

From Coulomb-Blockade to Nonlinear Quantum Dynamics in a Superconducting Circuit with a Resonator

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Motivated by recent experiments on superconducting circuits consisting of a dc-voltage-biased Josephson junction in series with a resonator, quantum properties of these devices far from equilibrium are studied. This includes a crossover from a domain of incoherent to a domain of coherent Cooper pair tunneling, where the circuit realizes a driven nonlinear oscillator. Equivalently, weak photon-charge coupling turns into strong correlations captured by a single degree of freedom. Radiated photons offer a new tool to monitor charge flow and current noise gives access to nonlinear dynamics, which allows us to analyze quantum-classical boundaries.

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Introduction.—The essence of quantum mechanics lies in the existence of pairs of conjugated, noncommuting variables linked by uncertainty relations. For a superconducting Josephson junction (JJ) the nonlinear dynamics of the number of Cooper pairs (CPs) transferred across the junction and the phase difference of the superconducting condensates are linked in that manner [1,2]. In a dc-biased circuit, the electromagnetic properties of the equilibrium environment and the coupling of the JJ determine, whether a “classical” ac Josephson current with a well-defined phase, an incoherent tunneling of single CPs due to dynamical Coulomb blockade (DCB), or the full quantum dynamic regime, where neither phase nor charge behaves like a classical variable, occurs [3].

In the last years, this setup has been extended by combining JJ devices with superconducting resonators, the electromagnetic modes of which act as dynamical degrees of freedom. This has led to an unprecedented control of quantum properties such as the creation of catlike states [4] and the observation as well as theoretical description of nonlinear dynamics [5–11]. While in these circuits no net charge flows through the JJ, dc-voltage-biased setups, implemented very recently in [12–15], offer new possibilities to study nonlinear quantum properties in a tunable photon-charge system *far from equilibrium* [16–18].

In this Letter we consider a circuit, realized experimentally in [13], where a JJ, biased by a voltage V , is placed in series to a resonator as displayed in Fig. 1. At low temperatures and voltages below the superconducting gap, the excess energy $2eV$ of tunneling CPs is completely transformed into photons exciting the resonator. There, photon leakage leads to radiation that can be detected. Two limiting scenarios are then possible: Either CP tunneling is slow compared to photon relaxation so that between subsequent tunneling events the cavity returns to its ground state, or charge transfer is fast so that photons accumulate and undergo backaction on the JJ giving rise to strong charge-photon correlations. The first regime, known as

DCB, has been analyzed in [13] and corresponds to an *incoherent* CP flow. However, what happens in the second regime and how the crossover between the two domains occurs, is not known yet. The goal of this work is to fill this gap and to provide detailed predictions for future experiments. As we will show, in the second regime, the JJ behaves according to the classical Josephson relations, based on *coherent* CP tunneling, and the circuit realizes a driven nonlinear oscillator governed by a single degree of freedom. By tuning its parameters, one may continuously switch between the two domains and thus access different dynamical properties. In a broader context, these results contribute to current efforts to achieve a deeper

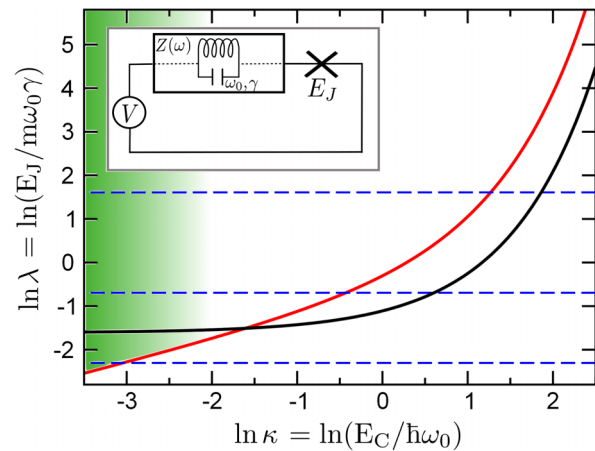


FIG. 1 (color online). Parameter space of a circuit consisting of a resonator with mode frequency ω_0 and damping γ in series with a voltage-biased JJ (inset) with dimensionless coupling $\lambda = E_J / (m\omega_0\gamma)$ and the scale for DCB $\kappa = E_C / \hbar\omega_0$: Classical behavior is seen in the green range; below (above) the red line charge flow through the JJ occurs incoherently (coherently) with the resonator being at $T = 0$ basically empty (excited); the black line separates the domains of linear (below) and nonlinear (above) dynamics. Horizontal lines (blue) refer to the parameter values of Fig. 2.

understanding of quantum-classical boundaries in non-equilibrium systems including superconducting (see [19] for a complementary setup to the one studied here), micro-mechanical [20], and cold-atom setups [21].

