Universal Quantum Fluctuations of a Cavity Mode Driven by a Josephson Junction

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We analyze the quantum dynamics of a superconducting cavity coupled to a voltage-biased Josephson junction. The cavity is strongly excited at resonances where the voltage energy lost by a Cooper pair traversing the circuit is a multiple of the cavity photon energy. We find that the resonances are accompanied by substantial squeezing of the quantum fluctuations of the cavity over a broad range of parameters and are able to identify regimes where the fluctuations in the system take on universal values.

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Recent progress in integrating superconducting resonators with Josephson junction devices [1], and in measuring quantum states in the microwave regime [2], has opened up many new ways of using such devices to study the quantum dynamics of nonlinear oscillators [3–7]. Significant attention has been devoted to the idea of using a few-level Josephson "artificial atom" to excite laserlike behavior in a superconducting resonator [5,8–12] and above threshold behavior has been observed in one such system [5]. An alternative way of exciting cavity modes which requires neither discrete levels nor an externally applied ac signal, is to harness the Josephson oscillations generated by a dc voltage. Though long studied [13–16], such systems are attracting renewed interest given the potential of current experiments to probe the quantum regime in a carefully controlled way.

In the last few years the properties of the photons emitted into a cavity mode by small voltage-biased Josephson junctions have been investigated both experimentally [17,18] and theoretically [19,20], within the regime where the cavity is close to equilibrium. However, a very recent experiment used an architecture in which a voltage-biased Cooper-pair transistor [12] is embedded within a superconducting microwave cavity [21] to achieve a far-from-equilibrium state with a large photon population [22].

In this Letter we investigate theoretically the quantum dynamics of a model circuit consisting of a voltage-biased Josephson junction and a superconducting cavity. We focus on the regime where a single cavity mode is strongly excited, deriving a Hamiltonian to describe the behavior close to the family of resonances which occur when the voltage energy lost by Cooper pairs traversing the circuit is an integer multiple of the mode frequency. The resulting Hamiltonian describes a nonlinear oscillator which is quite distinct from the Duffing oscillator and other commonly studied nonlinear systems [7]. The system exhibits quadrature and amplitude squeezing over a broad range of parameters. Surprisingly, there are regimes where the fluctuations take on values that are universal in the sense that they are independent of the system's parameters. We note that a study contemporary with ours [23] investigated a similar system quite independently, but in a very different regime.

Model system.—The system we consider is shown schematically in Fig. 1(a); it consists of two Josephson junctions in a SQUID geometry [16] mounted on a wire linking the central conductor and ground plane of a superconducting cavity. The SQUID acts as a single effective junction whose Josephson energy can be tuned by applying a suitable flux [24]. The dc bias is applied via a line which joins the center conductor at the midpoint of the cavity. This geometry (described in detail elsewhere [21,22,25,26]) allows the bias to be applied without affecting the Q factor of cavity modes with voltage nodes at the midpoint where the bias line joins the center conductor [Fig. 1(a)]. The high-Q factors of the modes in the system we consider are

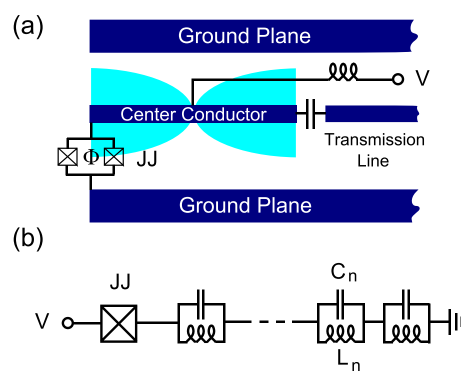

FIG. 1 (color online). (a) Schematic diagram and (b) effective circuit model of the system. It consists of a SQUID containing two Josephson junctions (JJs) in series with a microwave cavity, modeled as a set of LC oscillators, across which a voltage V is applied. The SQUID acts as a single effective junction whose Josephson energy can be tuned by application of a flux, Φ. Coupling of the cavity to a transmission line [shown in (a)] leads to dissipation. High-Q modes have voltage nodes at the point where the bias voltage line joins the center conductor, one such mode shape is indicated in (a). Definitions of the effective circuit parameters are given in [26].

crucial: they allow access to the far-from-equilibrium regime where the photon population is large.

