Ultraviolet Properties of $\mathcal{N} = 4$ Supergravity at Four Loops

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(Received 16 September 2013; published 3 December 2013)

We demonstrate that pure $\mathcal{N} = 4$ supergravity is ultraviolet divergent at four loops. The form of the divergence suggests that it is due to the rigid $U(1)$ duality-symmetry anomaly of the theory. This is the first known example of an ultraviolet divergence in a pure ungauged supergravity theory in four dimensions. We use the duality between color and kinematics to construct the integrand of the four-loop four-point amplitude, whose ultraviolet divergence is then extracted by standard integration techniques.

DOI: 10.1103/PhysRevLett.111.231302 PACS numbers: 04.65.+e, 11.15.Bt, 11.25.Db, 12.60.Jv

Recent years have seen enormous advances in our ability to obtain scattering amplitudes in gauge and gravity theories. Using these advances we can address basic questions on the ultraviolet properties of quantum gravity that had seemed relegated to the dustbin of undecidable questions. Power-counting arguments suggest that all pointlike theories of gravity should be ultraviolet divergent. However, such arguments can be misleading if there are additional hidden symmetries or structures. In particular, the duality between color and kinematics [1,2] has been shown to be responsible for improved ultraviolet behavior in the relatively simple two-loop case of half-maximal supergravity in five dimensions [3]. This example emphasizes the importance of carrying out more general investigations of the ultraviolet properties of supergravity theories to ascertain the full implications of new structures.

Pure Einstein gravity has long been known to be finite at one loop [4] but divergent at two loops [5]. It also diverges at one loop under the addition of generic matter [4,6]. However, the situation with pure ungauged supergravity is less clear. Such theories are known not to diverge prior to three loops [7]. The consensus from studies in the 1980s was that all supergravity theories likely diverge at three loops (see, for example, Ref. [8]), although with appropriate assumptions tighter bounds are possible [9]. However, it was not possible to check these arguments until the advent of the unitarity method [10,11]. For the most supersymmetric case of $\mathcal{N} = 8$ supergravity [12], explicit calculations have shown that the four-point amplitudes are finite at three loops for dimensions $D < 6$ [13] and at four loops for dimensions $D < 11/2$ [14]. These ultraviolet cancellations were subsequently shown to be a consequence of supersymmetry and the $E_{7(7)}$ duality symmetry of the theory [15,16]. However, a D^8R^4 counterterm appears to be valid under all standard symmetries, leading to predictions of a seven-loop divergence in $\mathcal{N} = 8$ supergravity in $D = 4$.

While seven loops is at present out of reach of direct computations, reducing the supersymmetry lowers the loop order at which nontrivial ultraviolet cancellations can be studied. As discussed in Ref. [16], the same type of symmetry argument used for $\mathcal{N} = 8$ supergravity at seven loops also implies the existence of an apparently valid three-loop R^4 counterterm in $\mathcal{N} = 4$ supergravity [17]. This suggests that pure $\mathcal{N} = 4$ supergravity should diverge at three loops. This is consistent with speculations based on the pattern of cancellations at one loop, suggesting that at least $\mathcal{N} \geq 5$ supergravity is needed to tame ultraviolet singularities [18].

However, as recently demonstrated, the coefficient of the potential three-loop four-point divergence of $\mathcal{N} = 4$ supergravity actually vanishes [19]. (See Ref. [20] for a string-theory argument.) Another related example is the unexpected finiteness of the two-loop four-point amplitude of half-maximal supergravity in five dimensions [3,20]. By assuming the existence of appropriate 16-supercharge superspaces, the observed finiteness can be understood as a consequence of standard symmetries [21]. However, these superspaces also lead to predictions in direct contradiction to explicit calculations when matter multiplets are added [22], implying that the assumption needs to be altered. There are also conjectures that certain structures or hidden symmetries may play a role [23]. In any case, these examples remain unexplained from standard symmetry considerations. This makes it important to investigate the next loop order. If there are no additional cancellations at four loops beyond the ones already identified at three loops, either in string theory or in field theory, it should diverge [20,21].

In this Letter, we compute the four-loop four-point divergence of $\mathcal{N} = 4$ supergravity following the same basic methods used in the corresponding three-loop computation [19] and described in some detail in Ref. [22]. We find that although $\mathcal{N} = 4$ supergravity does have an ultraviolet divergence, its form suggests that it is special and tied to the $U(1)$ duality anomaly of the theory.