Model.—The Hamiltonian of the circuit follows from the two subunits, the resonator part and the JJ part, where in the regime we are interested in only a single mode of the resonator impedance $Z(\omega)$ is relevant. One thus has $\tilde{H}_0 = (q^2/2C) + (\hbar/2e)^2(1/2L)\phi^2 - E_J \cos(\eta) - 2e(V - V_{\text{res}})N$ describing a harmonic oscillator with mass $m = (\hbar/2e)^2C$ and frequency $\omega_0 = 1/\sqrt{LC}$ in series to a JJ with phase η which is subject to a bias voltage V . Here, the two sets of conjugate variables obey $[q, \phi] = -2ie$ and $[N, \eta] = -i$, where in contrast to the charge operator q , the number operator N has discrete eigenvalues, counting the number of CPs that have transferred the JJ. The effective voltage at the JJ also contains the voltage drop at the resonator $2eV_{\text{res}} = -\hbar\dot{\phi}$, thus coupling the JJ and resonator dynamically.

Now, a straightforward calculation shows [22] that for weak detuning $\Delta/\omega_0 = (\omega_0 - \omega_J)/\omega_0 \ll 1$ with $\omega_J = 2eV/\hbar$, in the rotating frame the Hamiltonian \tilde{H}_0 takes the form

$$H_0 = \hbar\Delta a^\dagger a + i\frac{E_J^*}{2} : (a^\dagger e^{i\eta} - ae^{-i\eta}) \frac{J_1(2\sqrt{\kappa n})}{\sqrt{n}} : , \quad (1)$$

with standard photon annihilation(creation) operators of the resonator and $e^{\pm i\eta}$ invoking forward(backward) CP tunneling. Here, $: : \text{denotes normal ordering and the Bessel function } J_1 \text{ of the first kind contains the photon number operator } n = a^\dagger a. \text{ The dimensionless parameter}$

$$\kappa = E_C/\hbar\omega_0 \equiv \hbar/(2m\omega_0)$$

represents the scale for charge quantization through $E_C = 2e^2/C$ while $E_J^* = E_J e^{-\kappa/2}$ is a renormalized Josephson energy [3].

According to the experimental setting [13], the electromagnetic environment of the circuit consists of high-frequency modes acting as a heat bath and low-frequency voltage noise. The former leads to photon leakage from the resonator while the latter one can be seen as a fluctuating component of the bias voltage which thus couples to the charge N . At low temperatures, the dynamics of the reduced density operator of the JJ-resonator compound is then captured by a master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] + \frac{\gamma}{2}\mathcal{L}[a, \rho] + \frac{\gamma_J}{2}\mathcal{L}[N, \rho], \quad (2)$$

where dissipators $\mathcal{L}[x, \rho]$ model the impact of the respective environments. The rate γ determines the photon lifetime in the cavity via its Q factor, i.e., $Q = \omega_0/\gamma$, and the rate γ_J follows from the noise power of low-frequency voltage fluctuations; see [22]. This decreases with decreasing temperature so that $\gamma_J \ll \gamma$. Hence, we start by putting $\gamma_J = 0$ and discuss further details below.

The dynamics in (2) displays a complex interplay between charge transfer and photon emission or absorption. Restricting ourselves to the stationary state, this is particularly seen in the resonator population, which, according to Eq. (2), reads

$$\langle n \rangle_{\text{st}} = \frac{\lambda e^{-\kappa/2}}{4\kappa} \left\langle : (a^\dagger e^{i\eta} + ae^{-i\eta}) \frac{J_1(2\sqrt{\kappa n})}{\sqrt{n}} : \right\rangle_{\text{st}}. \quad (3)$$

