Bootstrapping Conformal Field Theories with the Extremal Functional Method

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The existence of a positive linear functional acting on the space of (differences between) conformal blocks has been shown to rule out regions in the parameter space of conformal field theories (CFTs). We argue that at the boundary of the allowed region the extremal functional contains, in principle, enough information to determine the dimensions and operator product expansion (OPE) coefficients of an infinite number of operators appearing in the correlator under analysis. Based on this idea we develop the extremal functional method (EFM), a numerical procedure for deriving the spectrum and OPE coefficients of CFTs lying on the boundary (of solution space). We test the EFM by using it to rederive the low lying spectrum and OPE coefficients of the two-dimensional Ising model based solely on the dimension of a single scalar quasiprimary—no Virasoro algebra required. Our work serves as a benchmark for applications to more interesting, less known CFTs in the near future.

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Introduction and preliminaries.—It is well known [1–8] that the global conformal symmetry group in D dimensions $SO(D+1,1)$ implies that the four-point function of four identical scalars with conformal dimension Δ_σ takes the form

$$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\sigma}x_{34}^{2\Delta_\sigma}}, \tag{1}$$

with $x_{ij}\equiv x_i - x_j$, and where $g(u,v)$ is a function of the conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2}$$

The existence of the operator product expansion (OPE) of the theory [9] implies that the function $g(u,v)$ can be written in two inequivalent ways, corresponding to expanding the correlator around $x_1\simeq x_2$, $x_3\simeq x_4$ or $x_1\simeq x_3$, $x_2\simeq x_4$ (i.e., the direct and crossed channels, respectively). For instance, in the direct channel we have

$$g(u,v) = 1 + \sum_{\Delta,L}\lambda^2_{\mathcal{O}_{\Delta,L}} G_{\Delta,L}(u,v). \tag{3}$$

The sum is over symmetric traceless even spin, L, primaries of the conformal group labeled by their conformal dimensions Δ. The restriction to even spin follows from the Bose symmetry of the scalars [10]. The contribution from each primary and its descendants is given by the function $G_{\Delta,L}(u,v)$, known as a conformal block [11–15]. The accompanying numbers $\lambda_{\mathcal{O}_{\Delta,L}}$ are the OPE coefficients appearing in the three-point function $\langle\sigma\sigma\mathcal{O}_{\Delta,L}\rangle$. The 1 in the sum above is the contribution of the conformal block of the identity which always appears in the four-point function of identical scalars.

Equivalence of the expansion in the direct and crossed channels implies the identity

$$\sum_{\Delta,L}\lambda^2_{\mathcal{O}} F^{(\sigma)}_{\Delta,L}(u,v) = 1, \tag{4}$$

with

$$F^{(\sigma)}_{\Delta,L}(u,v) \equiv \left(\frac{v^{\Delta_\sigma}G_{\Delta,L}(u,v) - u^{\Delta_\sigma}G_{\Delta,L}(v,u)}{u^{\Delta_\sigma} - v^{\Delta_\sigma}}\right).$$

Solving this relation is nontrivial: For instance, the contribution of a single conformal block in one channel must necessarily be matched by an infinite sum of conformal blocks in the cross channel. The idea of using relation (4), together with analogous ones for other correlation functions, to determine the spectrum and OPE coefficients of a conformal field theory (CFT) is known as the conformal bootstrap [16,17], and has been recently revived in the works [10,18–27]. A related bootstrap technique, using a somewhat different methodology, was recently introduced in [28].

The crucial handle into attacking the problem is to notice the positivity of the coefficients $\lambda^2_{\mathcal{O}_{\Delta,L}}$ (following from unitarity), which leads us to consider all possible positive linear combinations of the $F^{(\sigma)}_{\Delta,L}$. This problem is simplified by thinking of each $F^{(\sigma)}_{\Delta,L}$ as a *vector* in an infinite dimensional space with components given by Taylor coefficients (expanded around some judiciously chosen point as described in the next section). For more details on this method see, e.g., [25]. The set of all vectors form a semi-polyhedral cone, so we can rephrase the problem of solving the bootstrap constraints (4) as the question, under which assumptions does the cone generated by the $F^{(\sigma)}_{\Delta,L}$ vectors contain the constant function 1, the "identity vector"?

