Using Concatenated Quantum Codes for Universal Fault-Tolerant Quantum Gates

Tomas Jochym-O'Connor[1,2,*] and Raymond Laflamme[1,2,3]
[1]*Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*
[2]*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*
[3]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
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We propose a method for universal fault-tolerant quantum computation using concatenated quantum error correcting codes. The concatenation scheme exploits the transversal properties of two different codes, combining them to provide a means to protect against low-weight arbitrary errors. We give the required properties of the error correcting codes to ensure universal fault tolerance and discuss a particular example using the 7-qubit Steane and 15-qubit Reed-Muller codes. Namely, other than computational basis state preparation as required by the DiVincenzo criteria, our scheme requires no special ancillary state preparation to achieve universality, as opposed to schemes such as magic state distillation. We believe that optimizing the codes used in such a scheme could provide a useful alternative to state distillation schemes that exhibit high overhead costs.

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Introduction.—The ability to physically manipulate quantum mechanical systems promises to provide a means towards powerful quantum computing and simulation [1–4]. Understanding and controlling sources of noise during the manipulation of quantum systems is fundamental towards the development of scalable devices that could achieve such computing promises. The theory of quantum error correction has been developed to address the latter, protecting quantum systems through the use of additional ancillary systems. Quantum error correction has progressed rapidly to address multiple types of errors and situations, and provides a building block to large scale quantum devices using fault-tolerant quantum computation.

The goal of fault-tolerant quantum computation is to control quantum errors in a coherent way such that they do not propagate badly throughout the different quantum systems that are being coupled for the use of quantum computation. Any 2-qubit coupling gates can propagate errors and are typically avoided as multiple errors may lead to logical faults after the application of quantum error correction. However, in order for such schemes to provide universal quantum computation, additional resources are required, typically through the preparation of special quantum states [5–8]. Addressing quantum noise in this manner allows for the establishment of noise thresholds, levels of noise for which scalable quantum computation is achievable without exponential overhead in resources [5–7,9,10]. In certain cases, rigorous numerical values of the threshold have been established by calculating the exact propagation of errors given a fixed error model and method of encoding for logical computation [11–13].

Recently, one of the most widely used methods for fault-tolerant quantum computation is magic state distillation [8], which promotes transversal Clifford gate operations to universal quantum computation through gate teleportation. While providing a means to increase the fault-tolerance threshold, the overhead in the preparation scheme for magic state distillation remains one of the large bottlenecks for scalable quantum computing, estimated to account for up to 90% of the overall number of qubits in certain architectures [14]. As such, much effort is being invested into understanding and reducing the overhead associated with such schemes [15–18]. While such research has paved the way for the reduction of the overall cost of fault-tolerant quantum computation, this work will take a different approach by using concatenated quantum error correcting codes to provide universal fault tolerance, rather than state distillation. The scheme we propose uses two different quantum error correcting codes in concatenation. We argue that by sacrificing the full distance of the concatenated quantum error correcting code, we can exploit the transversal properties of both quantum codes to produce a set of operations that, while not globally transversal, provide a means for universal fault-tolerant quantum gates. In this work we shall focus on protecting against arbitrary single-qubit errors; however, we provide a brief description of how such a scheme could be generalized to correct against t errors. Recently, Paetznick and Reichardt [19] have proposed a similarly motivated work on universal quantum fault tolerance without the preparation of special ancillary states and their idea was further developed to a topological setting by Bombín [20]. In their scheme, additional transversal measurements and error correction are introduced after the action of the transversal Hadamard gate in order to recover the code space. The presented scheme differs from such a construction in that it does not require an additional round of error correction since the logical gates do not disrupt the code space as they are not necessarily transversal. This comes at the expense of requiring an additional concatenated code for protection. Additionally, there has been research that has focused on obtaining a set of fault-tolerant operations to transfer between different quantum error correcting codes, however such schemes have not yet yielded a set of universal operations [21–23].