Here, we introduced the coupling parameter

$$\lambda = E_J/(m\omega_0\gamma),$$

which, as shown below, determines together with the quantum parameter κ the dynamics of the circuit. The current through the JJ, i.e., $\langle I_J \rangle_{\text{st}} \equiv 2e\langle \dot{N} \rangle_{\text{st}}$, is obtained accordingly from (2) and turns out as $\langle I_J \rangle_{\text{st}} = 2e\gamma\langle n \rangle_{\text{st}}$. This reflects energy conservation between charge flow and photon absorption. Photon radiation in the steady state is, of course, also fixed by $\langle n \rangle_{\text{st}}$. Correlations of charges and photons are captured by $\langle ae^{-i\eta} \rangle_{\text{st}} = C_{a,\eta}\gamma/(\gamma - 2i\Delta)$ with

$$C_{a,\eta} = \frac{\lambda e^{-\kappa/2}}{2\sqrt{\kappa}} \left\langle : J_0(2\sqrt{\kappa n}) + \frac{a}{a^\dagger} J_2(2\sqrt{\kappa n}) e^{-2i\eta} : \right\rangle_{\text{st}}. \quad (4)$$

While in general explicit results must be obtained numerically, it is intriguing to first consider how limiting cases are recovered and what are their precise ranges of validity.

Incoherent charge transfer.—In the weak coupling regime $\lambda \ll 1$ (cf. Fig. 1), the Bessel functions can be linearized by assuming self-consistently $\kappa\langle n \rangle_{\text{st}} \ll 1$. One then arrives with (1) at the Hamiltonian for DCB [23], i.e., $H_{0,\text{DCB}} = \hbar\Delta n + (i/2)\sqrt{\kappa}E_J^*(a^\dagger e^{i\eta} - ae^{-i\eta})$. Likewise, we find for the photon occupation (3) with $C_{a,\eta} \approx \lambda e^{-\kappa/2}/(2\sqrt{\kappa})$ in leading order

$$\langle n \rangle_{\text{st}}^{\text{lin}} = \frac{\lambda^2 e^{-\kappa}}{4\kappa} \frac{\gamma^2}{\gamma^2 + 4\Delta^2}, \quad (5)$$

which in turn verifies the assumption. The current $\langle I_J \rangle_{\text{st}}^{\text{lin}} = 2e\gamma\langle n \rangle_{\text{st}}^{\text{lin}}$ is identical to the one derived within the golden rule treatment of DCB [$P(E)$ theory] [13,23] describing *incoherent* CP transport across the JJ. Subsequent tunneling processes are thus statistically independent and, in the low-temperature range, one may express the current also in terms of the forward tunneling rate Γ_f as $\langle I_J \rangle_{\text{st}} = 2e\Gamma_f$ with $\Gamma_f = \gamma\langle n \rangle_{\text{st}}^{\text{lin}}$. This implies that between subsequent tunneling events the resonator returns to its ground state (at $T = 0$) and there is no backaction onto the JJ. The condition for this scenario is that photon relaxation occurs sufficiently fast compared to CP tunneling, i.e., $\Gamma_f \ll \gamma \Rightarrow \langle n \rangle_{\text{st}}^{\text{lin}} \ll 1$. This relation defines the crossover between incoherent and coherent charge flow (red line in Fig. 1): For fixed κ , the coherent domain is approached by increasing λ and thus by either increasing the Josephson energy or the photon lifetime in the cavity.

We note that formally the linearization condition used in (5) $\kappa\langle n \rangle_{\text{st}} \ll 1$ is not identical to the condition for the incoherent-coherent transition $\langle n \rangle_{\text{st}} \ll 1$, (cf. Fig. 1). Physically, for small κ and λ the circuit may thus display linear dynamics even in the coherent regime.

Classical regime.—We now consider the situation where the photon occupation and in turn the JJ current are large, while the quantum parameter $\kappa \ll 1$, such that $\kappa\langle n \rangle_{\text{st}} = \text{const}$. Charging effects thus do not play any role and we may put $e^{i\eta} \rightarrow 1$ and replace operators by $2\sqrt{\kappa a} \rightarrow Z \exp(i\varphi)$ with real-valued amplitude Z and phase φ . Consequently, at resonance ($\Delta = 0$) the rotating frame Hamiltonian (1) reduces to

