangle),$$
$$|\underline{b}c\rangle = \frac{1}{2}(-i\sqrt{2}|1,1,1\rangle + |1,2,0\rangle - |1,0,2\rangle),$$
$$|a\underline{c}\rangle = \frac{1}{2}(-i\sqrt{2}|1,1,1\rangle + |0,1,2\rangle - |2,1,0\rangle).$$

It is clear, that in the Fock space representation the three events are complementary, since $|\langle\underline{a}b|\underline{b}c\rangle|^2 = |\langle\underline{b}c|a\underline{c}\rangle|^2 = |\langle\underline{a}b|a\underline{c}\rangle|^2 = 1/4$.

The pairwise exclusivity of the three events that enter inequality (2) is not physically testable. According to Specker, pairwise exclusivity of two events can be tested in the lab; i.e., in an experiment both events can be tested, but only one event can occur. In our case pairwise exclusivity is more counterfactual—we cannot test both events, but assumptions (i) to (iii) imply that when one event occurs, the other event could not have occurred if it had been tested instead.

The exclusivity stems from the assumptions (i), (ii), and (iii), and from the fact that each event is a composition of two single-photon events. This resembles the exclusivity of composite events discussed in [10], where exclusive events like (p AND q) and ($\underline{p}$ AND r) were defined for two independent experiments ($\underline{p}$ is exclusive to p). Note, that events p and $\underline{p}$ were exclusive events in one laboratory, whereas q and r were some events in the other laboratory. However, the events in each laboratory were represented by projectors Π_p, $\Pi_{\underline{p}}$, Π_q, Π_r, and composite events were defined as a tensor product of projectors corresponding to different laboratories. What is important, is that the tensor product structure takes care of compatibility, because although Π_q and Π_r may not be orthogonal, the projectors $\Pi_p \otimes \Pi_q$ and $\Pi_{\underline{p}} \otimes \Pi_r$ are orthogonal due to orthogonality of Π_p and $\Pi_{\underline{p}}$. The reason why our case is different is that a tensor product structure does not naturally occur for indistinguishable particles which is the root of complementarity of the events that are assumed to be exclusive.

In the beginning we showed that the violation of the Specker's inequality (1) up to $3/2$ is possible under the assumptions of complementarity and no-disturbance. The above arguments show that our model obeys the complementarity assumption. Moreover, it is easy to show that the no-disturbance assumption is also valid. Note that the probability that *a particular* photon is reflected (transmitted) does not depend on which other photon enters the other BS's input port. For example, the probability that photon A is reflected is the same independent of whether it is scattered together with photon B or C

$$p(\underline{a}) = p(\underline{a}b) + p(\underline{ab}) = p(\underline{a}c) + p(\underline{ac}) = 1/2.$$

Note, that in our case the no-disturbance is intertwined with the indistinguishability. Since photons are indistinguishable, the probabilities of reflection or transmission cannot depend on the choice of the photon in the other port.

KCBS-like scenario.—Recently Klyachko-Can-Binicioglu-Shumovsky (KCBS) [11] proved that for five cyclically exclusive events $\{A_1, \ldots, A_5\}$ (that is, at most one of events A_i and A_{i+1}, for $i = 1, \ldots, 5$ modulo 5, can happen) quantum mechanics does not allow joint probability distributions in accord with a noncontextual hidden variable model. KCBS derived an inequality for probabilities of these five events and showed that the sum of their probabilities cannot exceed 2 for any noncontextual hidden variable theory

$$\sum_{i=1}^{5} p(A_i = 1) \leq 2. \tag{3}$$

It was also shown [10–12] that in quantum mechanics the sum of probabilities for five cyclically orthogonal projective measurements can violate the bound of two, but can reach at most $\sqrt{5}$.

An analogical approach to the one used before can be used to formulate an alternative version of the KCBS-like scenario with five cyclically exclusive events. This time consider five photons in five optical fibers A, B, C, D, and E. In Fock space representation the state of the system is of the form $|1,1,1,1,1\rangle$. The five measurement scenarios utilize five detectors coupled to each fiber and a single BS that mixes modes A and B (M_1), B and C (M_2), C and D (M_3), D and E (M_4), or A and E (M_5). The five cyclically exclusive events are $\underline{a}b$, $\underline{b}c$, $\underline{c}d$, $\underline{d}e$, and $a\underline{e}$. Again, due to the bunching phenomenon the probability of each event is $1/2$ and, hence, the KCBS inequality is violated up to $5/2$.