Our construction of the four-loop four-point amplitude of $\mathcal{N} = 4$ supergravity starts with the corresponding pure Yang-Mills Feynman diagrams in Feynman gauge. To obtain $\mathcal{N} = 4$ supergravity, we also need the $\mathcal{N} = 4$ super-Yang-Mills diagram kinematic numerators listed in Ref. [24] that obey the duality between color and kinematics. In this form the kinematic-numerator factors n_i satisfy algebraic relations in one-to-one correspondence with relations satisfied by the color factors c_i. These factors are associated with 85 diagrams (plus permutations of external legs) containing only cubic vertices, as illustrated in Fig. 1. The $\mathcal{N} = 4$ supergravity integrands are obtained simply by replacing the color factors c_i in the pure-Yang-Mills integrand with the corresponding $\mathcal{N} = 4$ super-Yang-Mills kinematic-numerator factors,

$$c_i \to n_i. \tag{1}$$

The construction of the supergravity integrand via the duality between color and kinematics automatically satisfies the D-dimensional unitarity cut constraints, given that the input gauge-theory amplitudes have the correct cuts.

The $\mathcal{N} = 4$ super-Yang-Mills numerators [24] used in the construction are proportional to the color-ordered $\mathcal{N} = 4$ super-Yang-Mills tree-level amplitudes $A^{\text{tree}}_{\mathcal{N}=4}$, which can be conveniently expressed in an on-shell superspace formalism in four dimensions [25]. As an example, diagram 1 in Fig. 1 has a numerator given by $n_1 = s^4 t A^{\text{tree}}_{\mathcal{N}=4}$, where $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ are standard Mandelstam invariants. The remaining numerator factors are specified in Ref. [24] and are, in general, somewhat more complicated, depending also on loop momenta.

Using Feynman diagrams for the nonsupersymmetric pure Yang-Mills amplitude might seem inefficient, but for the problem at hand it is a reasonable choice. It automatically gives us local covariant expressions with no spurious singularities that could complicate loop integration. Moreover, only the relatively small subset of diagrams containing color factors matching those of the nonvanishing diagrams in the corresponding $\mathcal{N} = 4$ super-Yang-Mills theory are needed, otherwise the contribution vanishes as well in $\mathcal{N} = 4$ supergravity. Feynman diagrams also avoid subtleties associated with the bubble-on-external-leg diagrams, such as diagram 85 of Fig. 1.

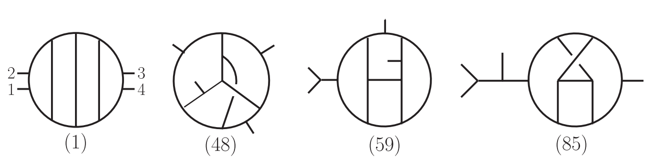

FIG. 1. Four of the 85 diagrams with cubic vertices used to organize the $\mathcal{N} = 4$ super-Yang-Mills amplitudes into a form that respects the duality between color and kinematics. The remaining diagrams are listed in Ref. [24].

After integration all such pure Yang-Mills Feynman diagrams are smooth in the on-shell limit, canceling the $1/k^2$ propagator as $k^2 \to 0$. In $\mathcal{N} = 4$ supergravity such contributions vanish because the color factors in the pure Yang-Mills diagrams are replaced by vanishing numerator factors independent of loop momentum [24].

The logarithmic ultraviolet divergence may be extracted by series expanding in small external momenta, or equivalently large loop momenta [26]. The resulting tensor integrals are then reduced to scalar integrals via Lorentz invariance. We regularize the integrals using dimensional reduction [27]. Further details of the procedure are given in Ref. [22].

The small-momentum expansion has the undesired effect of introducing new unphysical infrared singularities. To separate out all resulting infrared divergences from the ultraviolet ones, we use a mass regulator. A particularly convenient choice is to introduce a uniform mass into all Feynman propagators prior to expanding in external momenta [28]. For the case of pure $\mathcal{N} = 4$ supergravity with no matter multiplets, with this regulator, the subdivergences should all cancel amongst themselves because there are no one-, two- or three-loop divergences. This can be used to greatly simplify the computation since we do not need to compute subdivergences. However, we compute them regardless, using their cancellation as a nontrivial consistency check. More generally, the issue of infrared regularization is delicate because of regulator dependence. For example, if the mass regulator were introduced after the expansion in external momenta, it would ruin the cancellation of subdivergences between different integrals, and one would need to include all subdivergence subtractions to remove the regulator dependence.