A simple effective circuit model for the system is shown in Fig. 1(b), with the cavity modeled as a set of LC oscillators [26]. The cavity is subject to dissipation arising from couplings between its modes and those of a transmission line.

The Hamiltonian of the effective circuit shown in Fig. 1(b) takes the time-dependent form

$$H = \sum_i \hbar\omega_i a_i^\dagger a_i - E_J \cos\left[\omega_d t + \sum_i \Delta_i (a_i + a_i^\dagger)\right], \quad (1)$$

where a_i is the lowering operator for the ith oscillator with frequency $\omega_i = 1/\sqrt{L_i C_i}$, E_J is the effective Josephson junction energy and $\omega_d = 2eV/\hbar$ is the frequency associated with the bias voltage V [26]. The zero-point displacement of each of the oscillators is given by $\Delta_i = (2e^2\sqrt{L_i/C_i}/\hbar)^{1/2}$.

Resonances occur when the voltage energy lost by an integer number of Cooper pairs traversing the circuit matches the energy required to create photons in one or more of the cavity modes. Here we will explore resonances of the fundamental cavity mode (with frequency ω_0) which occur when $\omega_d \simeq p\omega_0$, with p an integer, and neglect the higher modes in the Hamiltonian (1).

We analyze the system by moving to a rotating frame defined by $U(t) = e^{i(\omega_d/p)a^\dagger a t}$ (dropping the subscript labeling the mode) and derive a time-independent effective Hamiltonian by making a rotating wave approximation (RWA). The RWA should describe the system faithfully when it is very close to resonance, $\omega_0 - \omega_d/p = \delta^{(p)} \ll \omega_0, \omega_d/p$, provided the couplings Δ_0 and E_J are not too strong. We proceed by expressing the sinusoidal term as exponentials and the Baker-Hausdorff formula [27] is used to rewrite the exponentials of $\Delta_0(a + a^\dagger)$ as a product of exponentials of $a^\dagger$ and a. This step leads to normal ordering of $a^\dagger$, a [27] and generates a factor of $e^{-\Delta_0^2/2}$. We then write out the series expansion of the exponentials, dropping terms with explicit time dependence. Finally, we simplify using the expansion of the pth Bessel function, $J_p(z) = \sum_n (-1)^n (z/2)^{2n+p}/n!(n+p)!$ This results in an effective Hamiltonian

$$H^{(p)} = \hbar\delta^{(p)} a^\dagger a - \frac{(-i)^p \tilde{E}_J}{2} : [(a^\dagger)^p + (-1)^p a^p] \frac{J_p\left(2\Delta_0\sqrt{a^\dagger a}\right)}{(a^\dagger a)^{p/2}} :, \quad (2)$$

where $\tilde{E}_J = E_J e^{-\Delta_0^2/2}$ [28] and the colons signify normal ordering.

Taking into account weak coupling between the cavity and the modes in the external microwave transmission line [26], we apply input-output theory [2,27,29] and obtain the Heisenberg equation of motion,

$$\dot{a} = -\left(i\delta^{(p)} + \frac{\gamma}{2}\right)a + \sqrt{\gamma}a_{in} + (-i)^{p-1} \times \frac{\tilde{E}_J\Delta_0}{2\hbar} : \left(\frac{a^\dagger}{a}\right)^{(p-1)/2} J_{p-1}(2\Delta_0\sqrt{a^\dagger a}) : + i^{p-1}\frac{\tilde{E}_J\Delta_0}{2\hbar} : \left(\frac{a}{a^\dagger}\right)^{(p+1)/2} J_{p+1}(2\Delta_0\sqrt{a^\dagger a}) :, \quad (3)$$

where γ and the operator a_{in} describe damping and noise, respectively, arising from the coupling to external modes. Assuming zero temperature, the noise operator is described by the correlation functions [27]: $\langle a_{in}(t)\rangle = \langle a_{in}^\dagger(t)\rangle = 0$, $\langle a_{in}(t)a_{in}(t')\rangle = \langle a_{in}^\dagger(t)a_{in}^\dagger(t')\rangle = \langle a_{in}^\dagger(t)a_{in}(t')\rangle = 0$, and $\langle a_{in}(t)a_{in}^\dagger(t')\rangle = \delta(t - t')$.

