

Clash of Kinks: Phase Shifts in Colliding Nonintegrable Solitons

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We derive a closed-form expression for the phase shift experienced by $(1+1)$ -dimensional kinks colliding at ultrarelativistic velocities ($\gamma v \gg 1$), valid for arbitrary periodic potentials. Our closed-form expression is the leading-order result of a more general scattering theory of solitary waves described in a related paper [Phys. Rev. D **88**, 105024 (2013)]. This theory relies on a small kinematic parameter $1/(\gamma v) \ll 1$ rather than a small parameter in the Lagrangian. Our analytic results can be directly extracted from the Lagrangian without solving the equation of motion. Based on our closed-form expression, we prove that kink-kink and kink-antikink collisions have *identical* phase shifts at leading order.

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Introduction and summary.—Scattering theory provides a crucial link between the Lagrangian specifying the properties of fields and outcomes of experiments. The scattering theory of solitary waves (localized waves that travel without distortion—sometimes colloquially called solitons [1]) is particularly interesting since such waves play an important role in many disparate fields from modeling fluxons in superconductivity [2], optics [3], high energy physics (e.g., Ref. [4]) to cosmology [5–9]. While isolated solitary waves have been studied in detail, the physics of their scattering is not well understood analytically, except in special integrable cases.

In this Letter, we will consider the simplest of such solitary waves, that of a single canonical scalar field governed by a periodic potential in $(1+1)$ dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - V(\phi), \quad (1)$$

$$V(\phi) = V(\phi + \Delta\phi). \quad (2)$$

The equation of motion for ϕ is

$$\partial_t^2 \phi - \partial_x^2 \phi + V'(\phi) = 0. \quad (3)$$

The potential $V(\phi)$ is a general periodic potential with multiple minima. A simple solitary wave in this theory is a kink: an interpolation between two adjacent minima which can travel at a constant velocity without any distortion. Apart from an isolated special case with $V(\phi) = (1 - \cos\phi)$, known as the sine Gordon model, the usual way to predict the outcome of collisions between kinks is to numerically evolve the equation of motion (e.g., Refs. [10,11]). On the analytical side, perturbative predictions have only been done for cases which are close the sine Gordon case (see reviews [12,13]).

In this Letter, we demonstrate an analytical calculation for the outcome of collisions between kinks. Our main

result is a closed-form expression for the phase shift (spatial translation) experienced by a stationary kink due to a collision with an incoming kink or antikink with velocity $v \rightarrow 1$:

$$\Delta x = \frac{1}{2(\gamma v)M} \int_0^{\Delta\phi} \int_0^{\Delta\phi} d\phi_1 d\phi_2 \times \left[\frac{V(\phi_1) + V(\phi_2) - V(\phi_1 + \phi_2)}{\sqrt{V(\phi_1)V(\phi_2)}} \right] + O((\gamma v)^{-2}), \quad (4)$$

where $\gamma = (1 - v^2)^{-1/2}$ and $M = \int_0^{\Delta\phi} d\phi \sqrt{2V(\phi)}$ is the energy of the stationary kink.

Remarkably, the phase shift is an explicit function of the incoming velocity of the colliding wave and the potential, and does not require evaluation of the equation of motion. We emphasize that the periodic potential $V(\phi)$ need not be perturbatively close to the sine Gordon case and hence encompasses a much wider class of models compared to previous studies [12].

This simple form of Eq. (4) allows us to immediately draw several insights into the nature of kink interaction. First, the leading-order phase shift is the same for both kink-kink and kink-antikink collisions. (While there is no contradiction with our result, it is known that at long ranges, kink-kink interaction is repulsive but kink-antikink interaction is attractive [4,14,15].) Second, Eq. (4) is not positive definite; hence, there exist models with negative phase shifts. In particular, this implies that there exists an entire class of models where $\Delta x = 0$. Third, even though the collision is dissipative, the lack of time dependence in the right-hand side of Eq. (4) at leading order implies that no velocity change occurs at this order. Hence, the collision is almost elastic.

Equation (4) is the leading-order result of a perturbative expansion in $(\gamma v)^{-1}$ described in the related paper [16]

which applies to a wider class of solitary waves and includes a prescription for an order by order calculation of higher-order terms. Here, we focus on the derivation of this leading-order result for the phase shift. As a check, we performed detailed numerical simulations and found excellent agreement with our analytic answer.