$$H_{0,\text{cl}} = E_J J_1(Z) \sin(\varphi). \quad (6)$$

The same result is also obtained directly from a classical description of the circuit in Fig. 1: The Kirchhoff rules impose $V = V_J + V_{\text{res}}$, so that the voltage V_J across the JJ is slaved to the dynamics of the resonator phase $V_{\text{res}} = -(\hbar/2e)\dot{\phi}$. Accordingly, the classical Josephson energy reads $-E_J \cos(\phi + \omega_J t)$ and acts as a nonlinear drive on the resonator. Near resonance $\omega_J \approx \omega_0$, the ansatz $\phi(t) = Z \cos(\omega_J t + \varphi)$ for the stationary orbit then leads in the rotating frame to (6). We emphasize that in contrast to most driven nonlinear oscillators, recently realized also with superconducting circuits (see, e.g., [10]), here, the nonlinearity is part of the drive and not part of a static potential. The classical treatment is based on the *coherent* flow of CPs and the circuit is described by the single degree of freedom ϕ of a driven nonlinear system. Amplitudes and phases of stationary orbits are determined by the static parts of the classical equations of motions [22], i.e.,

$$Z^2 = 2\lambda J_1(Z) \cos(\varphi), \quad 0 = \sin(\varphi) dJ_1(Z)/dZ. \quad (7)$$

For $\lambda \ll 1$, this set of equations has only one solution, namely, the orbit $Z_0 \approx \lambda$, $\varphi_0 = 0$ obtained by linearizing $J_1(Z)$. With increasing λ , the amplitude grows, nonlinearities become relevant, and at $\lambda = \lambda_c \approx 3$ a first bifurcation occurs. There, a new class of orbits appears with constant amplitude $Z_1 \approx 1.8$, given by $[dJ_1(Z)/dZ](Z_1) = 0$, and growing phase $\varphi_1(\lambda > \lambda_c) > 0$. The critical coupling λ_c follows from $Z_0(\lambda_c) = Z_1$.

Full quantum dynamics.—The master equation, Eq. (2), describes the full dynamics of the JJ-resonator system. The density ρ is conveniently found numerically in a base of product states that are eigenstates of the number operators n and N . However, in this basis $\rho(t)$ does *not* reach a stationary state due to a finite current $\langle I_J \rangle_{\text{st}} \sim \text{Tr}\{N\dot{\rho}(t)\}$. This problem is circumvented by introducing auxiliary densities $\rho_\chi = \text{Tr}_J\{e^{i\chi n} \rho\}$ with a partial trace over the JJ degrees of freedom and χ being an integer. Coherences between differing numbers of transferred Cooper pairs are captured for $\chi \neq 0$. This way, one arrives at a hierarchy of coupled equations of motions for the ρ_χ which is solved by proper truncation. Based on the quantum regression

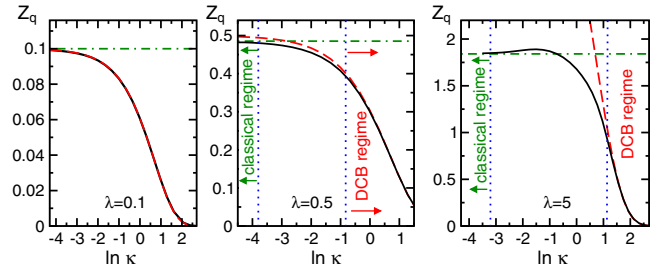


FIG. 2 (color online). Quantum amplitude $Z_q = 2\sqrt{\kappa\langle n \rangle_{\text{st}}}$ vs κ for different couplings λ . Green (red) lines show the classical (DCB), black lines the full nonlinear quantum results. Corresponding ranges of validity are indicated (cf. Fig. 1).

theorem [24], all relevant observables of cavity and JJ are then evaluated.

Results.—We are now able to investigate how the cross-over from the limiting regimes to the full nonlinear quantum case in Fig. 1 is encoded in various observables which are accessible experimentally.