By imposing various constraints on the spectrum, the identity may lie inside or outside the cone, but something very interesting happens when the identity vector lies exactly on the face of the polytope formed by the $F^{(\sigma)}_{\Delta,L}$. At precisely this point there is generically a single solution to crossing symmetry, since by convexity we are only allowed to use the vectors which make up the vertices of the face through which the identity vector is cutting. In the language of linear functionals, there exists a hyperplane containing the face of the cone along which the identity vector lies. The zeros of this unique functional are precisely the vectors which define the face through which the identity vector is passing. This uniqueness provides a recipe for constructing solutions to crossing symmetry, the extremal functional method (EFM): (i) Find the extremal linear functional ϕ. (ii) Compute the vectors $F^{(\sigma)}_{\Delta,L}$ which are zeros of ϕ. (iii) Solve for the linear combination of $F^{(\sigma)}_{\Delta,L}$'s which gives the identity vector. The coefficients are the square of the OPE coefficients. In this Letter, we test this method by reconstructing the spectrum of the two-dimensional Ising model to high accuracy. We hope to apply it in the future to more interesting cases, such as the three-dimensional Ising model.

Warm-up: Two derivatives.—In order to obtain a tractable problem, we discretize the infinite set of constraints in (4). We do this by first setting $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$, followed by an expansion around $z = \bar{z} = 1/2$. More concretely we set $z = \frac{1}{2}(1 + a + \sqrt{b})$, $\bar{z} = \frac{1}{2}(1 + a - \sqrt{b})$ and expand in a and b to some finite order. After discretization the constraints (3) can be thought of as demanding that the constant vector 1 lies in the cone spanned by positive linear combinations of the vectors $F^{(\sigma)}_{\Delta,L}$.

The simplest nontrivial example corresponds to considering the constant term and linear terms in a and b in the expansion, giving us three component vectors. That is, Eq. (4) now becomes the vector equation

$$\sum_{\Delta,L} \lambda^2_{\mathcal{O}_{\Delta,L}} \begin{pmatrix} F^{(\sigma)}_{\Delta,L}\left(\frac{1}{4},\frac{1}{4}\right) \\ \partial_a F^{(\sigma)}_{\Delta,L}\left(\frac{1}{4},\frac{1}{4}\right) \\ \partial_b F^{(\sigma)}_{\Delta,L}\left(\frac{1}{4},\frac{1}{4}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \qquad (5)$$

We are free to redefine the coefficients $\lambda^2_{\mathcal{O}_{\Delta,L}}$ to $\hat{\lambda}^2_{\mathcal{O}_{\Delta,L}} = \lambda^2_{\mathcal{O}_{\Delta,L}} F^{(\sigma)}_{\Delta,L}(\frac{1}{4},\frac{1}{4})$, and so the first equation becomes the normalization condition

$$\sum_{\Delta,L} \hat{\lambda}^2_{\mathcal{O}_{\Delta,L}} = 1. \qquad (6)$$

The rescaling is possible because the sign of $F^{(\sigma)}_{\Delta,L}(\frac{1}{4},\frac{1}{4})$ is always positive in the region of interest. This reduces the dimension of the vector space by one so that we are now looking at a slice of the cone corresponding to the intersection with the plane transverse to (1,0,0). This yields a two-dimensional convex polytope, namely the convex hull of all possible vectors $(\partial_a F/F, \partial_b F/F)$ and we are interested in determining the circumstances under which the origin is contained in it. Figure 1 shows the polytope in $(\partial_a F/F, \partial_b F/F)$ space corresponding to

$$\Delta \geq L + D - 2, \quad (L \geq 2), \quad \Delta \geq \Delta_\epsilon, \quad (L = 0),$$

where $\Delta_\epsilon > 0$ is an arbitrary gap we impose (and vary) in the scalar spectrum.

This is done for $d = 2$ and $\Delta_\sigma = 0.125$, as we wish to compare with the two-dimensional Ising model.