Deriving the phase shift.—A stationary kink $\phi_K(x)$ is a solution to the equation of motion for the Lagrangian in Eq. (1)

$$\phi_K''(x) = V'(\phi_K(x)), \quad (5)$$

such that $\phi_K(-\infty) = 0$ and $\phi_K(\infty) = \Delta\phi$. The profile exponentially approaches vacuum values beyond some region $\sim L$ from its center—see the part of the curve labeled $\phi_K(x)$ in Fig. 2. Since the theory is Lorentz invariant, a kink moving to the left at a speed v is obtained by boosting the stationary solution: $\phi_K(\gamma(x + vt))$. We set up the initial condition for a collision at $t \rightarrow -\infty$ by linearly superposing two kink solutions

$$\phi(x, t) = \phi_K(x) + \phi_K(\gamma(x + vt)). \quad (6)$$

The outcome of the collision can be written as

$$\phi(x, t) = \phi_K(x) + \phi_K(\gamma(x + vt)) + h(x, t), \quad (7)$$

where $h(x, t)$ includes all the perturbations generated by the collision. As we will see, h is small because of the suppression of the space-time area of interaction of the two solitary waves: $\mathcal{A}_{\text{int}} \propto 1/(\gamma v)$ (see Fig. 1 for details). For ultrarelativistic collisions, $1/(\gamma v) \ll 1$.

After the fast-moving solitary wave has moved away from the stationary one, we are essentially left with the stationary solitary wave plus perturbations generated by the collision. This spectrum of perturbations includes the shift in the position Δx of the stationary solitary wave

$$\phi_K(x + \Delta x(t)) = \phi_K(x) + \Delta x(t)\phi_K'(x) + \dots \quad (8)$$

To extract the phase shift from $h(x, t)$, we expand it as

$$h(x, t) = \phi_K'(x)\Delta x(t) + \dots, \quad (9)$$

where the ellipses represent terms orthogonal to ϕ_K' . The phase shift $\Delta x(t)$ can then be projected from $h(x, t)$ using

$$\Delta x(t) = \frac{\int dx h(x, t)\phi_K'(x)}{\int dx \phi_K'^2(x)}. \quad (10)$$

To find $h(x, t)$, we linearize the equation of motion (EOM) Eq. (3) to obtain

$$[\partial_t^2 - \partial_x^2 + W_0(x)]h = -\Delta W(x, t)h - S(x, t), \quad (11)$$

where

$$W_0(x) \equiv V''(\phi_K(x)), \quad (12)$$

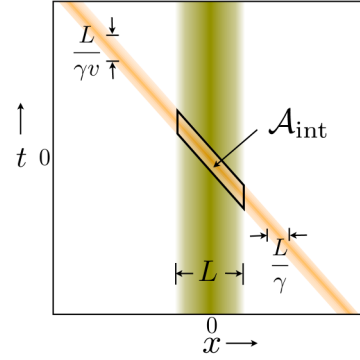


FIG. 1 (color online). The overall space-time is occupied by the two solitary waves; i.e., the area where their field values deviate significantly from vacuum is shown above. The green strip represents the stationary solitary wave, whereas the orange one is the incoming solitary wave. Assuming that the solitary waves simply pass through each other, the area of overlap between the solitary waves \mathcal{A}_{int} is denoted by the black parallelogram. Elementary geometry yields $\mathcal{A}_{\text{int}} \propto 1/(\gamma v)$. For ultrarelativistic collisions, the Lorentz contraction of the incoming solitary waves as well as the short time of collision are responsible for a suppressed \mathcal{A}_{int} .

$$\Delta W(x, t) \equiv V''(\phi_K(x) + \phi_K(\gamma(x + vt))) - V''(\phi_K(x)),$$

$$S(x, t) \equiv V'(\phi_K(x) + \phi_K(\gamma(x + vt))) \quad (13)$$

$$-V'(\phi_K(x)) - V'(\phi_K(\gamma(x + vt))). \quad (14)$$

$W_0(x)$ is the mass term for perturbations around an isolated stationary kink, and $\Delta W(x, t)$ is the change in mass due to the incoming kink, whereas $S(x, t)$ is the external source which is active only when the two solitary waves overlap. We will treat the incoming kink as a time dependent perturbation in the background of the stationary kink.

Before the collision, $h = 0$. During the overlap, as can be seen from Eq. (11), both $\Delta W h$ and S become nonzero, sourcing the perturbation h . However, since ΔW is multiplied by h , its effect is suppressed compared to the effect of S . Now, since S is only active within $\mathcal{A}_{\text{int}} \propto 1/(\gamma v)$, the perturbation h generated by it must also be suppressed by $1/(\gamma v)$. Thus, we expect the leading-order result of the collision to be $\propto 1/(\gamma v)$.

To solve Eq. (11), we expand h as

$$h(x, t) = \sum_a G_a(t)f_a(x), \quad (15)$$

where $\{f_a(x)\}$ is an orthonormal basis obtained from the eigenvalue equation

$$[-\partial_x^2 + W_0(x)]f_a(x) = E_a f_a(x). \quad (16)$$

As can be easily checked, the ground state is the zero energy mode $E_0 = 0$ given by $f_0(x) = M^{-1/2}\phi_K'(x)$. In the previous section, we used this zero mode $\propto \phi_K'(x)$ to project out the phase shift Δx from the general perturbation h .