We can use Eq. (3) to obtain an approximate description of the average behavior of the system together with the corresponding fluctuations. We make the replacement $a = \alpha + \delta a$ (and a corresponding one for $a^\dagger$), where $\alpha = \langle a\rangle$ is a complex number and the operator δa describes quantum fluctuations. From the definition of α, we see that the average of the fluctuations must vanish, $\langle\delta a\rangle = 0$, and provided they are small we can discard powers of these operators beyond linear order. This amounts to a semiclassical description which also incorporates the zero point fluctuations of the mode [27].

Average dynamics and fluctuations.—The equation of motion for α is obtained by making the replacement $a = \alpha + \delta a$ in Eq. (3), retaining only terms of linear order in δa, and taking the expectation value. Using the definition $\alpha = Ae^{-i\phi}$ to introduce real variables for amplitude A and phase ϕ we find

$$\dot{A} = -\frac{\gamma}{2}A - \frac{\tilde{E}_J\Delta_0}{2\hbar}\sin[p(\phi - \pi/2)][J_{p-1}(2A\Delta_0) + J_{p+1}(2A\Delta_0)], \quad (4)$$

$$\dot{\phi} = \delta^{(p)} - \frac{\tilde{E}_J\Delta_0}{2A\hbar}\cos[p(\phi - \pi/2)][J_{p-1}(2A\Delta_0) - J_{p+1}(2A\Delta_0)]. \quad (5)$$

The system possesses a rich variety of fixed points [13] whose locations (A_0, ϕ_0) follow from Eqs. (4) and (5). Focusing for simplicity on the cases where the system is on resonance ($\delta^{(p)} = 0$), these points can be divided into three classes. For $p > 1$, there is always a fixed point at zero amplitude (though it may not be stable). Beyond this, there are fixed points which owe their existence purely to the presence of dissipation in the system (which we shall refer to as type-I fixed points). These fixed points have phases $\phi_0 = \phi_{\rm I}^{(p)}$ with $\cos[p(\phi_{\rm I}^{(p)} + \pi/2)] = 0$, and the amplitudes $A_0 = A_{\rm I}^{(p)}$ are solutions of

$$A_{\rm I}^{(p)} = \pm\frac{\tilde{E}_J\Delta_0}{\gamma\hbar}[J_{p+1}(2\Delta_0 A_{\rm I}^{(p)}) + J_{p-1}(2\Delta_0 A_{\rm I}^{(p)})]. \quad (6)$$

Finally, there is a set of points related to the extremal points in the underlying Hamiltonian whose amplitudes $A_0 = A_{\rm II}^{(p)}$ are determined by turning points of Bessel functions $J_p'(2\Delta_0 A_{\rm II}^{(p)}) = 0$ with phases given by

$$\sin[p(\phi_{\rm II}^{(p)} - \pi/2)] = -\frac{A_{\rm II}^{(p)}\hbar\gamma}{2\tilde{E}_J\Delta_0 J_{p+1}(2\Delta_0 A_{\rm II}^{(p)})}. \quad (7)$$