The fact that the resonator occupation determines in the classical limit the amplitude of the orbit via $\hbar\omega_0\langle n \rangle_{\text{st}} \rightarrow m\omega_0^2 Z^2/2$, suggests to formally define $Z_q = 2\sqrt{\kappa\langle n \rangle_{\text{st}}}$. We may then study how for fixed λ this “quantum amplitude” approaches the classical domain for small κ and the DCB domain for large κ ; see Fig. 2. In the case of $\lambda \ll 1$, the DCB result (5) provides a fairly accurate description over the full range where quantum effects only appear in the renormalized parameter $\lambda e^{-\kappa/2}$. In contrast, for larger couplings, nonlinearities are relevant in the classical regime (smaller κ) as well as in the nonperturbative quantum domain where CP transfer is coherent (intermediate values of κ). These results verify the domains in parameter space depicted in Fig. 1. Moreover, we recover from the simulations the experimental observations of Ref. [13] obtained in the DCB regime (see Fig. 3) [25]. Since $\langle I_J \rangle_{\text{st}} \propto \langle n \rangle_{\text{st}}$ and ω_J varies with the voltage V , the occupation $\langle n \rangle_{\text{st}}$

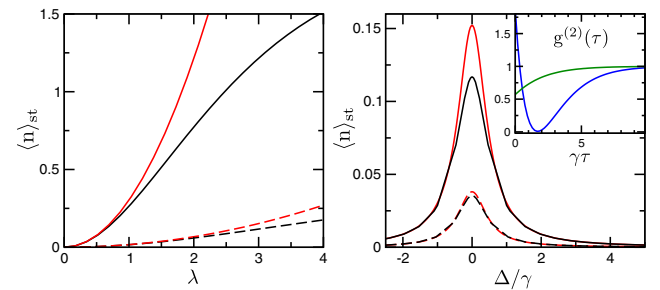


FIG. 3 (color online). Photon occupation $\langle n \rangle_{\text{st}}$ (black) vs coupling λ (left) and detuning Δ (right). Exact results (black) are shown together with the DCB result (5) (red). Left: $\kappa = 0.5$ (solid line), $\kappa = 2$ (dashed line). Right: $\lambda = 1.5$ (dashed line), $\lambda = 3$ (solid line). Inset: Green (blue) lines reveal (anti)bunching for $\kappa = 0.5$ ($\kappa = 4.7$) visible in $g^{(2)}(\tau)$ for $\lambda \ll 1$.

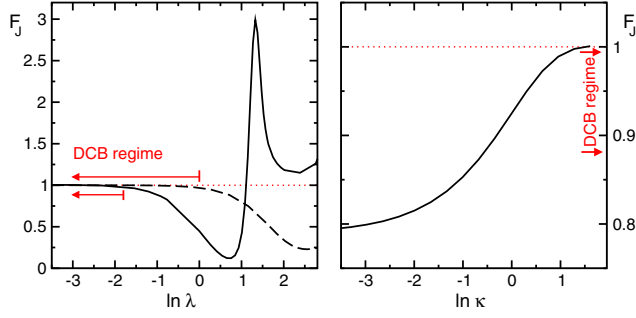


FIG. 4 (color online). Left: Fano factor F_J for the current-current noise vs λ for $\kappa = 0.05$ (solid line) and $\kappa = 3$ (dashed line). Right: F_J vs κ for $\lambda = 0.5$; incoherent single-CP transfer corresponds to $F_J = 1$ (dots).

as a function of the detuning (cf. Fig. 3 right) provides also the IV curve of the JJ.

Experimentally of particular relevance are photon correlations such as $g^{(1)}(\tau) = \langle a^\dagger(t)a(t+\tau) \rangle / \langle a^\dagger a \rangle$. Its Fourier transform provides for long times t the spectral distribution of the photon radiation. We find that its width sensitively depends on the low-frequency voltage fluctuations which have been neglected so far. Upon comparing experimental data with numerical predictions for $\gamma_J \neq 0$, one gains $\gamma_J/\gamma \approx 0.04 \ll 1$, verifying that they are relevant only for those quantities which are broadened solely by γ_J . The next order correlation $g^{(2)}(\tau) = \langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t) \rangle / \langle a^\dagger a \rangle^2$ carries information about correlations between emitted photons and thus indicates (anti-)bunching; see Fig. 3. Even though charge flow is incoherent in the DCB regime, $g^{(2)}(\tau)$ reveals non-Poissonian photon correlations $g^{(2)}(\tau) \neq 1$. For weak driving $\lambda \ll 1$, $g^{(2)}(0)$ is related to the probability of finding the resonator excited by a second tunneling event, before it relaxes to its ground state, with the result $g^{(2)}(0) = (1 - \kappa/2)^2$.