At the end of this process, we obtain a large number of vacuum integrals with the two basic diagrammatic structures shown in Fig. 2. These are of the form

$$\int \prod_{j=1}^{4} \frac{d^D p_j}{(2\pi)^D} \frac{P(m^2, p_1 \cdot p_2)}{\prod_{i=1}^{9} (p_i^2 - m^2)^{a_i}}, \tag{2}$$

where P is a numerator polynomial in the mass and the irreducible dot product formed from the momenta flowing through propagators 1 and 2, indicated in Fig. 2. (By irreducible we mean that it cannot be expressed as a linear combination of inverse propagators and masses.) The 9 p_i correspond to the 9 propagators in each of the vacuum diagrams of Fig. 2, with the first four being independent loop momenta. The indices a_i are integers.

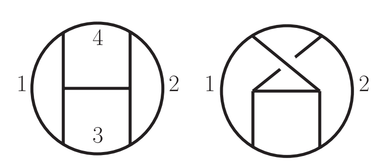

FIG. 2. The two basic vacuum graphs.

The standard modern way to evaluate these vacuum integrals is to use integration-by-parts relations [29] within dimensional regularization. This allows us to write down any given integral as a linear combination of so-called master integrals which can then be evaluated. For four-loop Feynman vacuum integrals, this was done in Ref. [30]. In our calculation, the reduction to master integrals turns out to be complicated because high powers of numerator loop momenta are involved. To deal with this, we use the C++ version of the code FIRE [31], implementing the Laporta algorithm [32]. We use the same master-integral basis set as in Ref. [33]. (See Ref. [34] for a high-precision numerical evaluation.)

Each state of pure $\mathcal{N}=4$ supergravity is a direct product of a color-stripped state of $\mathcal{N}=4$ super-Yang-Mills theory and of pure nonsupersymmetric Yang-Mills theory. Pure $\mathcal{N}=4$ supergravity contains two multiplets that do not mix under linearized supersymmetry: one contains the negative-helicity graviton and the other the positive-helicity graviton. We find that all amplitudes in pure $\mathcal{N}=4$ supergravity are divergent at four loops,

$$\mathcal{M}^{4-\text{loop}}\Big|_{\text{div.}} = \frac{1}{(4\pi)^8}\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^{10}\frac{1}{144}(1-264\zeta_3)\mathcal{T}, \quad (3)$$

where $\epsilon=(4-D)/2$ is the dimensional-regularization parameter, and

$$\mathcal{T} = stA^{\text{tree}}_{\mathcal{N}=4}(\mathcal{O}_1 - 28\mathcal{O}_2 - 6\mathcal{O}_3), \quad (4)$$

where

$$\begin{aligned} \mathcal{O}_1 &= \sum_{\mathcal{S}_4}(D_\alpha F_{1\mu\nu})(D^\alpha F_2^{\mu\nu})F_{3\rho\sigma}F_4^{\rho\sigma}, \\ \mathcal{O}_2 &= \sum_{\mathcal{S}_4}(D_\alpha F_{1\mu\nu})(D^\alpha F_2^{\nu\sigma})F_{3\sigma\rho}F_4^{\rho\mu}, \\ \mathcal{O}_3 &= \sum_{\mathcal{S}_4}(D_\alpha F_{1\mu\nu})(D_\beta F_2^{\mu\nu})F_{3\sigma}{}^{\alpha}F_4^{\sigma\beta}. \end{aligned} \quad (5)$$

The sum runs over all 24 permutations of the external legs. The linearized field strength for each leg j is given in terms of polarization vectors for that leg,

$$\begin{aligned} F_j^{\mu\nu} &\equiv i(k_j^\mu\varepsilon_j^\nu - k_j^\nu\varepsilon_j^\mu), \\ D^\alpha F_j^{\mu\nu} &\equiv -k_j^\alpha(k_j^\mu\varepsilon_j^\nu - k_j^\nu\varepsilon_j^\mu). \end{aligned} \quad (6)$$