Preliminaries.—Let $\mathcal{G}_1$ be a finite set of unitary operators on a single-qubit Hilbert space $\mathcal{H}_1$. We say that the set $\mathcal{G}_1$ is a universal gate set on $\mathcal{H}_1$ if any unitary transformation U in $\mathcal{H}_1$ can be approximated using gates from the gate set $\mathcal{G}_1$, that is, given a target fidelity ϵ the unitary U can be approximated using $O(\log^c 1/\epsilon)$ gates from the gate set $\mathcal{G}_1$, where c is a constant [24,25]. Given a Hilbert space $\mathcal{H}$ composed of multiple qubits, any universal set of quantum gates on each of the individual qubits along with entangling gates coupling the qubits form a universal gate set for the full Hilbert space $\mathcal{H}$ [26]. The universal gate set that we shall focus on in this work's example will be the set $\mathcal{G} = \{H, T, \text{CNOT}\}$, where H is the Hadamard gate, T is the $\pi/8$ gate ($T = e^{-i\pi/8}|0\rangle\langle 0| + e^{i\pi/8}|1\rangle\langle 1|$), and CNOT is the 2-qubit controlled-not gate [27].

Researchers focus on universal gate sets since developing techniques to deal with errors associated with a finite set of gates is a much more tractable task than correcting for faults for arbitrary unitary gates. As such, quantum error correcting codes are constructed to best protect against errors in the implementation of logical gates for a chosen universal gate set. We shall denote the weight of an error as the number of locations where a given error acts nontrivially (not the identity). One of the simplest methods to construct fault-tolerant schemes is by applying gates transversally. A logical gate g on a quantum error correcting code $\mathcal{C}$ is called t-transversal if g interacts with at most t locations of the underlying qubits composing the code $\mathcal{C}$. Unfortunately, it has been shown that no quantum error correcting code contains a universal set of transversal gates [28,29]. This motivates the search for different fault-tolerant methods to implement universal quantum logic.

Concatenated quantum error correction.—The general concatenated error correcting scheme is as follows: the qubits that we desire to protect against errors are encoded into a quantum error correcting code $\mathcal{C}_1$. In this work, we shall require the code distance of $\mathcal{C}_1$ to be at least three, so that it can correct arbitrary single-qubit errors. The qubits that make up the code $\mathcal{C}_1$ are subsequently encoded into a second code $\mathcal{C}_2$, which again will be required to have distance of at least three. The general layout of the scheme is summarized in Fig. 1. As we are focusing on codes that correct for an arbitrary single-qubit error, we shall refer to a transversal gate for a given code as a gate which is 1-transversal and any gate not having this form as nontransversal.

The important properties for the quantum error correcting codes $\mathcal{C}_1$ and $\mathcal{C}_2$ for the implementation of universal fault-tolerant quantum logic are as follows: (1) For any logical gate that is nontransversal in $\mathcal{C}_1$, there must exist an

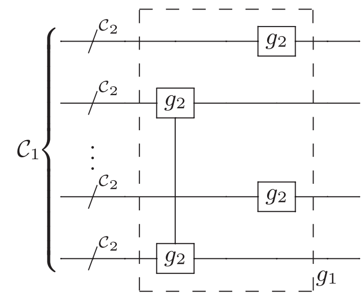

FIG. 1. General construction of a logical gate for a concatenated error correction scheme. The qubit of information is encoded in a quantum error correcting code $\mathcal{C}_1$, whose qubits are in turn encoded into a code $\mathcal{C}_2$. As such, the logical gate g_1 (given by the boxed region) on the encoded space $\mathcal{C}_1$ will be composed of multiple logical gates g_2 on the $\mathcal{C}_2$ code blocks.

application of this logical gate using gates that are transversal in $\mathcal{C}_2$. (2) The recovery operations (syndrome measurement and error correction operations) on $\mathcal{C}_1$ and $\mathcal{C}_2$ must be globally transversal (in the full concatenated code space).

Since there exists no quantum error correcting code that exhibits a full set of transversal quantum gates [28,29], there will always be at least one gate in a given universal gate set that will couple qubits that make up the error correcting code, leading to the possibility of bad error propagation. Consider the first level of encoding $\mathcal{C}_1$, the nontransversal gate can lead to a propagation of errors; however, if we are now encoding each of the qubits making up the code $\mathcal{C}_1$ into a further error correcting code, the propagation from a single to multiple physical faults will not necessarily lead to a propagation of logical faults if the errors are sufficiently sparse.