The equations of motion for the fluctuations about a given fixed point (A_0, ϕ_0) take the form

$$\begin{pmatrix}\dot{\delta a}\\ \dot{\delta a}^\dagger\end{pmatrix} = \begin{pmatrix} -i[\delta^{(p)} + \nu_{(p)}(A_0,\phi_0)] - \gamma/2 & g_{(p)}(A_0,\phi_0) \\ g^*_{(p)}(A_0,\phi_0) & +i[\delta^{(p)} + \nu_{(p)}(A_0,\phi_0)] - \gamma/2\end{pmatrix}\begin{pmatrix}\delta a\\ \delta a^\dagger\end{pmatrix} + \sqrt{\gamma}\begin{pmatrix}a_{in}\\ a^\dagger_{in}\end{pmatrix}, \quad (8)$$

where

$$\nu_{(p)}(A_0,\phi_0) = \frac{\tilde{E}_J\Delta_0^2}{\hbar}J_p(2\Delta_0A_0)\cos[p(\phi_0 - \pi/2)], \quad (9)$$

$$g_{(p)}(A_0,\phi_0) = -i\frac{\tilde{E}_J\Delta_0^2}{2\hbar}\{J_{p-2}(2\Delta_0A_0)e^{i(p-2)(\phi_0-\pi/2)} + J_{p+2}(2\Delta_0A_0)e^{-i(p+2)(\phi_0-\pi/2)}\}. \quad (10)$$

The eigenvalues of the matrix in (8) determine the stability of the corresponding fixed point; the solution of the coupled equations allows the stationary state fluctuations of the system to be obtained,

$$\langle\delta a\delta a^\dagger + \delta a^\dagger\delta a\rangle = \frac{(\delta^{(p)} + \nu_{(p)})^2 + \gamma^2/4}{(\delta^{(p)} + \nu_{(p)})^2 + \gamma^2/4 - |g_{(p)}|^2}, \quad (11)$$

$$\langle\delta a^2\rangle = \frac{g_{(p)}}{\gamma + 2i(\delta^{(p)} + \nu_{(p)})}\langle\delta a\delta a^\dagger + \delta a^\dagger\delta a\rangle. \quad (12)$$

Fluctuations in the energy of the system are described by the Fano factor $F = (\langle n^2\rangle - \langle n\rangle^2)/\langle n\rangle$, where $n = a^\dagger a$. Because the system has a tendency to possess multiple fixed points with the same amplitude, but different phases, amplitude squeezing (characterized by $F < 1$) occurs more widely than quadrature squeezing. For fixed points where $A_0 \gg 1$, corrections of order $1/A_0$ can be neglected, leading to

$$F = \langle\delta a^\dagger\delta a + \delta a\delta a^\dagger\rangle + e^{2i\phi_0}\langle\delta a^2\rangle + e^{-2i\phi_0}\langle(\delta a^\dagger)^2\rangle. \quad (13)$$

The Fano factor depends on the particular fixed point the system is at (as we discuss below). The most interesting behavior is seen when the system is at one of the type-II fixed points for which we find (on resonance)

$$F = \frac{z_pJ_p(z_p)}{2[z_pJ_p(z_p) - pJ_{p+1}(z_p)]}, \quad (14)$$

where $z = z_p$ is a solution of $J_p'(z) = 0$. Remarkably, these values depend only on the resonance and fixed point involved and are universal in the sense that they are independent of the system's parameters.

It should certainly be possible to measure the quantum fluctuations in experiment. Squeezing has already been measured in several microwave systems [30–33]. Furthermore, reconstruction of the full Wigner function of a microwave field using quadrature measurements (an approach well suited to the large photon numbers states relevant here) was demonstrated recently [2].

One-photon resonance.—We now examine in detail the one-photon resonance ($p = 1$) at $\omega_d = \omega_0$, focusing on the behavior as a function of E_J [34] (since this could in practice be varied via the flux applied to the SQUID [24]). Figure 2 shows how the average energy of the steady state $\langle n\rangle$ evolves with E_J when the system is on resonance ($\delta^{(1)} = 0$). The analytical results, given by the square of the corresponding stable fixed point amplitude, are compared with a numerical solution [35] of the Lindblad master equation equivalent to Eq. (3), which provides a check on the validity of the analytical approach [36].