The circuit considered here provides not only a new tool to analyze charge flow by detecting emitted photons, but also to monitor nonlinear dynamics by detecting current correlations as, e.g., the Fano factor $F_J = S_{I_J I_J} / (4eI_J)$ of the JJ-current-current noise $S_{I_J I_J}$; see Fig. 4. As expected, we find $F_J = 1$ in the incoherent, single-CP transport regime (shot noise) for weak coupling. The onset of a coherent charge flow through the JJ results in a substantial drop of F_J due to a reduction of shot noise. However, most strikingly the Fano factor approaches a minimum followed by a pronounced peak exactly at those values for λ where according to (7) new classical orbits emerge. As this class of new orbits is a mere consequence of the nonlinearity and exists even in absence of dissipation (formally $\lambda \rightarrow \infty$), the resonator gains on average (over one driving period) no net energy from the driving source. Physically, this means that a new channel for a correlated two-CP transfer opens: The energy quantum $2eV = \hbar\omega_0$ deposited in the cavity by a forward CP transfer is used to promote a backward

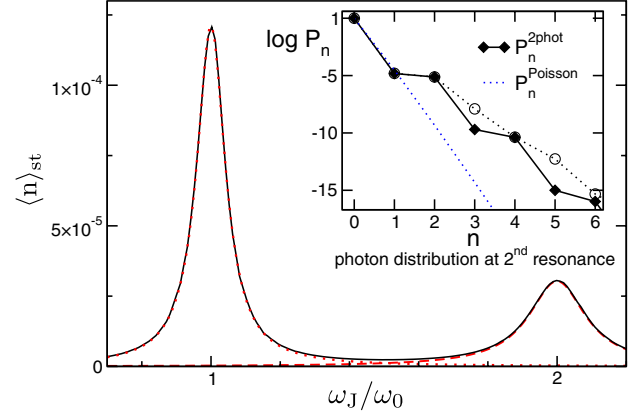


FIG. 5 (color online). One and two photon resonances in the cavity occupation for $\kappa=0.5$, $\lambda=0.02$, $Q=10$ (black) together with DCB results (red). Two photon processes lead to a non-Poissonian photon distribution (inset) for $Q=10$ (circles) and $Q \rightarrow \infty$ (diamonds).

transfer leading in turn to no net current. Around the classical bifurcation point the competition between two sets of classical orbits is then observable as a substantial increase in the charge noise. For larger κ , the bifurcation point is shifted according to $\lambda \rightarrow \lambda e^{-\kappa/2}$ and features are smeared out by quantum fluctuations (Fig. 4, left). These findings open fascinating avenues to study signatures of classical bifurcations in the deep quantum regime experimentally.

We conclude this analysis by highlighting that rich physics is also present *beyond* the fundamental resonance $\omega_J \approx \omega_0$. For this purpose, one relaxes the rotating-wave approximation in the numerical approach described above which then gives access to further resonances in the resonator occupation (respectively, the JJ current) when the applied dc voltage is varied [26]. As illustrated in Fig. 5, resonances occur indeed for $\omega_J = p\omega_0$, $p \in \mathbb{Z}$. In a generalization of (5), within the DCB regime one shows that the p th resonance scales as $e^{-\kappa} \kappa^{|p|-2} / |p|!$. As displayed for $p=2$ in Fig. 5, the generation of p photons by a single CP leads to strong correlations in the photon output and thus to a strongly non-Poissonian resonator occupation; see Fig. 5. Accordingly, for weak driving the correlation $g^{(2)}(0) = 1/(2\langle n \rangle_{st}) \gg 1$ diverges (cf. Ref. [17]).

To summarize, we have analyzed the quantum dynamics of a superconducting circuit consisting of a voltage-biased JJ in series with a resonator in strong nonequilibrium. Analytical findings and numerical simulations provide detailed information about the crossover from the regime of sequential tunneling with weak photon-charge correlations to the one where the circuit behaves as a driven nonlinear oscillator with a single degree of freedom. This also implies a quantum-classical transition. Charge flow is detectable via photon radiation and current-current correlations display quantum signatures of classical bifurcations in the nonlinear regime. Multiphoton resonances reveal

complex charge-photon interaction including photon bunching and are thus of great interest for future theoretical and experimental studies.

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