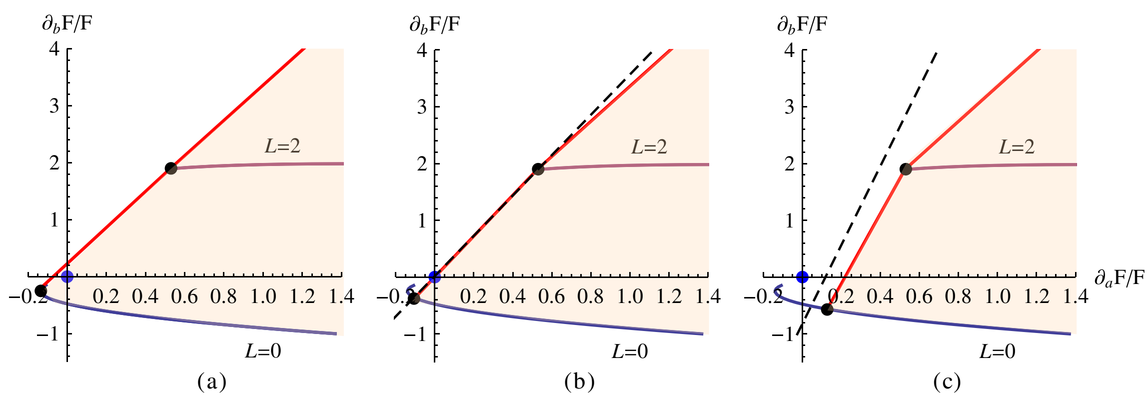

FIG. 1 (color online). Convex hull formed by the $F^{(\sigma)}_{\Delta,L}$ vectors. In blue and purple the spin-0 and spin-2 vectors, respectively. The lines start at the unitarity bounds ($\Delta = 0$, 2, respectively, on the left and Δ increases to the right). Higher spin curves lie outside the plot. The red curve shows part of the boundary of the polytope, the other being the spin-0 line in blue. A solution to crossing symmetry exists whenever the origin (in blue) is contained inside the polytope. In the critical case (b) a solution must involve the vectors marked with black dots. The dashed line is the linear functional separating the origin from the polytope, and it overlaps with a polytope face in the critical case.

Represented are the scalar and spin-2 lines, corresponding to the sets of vectors with spins 0 and 2, and arbitrary Δ. Increasing the conformal dimension shifts the vectors to the right-hand side of the plot, tracing the lines shown there in blue and purple (i.e., each point along the colored lines corresponds to a given Δ with $L = 0$, 2, respectively). There are also other lines corresponding to higher spin fields lying outside the range of the plot above. Altogether, the convex hull of all these vectors forms an unbounded polytope, one of whose faces runs from the tips of the scalar and spin-2 lines, another being the scalar line in blue.

If we allow all possible spin-0 fields consistent with unitary (i.e., $\Delta_\epsilon = 0$), it is clear that the origin lies well inside the polytope [Fig. 1(a)]. This means that within our approximation, we cannot rule out CFTs with this unrestricted spectrum. As we increase the gap in the scalar spectrum (i.e., allow only $\Delta > \Delta_\epsilon$ for increasing values of Δ_ϵ), the boundary of the polytope shifts, following the scalar line up to a critical point Δ_ϵ^*. At this critical point the face which runs from the tip of the scalar line to the tip of the spin-2 line cuts across the origin [Fig. 1(b)]. Finally, increasing Δ_ϵ further in Fig. 1(c) we see that the origin is not contained in the polytope and hence there is no solution to the crossing constraints. Accordingly, it is possible to find a linear functional separating the identity vector from the polytope, as shown by the dashed line in the same figure.

At the critical point in Fig. 1(b) the solution is unique (generically). By convexity it is clear that the solution to crossing symmetry must include the critical scalar of dimension Δ_ϵ^*, and the field at the tip of the spin-2 line, which is of course none other than the stress tensor, and no other operator. So, at this point we have found the two unique operators which must be in the spectrum within this two-derivative approximation. Numerically one finds that the critical scalar has dimension $\Delta_\epsilon^* \simeq 1.03$.

To compute the OPE coefficients we simply solve (5) for $\lambda_{\mathcal{O}_{\Delta_\epsilon,0}}$ and $\lambda_{\mathcal{O}_{2,2}}$ (with all other λ vanishing), yielding a system of two equations for two variables, whose solution is

$$\Delta_\sigma = 0.125, \qquad \Delta_\epsilon \simeq 1.03, \qquad \lambda_{\sigma\sigma\epsilon} \simeq 0.24, \qquad c \simeq 0.45, \tag{7}$$

with c the central charge, related to the $\sigma\sigma T$ OPE coefficient $\lambda_{\sigma\sigma T}$ by $c = \Delta_\sigma^2/\lambda_{\sigma\sigma T}^2$. The two-dimensional Ising model satisfies

$$\Delta_\sigma = 0.125, \qquad \Delta_\epsilon = 1 \qquad \lambda_{\sigma\sigma\epsilon} = 0.25, \qquad c = 0.5, \tag{8}$$

and so even with this very basic two derivative approximation we already get something quite reasonable.