For very small E_J there is a single stable (type-I) fixed point whose amplitude grows linearly with E_J, hence the

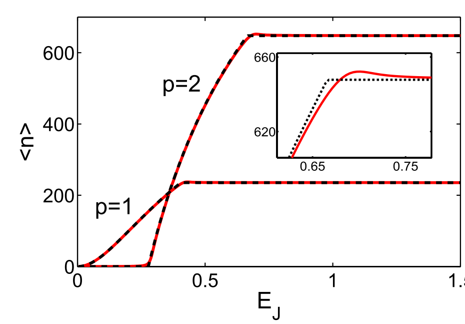

FIG. 2 (color online). Average energy $\langle n\rangle$ calculated numerically for the full quantum problem (full curves) compared with stable fixed point amplitudes (dashed curves) for the $p = 1, 2$ resonances. The stable fixed point changes from $A_{\rm I}$ to $A_{\rm II}$ at $E_J = 0.405$ (0.666) for $p = 1(2)$ and the threshold where $A = 0$ becomes unstable for $p = 2$ is $E_J^c = 0.278$. The inset is a magnified part of the $p = 2$ curves. Adopting units where $\hbar\omega_0 = 1$, we take $\hbar\gamma = 10^{-3}$, $\Delta_0 = 0.06$, $\delta^{(1)} = \delta^{(2)} = 0$, values which are used throughout.

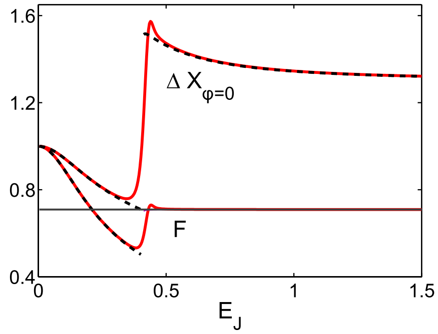

FIG. 3 (color online). Fluctuations in the energy F (lower full curve) and quadrature $\Delta X_{\phi=0}$ (upper full curve) calculated numerically as a function of E_J for $p = 1$. Analytic results for the fixed points $A_{\rm I}^{(1)}$ (for $E_J < 0.405$) and $A_{\rm II}^{(1)}$ (for $E_J \geq 0.405$) are shown as dashed curves. The horizontal line is the value of F given by Eq. (14).

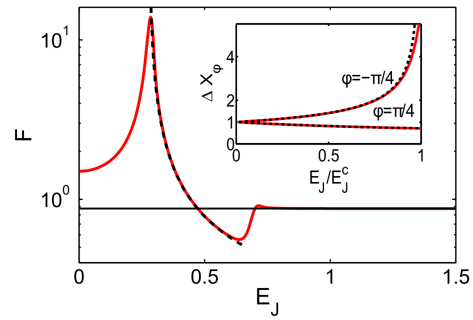

FIG. 4 (color online). Fluctuations at the $p = 2$ resonance. The main plot shows F, numerics (full curve) are compared with analytic results when $A_0 > 0$: the dashed curve is for the $A_{\rm I}$ fixed points and the horizontal line, given by Eq. (14), describes the $A_{\rm II}$ fixed points. Threshold is $E_J^c = 0.278$ and the bifurcation between $A_{\rm I}$ and $A_{\rm II}$ fixed points is at $E_J = 0.666$. The inset compares analytic (dashed curves) and numerical (full curves) calculations of $\Delta X_{\phi=\pm\pi/4}$ below threshold.

energy initially grows quadratically (see Fig. 2). This behavior is easily understood by expanding Eq. (2) for $p = 1$ to lowest order in Δ_0; the resulting Hamiltonian $H^{(1)} \simeq i(E_J\Delta_0/2)(a^\dagger - a)$ describes a (linearly) driven harmonic oscillator. As E_J increases further nonlinear effects become important and this approximate Hamiltonian becomes inadequate.