Specifically, the first requirement of the concatenated quantum error correction scheme stipulates that every nontransversal gate in the code $\mathcal{C}_1$ can be implemented using transversal gates in the code $\mathcal{C}_2$. The nontransversality of a given gate will cause the propagation of a single physical fault between different logical qubits in $\mathcal{C}_1$. The implementation of the nontransversal $\mathcal{C}_1$ gates will govern the propagation of the physical errors between the qubits. Therefore, we require the gates that make up the logical gate on $\mathcal{C}_1$, themselves logical gates for the code $\mathcal{C}_2$, to be transversal in the encoded space $\mathcal{C}_2$. By imposing such a restriction, a single error occurring in the nontransversal gate application in $\mathcal{C}_1$ will propagate to at most a single physical error in each of the logical qubits forming $\mathcal{C}_1$, which themselves are encoded blocks of $\mathcal{C}_2$. This is precisely the property for which one is searching in a fault-tolerant quantum computation, that a single physical error will propagate to at most a single physical error on encoded code blocks, allowing for the correction of such errors.

Given a choice of codes $\mathcal{C}_1$ and $\mathcal{C}_2$, not all gates of the universal gate set will be transversal in $\mathcal{C}_2$. By the properties outlined above, any logical gate in $\mathcal{C}_1$ that uses gates from

$\mathcal{C}_2$ that are not transversal in its construction must be transversal in $\mathcal{C}_1$. In performing such a gate, a single fault on a particular $\mathcal{C}_2$ code block could propagate to multiple errors within this code block and could lead to a logical $\mathcal{C}_2$ fault in the code block where the error occurred. However, a single logical fault on one of the $\mathcal{C}_2$ code blocks will not yield a global logical fault on $\mathcal{C}_1$, as such a code can correct for arbitrary logical faults on one of its encoding logical qubits.

The concatenation scheme therefore protects all gates in the universal gate set. The scheme circumvents the result claiming that no universal gate set can be implemented transversally [28,29] by not implementing the gates in a strict transversal manner. Rather, the gates are implemented such that errors spread to locations that are further protected by an additional code through concatenation.

How is error correction then applied? We shall describe the error correction properties that are required after the application of two types of logical quantum gates, those that are nontransversal in $\mathcal{C}_1$ yet use an application of transversal $\mathcal{C}_2$ gates, and the application of logical gates that are transversal in $\mathcal{C}_1$, whose individual block gates are nontransversal in $\mathcal{C}_2$. In the case of the former, the important property of the error correction is that it does not couple qubits within the code blocks of $\mathcal{C}_2$, as the application of the gate could propagate a single fault into multiple single faults on each of the $\mathcal{C}_2$ code blocks. If the error correction procedure propagates errors within the $\mathcal{C}_2$ code blocks, then single errors on each code block will propagate to multiple errors on each code block, thus possibly leading to logical errors on multiple code blocks, therefore causing a global logical fault. As such, it is very important that the error syndrome measurement and correction be performed transversally on each of the $\mathcal{C}_2$ code blocks. Error correction at the $\mathcal{C}_1$ level is not necessary after the application of this type of logical gate, as the errors propagate within the code blocks and the scheme is constructed in a way that all such errors on the code blocks are recoverable as long as only a single fault occurs in the application of the gate.

The error correction procedure after the implementation of the transversal gate in $\mathcal{C}_1$ (using nontransversal $\mathcal{C}_2$ gates) will require an additional level of error correction. As in the application of the nontransversal $\mathcal{C}_1$ gates, error correction on each of the $\mathcal{C}_2$ code blocks is first applied. As the application of the logical gate on $\mathcal{C}_1$ uses nontransversal $\mathcal{C}_2$ gate applications, a single (correctable) error on a particular $\mathcal{C}_2$ code block can propagate to a noncorrectable set of errors on that given code block. As such, performing the $\mathcal{C}_2$ error correction on that code block will introduce a logical error (if the error were to occur during the $\mathcal{C}_2$ error correction process itself then this error will be weight one). However, as mentioned above, if only a single logical $\mathcal{C}_2$ error has occurred, the logical fault introduced by the error correction will be correctable using an error correction procedure on $\mathcal{C}_1$. However, it is important that the error correction procedure on $\mathcal{C}_1$, which is a logical error correction procedure as it acts on logically encoded states in $\mathcal{C}_2$, is itself globally transversal. As such, errors that could occur during error correction will not propagate to multiple physical errors that could be detrimental upon the application of further logical computation.