We have also included contributions from $\mathcal{N}=4$ matter multiplets in the loops. As discussed in Refs. [22,35], amplitudes with matter multiplets are straightforwardly obtained via dimensional reduction from higher-dimensional pure half-maximal supergravity without matter. After including the contribution of n_V matter multiplets, with all four external states belonging to the two graviton multiplets, the divergence is

$$\mathcal{M}^{4\text{-loop}}_{n_V}\Big|_{\text{div.}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10}\frac{n_V+2}{2304}\left[\frac{6(n_V+2)n_V}{\epsilon^2} + \frac{(n_V+2)(3n_V+4)-96(22-n_V)\zeta_3}{\epsilon}\right]\mathcal{T}. \quad (7)$$

In this expression n_V is independent of ϵ, a restriction that arises from imposing this on subdivergence subtractions. The two- and three-loop subdivergences, and subdivergences thereof, all cancel amongst themselves when we use a uniform mass regulator, as happened for the $n_V=0$ case. These cancellations are analogous to similar cancellations that occur at three loops and are surprising because there are subdivergences when matter multiplets are included [22,36]. However, the one-loop subdivergences do not cancel when $n_V\neq 0$. Instead, these enter nontrivially to make the divergence gauge invariant and proportional to $\mathcal{T}$.

By taking linear combinations,

$$\mathcal{O}^{--++} = \mathcal{O}_1 - 4\mathcal{O}_2, \qquad \mathcal{O}^{-+++} = \mathcal{O}_1 - 4\mathcal{O}_3, \qquad \mathcal{O}^{++++} = \mathcal{O}_2, \quad (8)$$

each of the obtained operators are nonvanishing only for the indicated helicity configurations and their parity conjugates and relabelings. Here the helicity labels refer to those of the polarization vectors used in Eq. (6) and not the supergravity states which are obtained by tensoring these states with those of $\mathcal{N}=4$ super-Yang-Mills theory. Using explicit helicity states in $D=4$, we have

$$\begin{aligned} \mathcal{O}^{--++} &= 4s^2t\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \\ \mathcal{O}^{-+++} &= -12s^2t^2\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}, \\ \mathcal{O}^{++++} &= 3st(s+t)\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}, \end{aligned} \quad (9)$$

using spinor-helicity notation. (See Ref. [37] for a recent review.) The divergence is thus present in all nonvanishing four-point amplitudes of $\mathcal{N}=4$ supergravity. Linearized supersymmetry acts only on the $A^{\text{tree}}_{\mathcal{N}=4}$ factor in Eq. (4), so each of these three configurations will *not* mix under this symmetry.

The appearance of the divergences in all three independent helicity configurations in Eq. (8) is surprising. In general, the analytic structure of amplitudes in the $--++$ sector is rather different from those of the other two sectors. This follows from generalized unitarity, where we decompose the supergravity loops into sums of products of tree amplitudes. In the $-+++$ and $++++$ sectors, all generalized cuts vanish in four dimensions because at least one tree amplitude will vanish. The same does not hold in the $--++$ sector. In particular, at one loop this implies that amplitudes in the $--++$ sector contain logarithms while amplitudes in the other two sectors are pure rational functions. The rational functions appearing in these sectors have been directly interpreted [35] as due to the $U(1)$ duality-symmetry anomaly [38]. We can understand the similarity of the four-loop ultraviolet divergence in all three sectors if we assume that it is due to the anomaly. As already noted in Ref. [35], unitarity implies that the anomaly contributes to higher-loop divergences in the $--++$ sector as well (unless canceled from another source). The similarity of the divergence in all three sectors would be a consequence of it arising from the same source. Another helpful clue comes from the fact that the divergence in Eq. (7) is proportional to $n_V + 2$. As explained in Ref. [35], the anomaly terms are proportional to this factor, providing further nontrivial evidence that the four-loop divergence is due to the anomaly.