It can be also shown that the events which are considered to be exclusive are also complementary.

Discussion.—Because the above results seem to be contradictory to the recent proof by Cabello [10] that exclusivity forbids the violation of the KCBS inequality to be greater than $\sqrt{5}$, it was argued that our bosonic schemes do not test contextuality [13]. However, we argue that there is no contradiction at all, because the contextuality discussed in this work differs from the one that is usually tested by noncontextuality inequalities and that our scheme tests contextuality of a different type than the one defined by Kochen and Specker [6].

In Ref. [13] it was argued that an experiment that tests some noncontextual inequality should have the following properties: (a) all measurements are performed on a system in the same state, (b) experiments should involve only compatible (repeatable) tests, (c) each test has to appear in more than one set of different compatible tests. Our scheme satisfies the first condition, since the state of the system on which a measurement is performed is always the same. In the case of inequality (2) it is $|1,1,1\rangle$ and in the case of the KCBS-like scenario it is $|1,1,1,1,1\rangle$. However, the last two conditions are not fulfilled.

Because of the indistinguishable nature of particles compatibility and repeatability do not occur in our proposal. After the scattering event the two photons cannot be distinguished. Note, that this problem also occurs in other types of contextuality. For example, contextuality using generalized measurements (POVMs) [14] also involves tests that are not repeatable [15].

Moreover, the notion of contextuality, presented in this work, refers to the fact that one can choose whether to scatter photon A with B (M_1) or with C (M_3) and to the fact that it is not possible to assign properties to individual bosons independently of this choice. We would like to highlight that the above notion of contextuality does not mean the multiplicity of measurement contexts for the two-photon events that we are testing.

Finally, the application of the exclusivity principle strongly relies on the independence of two (or more) systems [10], whereas in case of indistinguishable particles such independence never occurs—any boson bunches with any other boson of the same type even if they did not interact in the past.

Outlook.—The HOM experiment is often considered as a test of bosonic nature; however, note that the bunching phenomenon between two photons on a single BS can be explained using the outcome assignment model presented in this work. If one assigned values (transmitted or reflected) to two distinguishable particles A and B it would be possible to simulate bunching statistics. One simply assigns $p(a\underline{b}) = p(\underline{a}b) = 1/2$. On the other hand, the addition of the third particle and the ability to make a choice which two particles to send on a BS results in the inequality (2), that itself can be considered as a more rigorous test of the bosonic nature. We conjecture that in a similar way it is possible to extend our result to create more rigorous tests of the fermionic nature. Note that other new tests of indistinguishability have been recently proposed in [16].

Bosonic effects attract much attention due to the new idea of boson sampling [17] in which particle statistics is applied to solve problems that cannot be efficiently solved using classical resources. It is therefore natural to ask whether the power of boson sampling is related to the contextuality discussed in this work. If this is the case, boson sampling would be a powerful application of this new type of contextuality and one may hope to extend it further to contextuality of the KS type. Moreover, we are currently able to amplify randomness using two local boxes [18]. However, it would be more practical if we had only one box for this purpose. It is argued that boson sampling can be simulated classically for all practical purposes [19]. If we could use our test to guarantee lack of classical simulation then we will have a quantum box doing boson sampling and therefore producing quantum random numbers.

Conclusions.—In this Letter we introduced a system of bosonic particles and a set of measurement events that are capable of violating a variant of Specker's inequality which cannot be violated using standard quantum events described by projectors. The derivation of this inequality assumes that these particles are in principle distinguishable and that one can assign to each particle a binary variable that determines whether the particle is reflected or transmitted through BS. In this case BS is assumed to be a deterministic device whose action only depends on values of the variables assigned to individual particles. An alternative approach, which would explain the violation of our inequality, could assume that BS has some mechanism to direct both photons to the same random output port. However, such a device would require a source of intrinsic randomness, which we find implausible. Our results established a fundamental link between contextuality and indistinguishability and argue that noncontextuality inequalities can be used to test bosonic nature.

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