Example: A 105-qubit quantum error correcting code.—A simple example of the scheme outlined in this work involves two of the most well studied quantum error correcting codes. $\mathcal{C}_1$ will be the 7-qubit Steane code [30] and $\mathcal{C}_2$ the 15-qubit Reed-Muller code [31]. The 15-qubit Reed-Muller code has the following set of transversal gates: $\{T, \text{CNOT}\}$, where each logical gate is achieved by applying the gate itself to each of the qubits (or $T^\dagger$ in the case of the logical T gate). The missing gate from the universal gate set is the Hadamard gate. The 7-qubit Steane code (corresponding to $\mathcal{C}_1$) has the following set of transversal gates: $\{S, H, \text{CNOT}\}$, where $S = |0\rangle\langle 0| + i|1\rangle\langle 1|$. Each logical gate is achieved by applying an individual gate to each of the qubits, or pair of qubits (in the case of applying logical S, one applies $S^\dagger$ to each of the qubits). As such, $\mathcal{C}_1$ can implement gates from the Clifford group transversally, yet is missing the T gate from the universal gate set that can be implemented transversally.

The concatenated code is seven blocks of 15 qubits, totalling 105 qubits, encoding 1 qubit of information. As both quantum codes share the property that all Pauli gates, the S phase gate, and the CNOT gate can be implemented logically by applying the gate to each qubit, or pair of qubits, then the globally logical version of these gates for the 105-qubit code are also achieved by applying the corresponding gate to each qubit, or pair of qubits, of the full 105-qubit code. Additionally, all syndrome measurements (which will correspond to the measurement of Pauli observables) will be transversal within the code, as well as the Pauli corrections.

The logical T gate is achieved by combining logical gates on the different $\mathcal{C}_2$ code blocks, which, as shown in Fig. 2, is not transversal in the $\mathcal{C}_1$ code, yet uses gates that are all transversal within the 15-qubit $\mathcal{C}_2$ code blocks. As explained in the previous section, a single error in the implementation of the logical gate can propagate to multiple errors (a maximum of 3 for this particular gate application) yet will be distributed such that there are only single errors on each $\mathcal{C}_2$ code block. The error correction procedure measures the syndromes on each of the $\mathcal{C}_2$ code blocks individually, which corresponds to measuring the 14 syndromes corresponding to the 15-qubit code. The Pauli error correction operations are then applied to correct for the errors that occurred during the application of the logical T gate. As such, the concatenated code can correct for an arbitrary weight-one error that occurs during the implementation of the logical T gate.

In order to implement the logical H, one applies the logical H_{15} on each of the $\mathcal{C}_2$ code blocks; as such it is

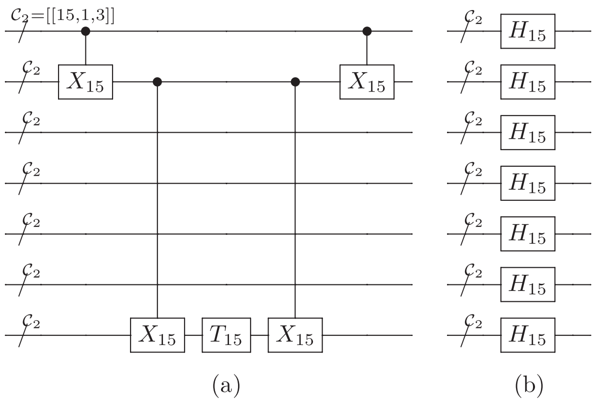

FIG. 2. (a) Logical T gate for the Steane code $\mathcal{C}_1$, composed of logical CNOT_{15} and T_{15} gates on the $\mathcal{C}_2$ code blocks. These gates are transversal in $\mathcal{C}_2$, and therefore only propagate errors to different code blocks, without propagating within a given $\mathcal{C}_2$ code block. (b) Logical H gate for the Steane code $\mathcal{C}_1$, implemented using logical H_{15} gates on each of the $\mathcal{C}_2$ code blocks. Note that the individual H_{15} are nontransversal on each code block.