We can reexpress the divergences in terms of counterterms involving the Riemann tensor. If we restrict the external states to four dimensions, numerical analysis reveals that the four-external-graviton counterterm can be reduced to a rather simple expression,

$$C = -\frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^6 \frac{1}{72\epsilon}(1 - 264\zeta_3)(T_1 + 2T_2), \qquad (10)$$

where

$$\begin{aligned} T_1 &\equiv (D_\alpha R_{\mu\nu\lambda\gamma})(D^\alpha R_{\rho\sigma}{}^{\lambda\gamma}) R^{\nu\rho}{}_{\delta\kappa} R^{\sigma\mu\delta\kappa}, \\ T_2 &\equiv (D_\alpha R_{\mu\nu\lambda\gamma})(D^\alpha R_{\rho\sigma}{}^{\lambda\gamma}) R^{\mu\nu}{}_{\delta\kappa} R^{\rho\sigma\delta\kappa}. \end{aligned} \qquad (11)$$

Using the divergence given in Eq. (3), one can also obtain the explicit counterterms for any other external states of the theory.

In any calculation of this type, it is important to have nontrivial consistency checks on the results. The most obvious one is the gauge invariance of the results (3) and (7). This requires intricate cancellations among the terms. We also find a required cancellation of poles in ϵ, as well as an expected [29] cancellation of various transcendental constants. Because there are no lower-loop divergences in pure $\mathcal{N} = 4$ supergravity, only a $1/\epsilon$ pole can remain at four loops. As an illustration, consider the basis integral corresponding to the first integral in Fig. 2, with all propagators having unit indices, except for the ones labeled by 3 and 4 which have vanishing indices (PR9 in the notation of Ref. [33]). Up to an overall factor, the divergent parts of this basis integral are

$$\begin{aligned} \mathrm{PR}\,9 = {} & \frac{1}{4\epsilon^4} + \frac{7}{3\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{169}{12} - \frac{27}{2}S2 + \frac{1}{2}\zeta_2 + \zeta_3\right) + \frac{1}{\epsilon} \\ & \times \left(\frac{143}{3} - \frac{135}{2}S2 - T1\mathrm{ep} + \frac{1}{6}\zeta_2 - \frac{4}{3}\zeta_3 + \frac{3}{2}\zeta_4\right), \end{aligned} \qquad (12)$$

where $S2$ and $T1$ep are transcendental constants specified in Ref. [33]. Besides finding the required cancellation of all poles down to the $1/\epsilon$ level in Eq. (3), the transcendental constants other than ζ_3 also cancel.

Another cross check on our procedure comes from computing the coefficient of an analogous potential divergence in pure Yang-Mills theory. By renormalizability, the divergences are proportional to tree-level color tensors, so all divergences containing independent color tensors other than the tree-level ones must vanish. Using identical methods as for the supergravity case, we have confirmed the ultraviolet finiteness of terms multiplying the two independent four-loop color tensors listed in Appendix B of Ref. [39].

Instead of providing definitive answers for the ultraviolet behavior of supergravity theories, our calculation raises additional interesting questions. We showed that the nonvanishing four-loop divergence of $\mathcal{N} = 4$ supergravity has a form suggesting that it is caused by the $U(1)$ duality-symmetry anomaly. It would be important to demonstrate this directly either via the counterterm structure or by tracking the contributions of the anomaly to the amplitudes. One may also wonder whether it is possible to remove the divergence by adding a finite term to the action so that an appropriate symmetry is preserved. A key issue is to find the higher-loop ultraviolet behavior of $\mathcal{N} \geq 5$ supergravity theories, since these should be free of duality anomalies and therefore free of potential divergences from this source. An important step towards this goal would be to develop improved means for constructing representations of super-Yang-Mills amplitudes that satisfy the duality between color and kinematics. Another interesting problem is that at present there is no complete symmetry explanation for the cancellation of the four-point ultraviolet divergences at three loops in four-dimensional $\mathcal{N} = 4$ supergravity or at two loops in five-dimensional half-maximal supergravity. It would be desirable to investigate this further. If history is any guide, further surprises await us as we probe supergravity theories to ever deeper levels.

We especially thank Guillaume Bossard, Kelly Stelle, and Radu Roiban for detailed discussions and for important comments on the manuscript. We also thank Simon Badger, John Joseph Carrasco, Kostja Chetyrkin, Yu-tin Huang, Henrik Johansson, Donal O'Connell, Pierre Vanhove, and Valery Yundin for helpful discussions. This material is based upon work supported by the Department of Energy under Awards No. DE-SC0009937 and No. DE-SC0008279. The work of A. V. S. and V. A. S. was supported by the Russian Foundation for Basic Research through Grant No. 11-02-01196. We thank Academic Technology Services at UCLA for computer support.

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