We can rephrase the problem in terms of linear functionals by demanding the existence of such a functional which is positive when acting on all vectors except the identity, where it is constrained to be negative. If we can find such a functional then we have proven that there is no solution to crossing symmetry. In the extremal case the functional will be positive on all vectors except a minimal set required to span a codimension one hyperplane. It is precisely these vectors that must appear in the solution to crossing symmetry.

To see this in our simple example recall that in the polytope picture the plane becomes a line [i.e., the intersection of the plane with the plane transverse to (1,0,0)]. In Fig. 1(b) we see that this extremal line overlaps with the face spanned by the critical scalar and the stress tensor. These two vectors are therefore zeros of the original linear functional, and the functional should be positive when acting on all other vectors.

Applying EFM to the $D = 2$ Ising model.—Motivated by these results, we add more derivatives of the F functions making the vectors larger, and thereby enlarging the dimension of our search space. In this way we are capturing more and more information about the shape of the full cone, and therefore about the spectrum. A natural (and it turns out good) way to parametrize this approach is the number N of components in our vectors or simply the dimension of the search space.

To study the $D = 2$ Ising model we set $\Delta_\sigma = 0.125$ and look for an extremal functional by decreasing the value of Δ_ϵ to just above the value where it first starts being possible to solve crossing symmetry (i.e., the largest possible gap in the scalar spectrum where a solution exists). This choice of Δ_σ is an input (borrowed from known exact results) rather than an output of our method; given the dimension of the external scalar the EFM computes all other operators appearing in its OPE. Ignorance of Δ_σ in, e.g., the $D = 3$ Ising model is the main reason our method does not immediately generalize to solving that theory. On the other hand, by making additional assumptions, such as the minimization of the central charge as a function of Δ_σ (see, e.g., [29]), we could indeed determine even this scaling dimension to quite high precision (and then use it as an input in the rest of the procedure outlined below). This approach to fixing Δ_σ (for the $D = 3$ Ising model) will be the subject of a future study [30].

Consider starting with small N and gradually increasing it up to a maximum of 60 components. As the number of derivatives is increased, the boundary of the allowed region shrinks towards lower values of Δ_ϵ^*. We are guaranteed to improve our bound as N increases since, for a given N, we can always use the functional at $N - 1$ by simply adding an extra zero component. We find empirically that indeed the value of Δ_ϵ^* systematically decreases towards $\Delta_\epsilon^* = 1$. For the maximum value of N we considered here we obtain the correct value $\Delta_\epsilon = 1$ to about six decimal places.

In Fig. 2 we show a plot of the extremal functional dotted into the conformal block vectors of spin 2. Notice the logarithmic scale. The sharp downward spikes correspond to the zeros of the functional, so for instance from

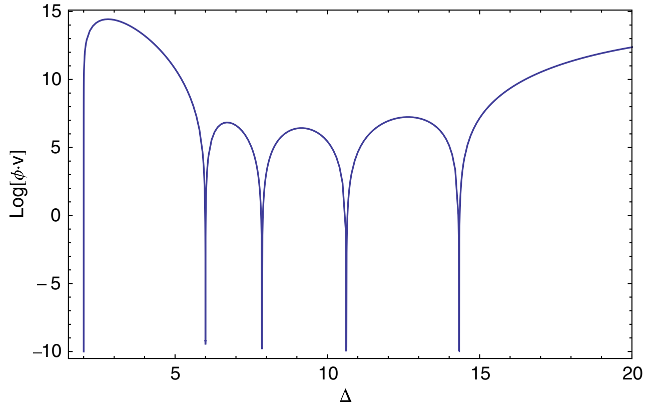

FIG. 2 (color online). The extremal functional acting on the spin-2 vectors.

this spin-2 plot we read off operators with dimensions $\simeq 2, 6, 7.9, \ldots$. Similar plots can be made for other spins. We will compare these spectra with those of the two-dimensional Ising model shortly but first we need to address an important question: which operators should we trust to be in the true spectrum? Indeed, much as Δ_ϵ converges to the correct value only for a high enough number of derivatives, the same is true for the other operators. We therefore need a criterion for deciding whether a given operator has stabilized or not.

The idea is to estimate the error from the variation of the operator dimensions as N increases. As an example we show the spectrum in the scalar channel in Fig. 3. The pattern is that as N increases, the functional develops new zeros and shifts the positions of old ones until they eventually stabilize. This leads us to the following:

(i) *Criterion 1*: An operator has converged and should be trusted to appear in the actual correlator if $\delta\Delta/\Delta < 1\%$. In this note $\delta\Delta \equiv \Delta_{N=60} - \Delta_{N=58}$.