transversal in the encoded states that form the code $\mathcal{C}_1$, yet each individual H_{15} is not transversal in its implementation on the $\mathcal{C}_2$ code blocks. A single error that occurs in one of the individual applications of the H_{15} gates could propagate to multiple errors within the code block, leading to possible logical errors. However, if only one such error occurs, the full quantum code will still be protected. After the action of the gate, error correction is applied to each of the $\mathcal{C}_2$ code blocks, possibly resulting in correction causing a $\mathcal{C}_2$ logical fault. However, if only one such logical $\mathcal{C}_2$ fault occurs, subsequent global error correction at the logical $\mathcal{C}_1$ level will detect such an error. The $\mathcal{C}_1$ error correction involves measuring the 6 stabilizers of the 7-qubit Steane code, where each stabilizer is now a logical stabilizers composed of X_{15} or Z_{15} operators. However, as such operators are transversal for the 15-qubit code, they can be measured in a transversal way. The maximal weight of the stabilizers measured for the $\mathcal{C}_1$ code is 28, since each of the logical X_{15} gates involve 7 X gates on the $\mathcal{C}_2$ code block, and the weight of the 7-qubit X stabilizers is 4. Error correction for the $\mathcal{C}_1$ level will then be completed by performing logical Pauli error correcting operations on affected $\mathcal{C}_2$ code blocks. The measurement of the $\mathcal{C}_1$ stabilizers will be the most expensive error correction step due to the high weight of the stabilizers.

As described, the concatenated code can correct for any weight-one error. However, it is worth noting that if one used a straight concatenation of the two codes to protect against quantum noise, the concatenated code will be a [[105,1,9]] quantum error correcting code; that is, it would protect against 4 arbitrary errors. In this fault-tolerant scheme, we are sacrificing the larger distance of a straight concatenation scheme in order to protect against arbitrary single qubit errors when performing logical gates.

Conclusion.—In this work, we have proposed a method for universal quantum fault tolerance using concatenated error correcting codes. The full distance of the concatenated scheme is sacrificed in order to establish a set of universal quantum gates that are robust to a smaller set of errors. The transversal properties of the two different error correction schemes are exploited to limit the propagation of errors to either be sufficiently sparse, only a small number of errors per encoded code block, or limiting all errors to be contained within a single code block.

The scheme described in this work could be adapted to account for quantum error correcting codes $\mathcal{C}_1$ and $\mathcal{C}_2$ that correct against arbitrary weight t errors. The key properties of universal gate sets developed for such a concatenation scheme would be modified such that given a gate which is not t-transversal in $\mathcal{C}_1$, the logical gates in $\mathcal{C}_2$ which form such a gate must be t-transversal in $\mathcal{C}_2$ when applied in composition. Additionally, similar requirements for the quantum error correction operations would be necessary. The error correction operations should be 1-transversal as to not possibly propagate errors that occur during the error correction process to multiple errors that could be detrimental at the next stage of computation.

The fault-tolerance threshold of the 105-qubit concatenated code will likely be quite low due to the high number of fault locations and the globally small distance of the code. However, exploring methods to implement sets of universal fault-tolerant gates without the additional overhead of special state preparation is an important question with regards to future physical implementations of fault-tolerant quantum computation. Could the overall number of qubits in such a scheme be reduced in order to improve the threshold rate? Perhaps more interestingly, could a family of such codes be established that increases the code distance of the overall scheme while keeping the rate of the code constant? Additionally, the set of conditions for the concatenated construction does not limit the scheme to CSS codes, yet they are relatively simple to analyze due to their transversal CNOT implementation. Finding examples of the concatenation scheme for non-CSS codes would be an interesting research direction.

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*trjochym@uwaterloo.ca

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