A bifurcation occurs when $\tilde{E}_J = \hbar\gamma z_1/[4J_0(z_1)\Delta_0^2]$, where $z_1 \simeq 1.841$ is the first maximum of $J_1(z)$, at which the type-I fixed point becomes unstable. Above the bifurcation there are two stable type-II fixed points which both have amplitude $A_{\rm II}^{(1)} = z_1/(2\Delta_0)$ (independent of both E_J and γ).

The fluctuations of the cavity mode are shown in Fig. 3. Amplitude squeezing occurs across the whole parameter regime studied and quadrature squeezing below the bifurcation between the type-I and type-II fixed points. Approaching the bifurcation from below, the linearized theory predicts $F \to 0.5$. For $p = 1$ amplitude squeezing coincides with quadrature squeezing at the type-I fixed point with $\Delta X_{\phi=0}^2 = F$, where we define the quadrature $X_\phi = ae^{-i\phi} + a^\dagger e^{i\phi}$. Above the bifurcation F saturates rapidly to the universal value 0.7092 given by Eq. (14) for $p = 1$.

Two-photon resonance—Next we turn to the behavior of the system at the two-photon ($p = 2$) resonance. The average energy of the cavity for $p = 2$ is shown as a function of E_J in Fig. 2. In this case there is a clear threshold after which the energy rises rapidly before leveling off and becoming independent of E_J.

The threshold arises because $A = 0$ is now a stable fixed point for sufficiently small E_J and significant occupation of the cavity only occurs when it becomes unstable at $\tilde{E}_J = \hbar\gamma/\Delta_0^2$. For weak couplings and low photon numbers we can again expand the Hamiltonian to lowest order in Δ_0; in this case $H^{(2)} \simeq E_J(\Delta_0/2)^2(aa + a^\dagger a^\dagger)$. This limiting form of the Hamiltonian is that of a degenerate parametric amplifier [37] and the two systems behave in the same way in the below-threshold regime [19].

Above threshold, a pair of type-I fixed points (which both have the same amplitude) is stable. The amplitude increases rapidly with E_J until these points in turn become unstable with a bifurcation at $\tilde{E}_J = \hbar\gamma z_2/[4J_1(z_2)\Delta_0^2]$, where $z_2 \simeq 3.054$ is the first maximum of $J_2(z)$. Above this bifurcation a new set of type-II stable fixed points emerge with amplitudes $A_{\rm II}^{(2)} = z_2/(2\Delta_0)$.

The corresponding fluctuations of the cavity mode are shown in Fig. 4. Below threshold the system behaves like the degenerate parametric amplifier [37] displaying squeezing of $X_{\phi=\pi/4}$. The linear theory predicts $\Delta X_{\phi=\pi/4}^2 \to 0.5$ as the threshold is approached from below while the uncertainty in the conjugate quadrature $\Delta X_{\phi=-\pi/4}$ diverges. The threshold is accompanied by a peak in F, which then drops abruptly and the linear theory again gives $F \to 0.5$ at the bifurcation between the type-I and type-II fixed points. Above the second bifurcation F goes to the universal value 0.8753 predicted by Eq. (14).

Conclusions.—We derived an effective Hamiltonian describing an experimentally accessible Josephson junction-cavity system close to resonances which occur when Cooper pairs crossing the junction excite photons in a cavity mode. The system displays amplitude and quadrature squeezing for a wide range of parameters. Furthermore, the amplitude fluctuations of the cavity mode can take universal values which are independent of the system's parameters.

Our work provides a starting point for a number of interesting future studies. The RWA Hamiltonian [Eq. (2)] can be used investigate the quantum dynamics of the system beyond the regime of linear fluctuations [7,38,39]. It would also be interesting to explore the behavior of the single mode system in the regime where E_J is large. Classically, the system can undergo period doubling bifurcations and become chaotic as E_J is increased, but in this regime the RWA will be inadequate and a more comprehensive description is required. Finally, the RWA approach can be extended in a straightforward way to analyze situations where the bias voltage is chosen to excite two modes [17] simultaneously [see Eq. (1)], where interesting correlations between the modes may be expected.

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