The next step is to compute the OPE coefficients. We select some subset of the total components and restrict ourselves to this lower dimensional vector space. By restricting to only these components we can then find a set of OPE coefficients which is optimal, in the sense of minimizing the value of the crossing symmetry constraints,

$$\text{OPE coeffs} = \min_{\{\lambda_i\}}\Big(\sum_{V_i:\ \phi\cdot V=0} \lambda_i V_i - \mathbb{1}\Big). \tag{9}$$

We minimize over the value of the OPE coefficients λ_i, where i labels vectors which are zeros of the extremal functional $\phi \cdot V_i = 0$. We take *all* zeros of the functional, not only those deemed to have stabilized. Moreover, all the vectors appearing above have been projected onto a lower dimensional subspace of the original set of derivatives (so they have less than N components).

The solution is sensitive to the choice of components used, but there is an independent metric for determining the quality of a set of OPE coefficients, namely the value of the minimum above, and it is this metric we use to select how many, and which components we keep from the original vector. In practice we have found that good results for the OPE coefficients can be obtained by minimizing the above taking into account only the first $Z + 1$ components of the Z zero vectors V.

At the end of this procedure we have a putative spectrum, comprising a set of operators—zeros of the extremal functional—and their respective OPE coefficients. We can do this for any N, and from this determine an approximate error:

(ii) *Criterion 2*: An OPE coefficient $\lambda_{\mathcal{O}}$ has converged, and should be trusted to be approximately correct if $\delta\lambda_{\mathcal{O}}/\lambda_{\mathcal{O}} < 15\%$. In this note $\delta\lambda_{\mathcal{O}} \equiv |\lambda_{\mathcal{O}_{N=60}} - \lambda_{\mathcal{O}_{N=58}}|$.

Notice that it only makes sense to talk about OPE convergence if there is an associated operator, and so criterion 2 can only be applied on operators satisfying criterion 1. The large disparity in our choice of cutoffs is mostly due to the fact that we find that operator dimensions converge more quickly than OPE coefficients, and so having a small cutoff for OPE coefficients would be too restrictive.

We would like to add that while our criteria involve an arbitrary choice of cutoff (1% and 15%), the methodology itself provides a natural error estimate, the variances $\delta\Delta/\Delta$ and $\delta\lambda/\lambda$ that *bound* the error when compared with the exact results.

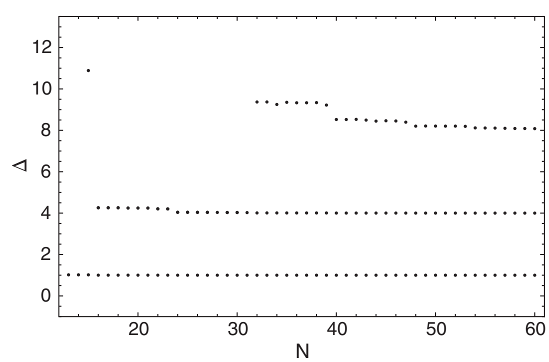

FIG. 3. Evolution of the zeros of the extremal functional in the scalar channel.

Results.—In Table I we present some of the operators and respective OPE coefficients obtained with our methods, side by side with those corresponding to the

TABLE I. Low-lying two-dimensional Ising spectrum.

Exact		EFM	
(Δ, L)	OPE	(Δ, L)	OPE
(1,0)	0.5	(1.0000025,0)	0.4999997
(4,0)	0.015625	(4.00030,0)	0.0156241
(2,2)	0.1767767	(2,2)	0.1767772
(6,2)	0.00262039	(5.99787,2)	0.00261753
(4,4)	0.0209531	(4,4)	0.0209626

two-dimensional Ising model. The agreement is impressive, especially for the low-lying operators. Our method yields, however, many more operators. Following is a summary of our total results: (i) The correct value for the dimension and OPE coefficient of the operator ϵ to 6 digits accuracy. (ii). The correct central charge to within 0.0005%. (iii) 7, 10, 15, and 19 operators with OPE coefficients correct within 0.01%, 0.1%, 1%, and 10%, respectively. Our highest operator has dimension 15, spin 14 with 30% error on the OPE coefficient. The estimated errors compare well with the actual ones.

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