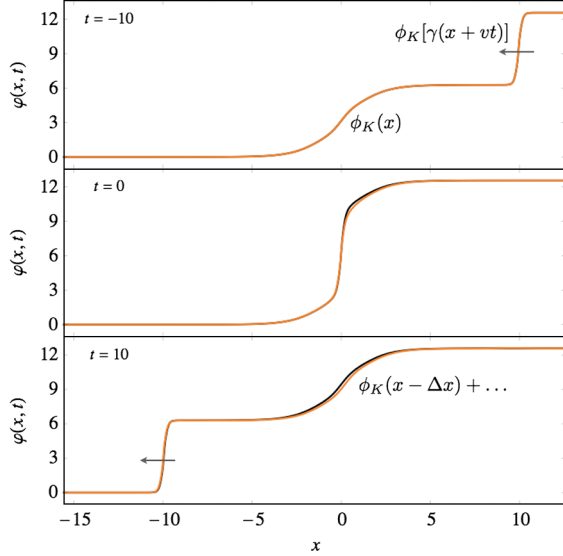


FIG. 2 (color online). The orange curves are the numerical field profiles before, during, and after collision. The black curve is the superposition solution which ignores all interactions. We can see a clear phase shift after the collision.

Since we are only interested in the phase shift Δx , instead of solving h from Eq. (11), we multiply Eq. (11) by ϕ'_K and integrate with respect to x to get an equation for Δx as a function of t :

$$\partial_t^2(\Delta x) = \frac{\int dx S(x, t) \phi'_K(x)}{\int dx \phi_K'^2(x)}. \quad (17)$$

In deriving the above equation, we used Eq. (16), the orthonormality of $\{f_a(x)\}$, and ignored the ΔWh term since we are only interested in the leading-order effects. We can similarly project onto other modes to obtain the entire spectrum, but here we will focus on the zero mode. With the initial condition $\Delta x(t \rightarrow -\infty) = 0$, Eq. (17) can be integrated to get

$$\Delta x(t) = \frac{1}{M} \int_{-\infty}^t d\tau (t - \tau) \int dx S(x, \tau) \phi'_K(x), \quad (18)$$

where $M = \int dx \phi_K'^2(x)$. The source $S(x, t)$ is only turned on during \mathcal{A}_{int} and is exponentially close to zero otherwise. Hence, the space-time area of integration is limited to \mathcal{A}_{int} , shown in Fig. 1. Now, consider the coordinate transformation

$$q = x, \quad p = \gamma(x + v\tau); \quad dx d\tau = (\gamma v)^{-1} dq dp. \quad (19)$$

Using these coordinate transformations, as well as the restriction of the integration range to \mathcal{A}_{int} , we have

$$\Delta x(t) = \frac{1}{(\gamma v)M} \int_{\mathcal{A}_{\text{int}}} dq dp [t - \tau(q, p)] S(q, p) \phi'_K(q), \quad (20)$$

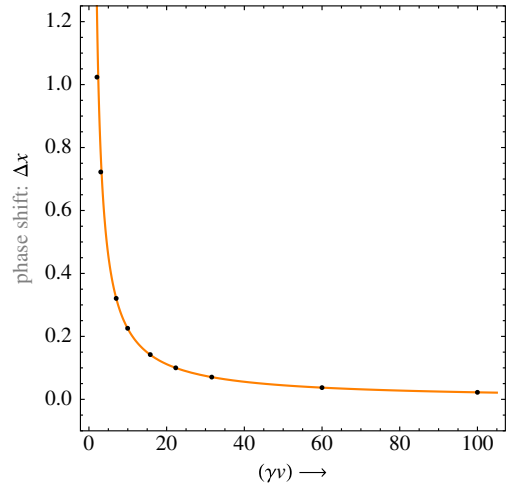


FIG. 3 (color online). We plot the numerically calculated phase shift undergone by a stationary kink colliding with an incoming kink as a function of (γv) . For this plot, the scalar field potential $V(\phi) = (1 - \cos\phi)(1 - 0.5\sin^2\phi)$. The orange curve is the theoretical prediction at leading order in $1/(\gamma v)$, and the black dots are the simulation results.

where $S(q, p) = V'(\phi_K(q) + \phi_K(p)) - V'(\phi_K(q)) - V'(\phi_K(p))$ and $\tau(q, p) = -(q/v) + p/(\gamma v)$. We now move to field space from q space using $\phi'_K(q) = \sqrt{2V(\phi_K(q))}$, which can be obtained from Eq. (5). A similar expression holds for p as well. Since ϕ is a monotonic function of q and p , inverses exist, and hence we can use $dq = d\phi_1/\sqrt{2V(\phi_1)}$ and $dp = d\phi_2/\sqrt{2V(\phi_2)}$. After a few integrations by parts, we get Eq. (4). Explicit integration shows that the term $\propto t$ in Eq. (18) is zero to $O(\gamma v)^{-2}$. Hence, we are finally left with a time independent phase shift, as asserted in Eq. (4). Since an antikink $\phi_{\text{AK}}(x) = \phi_K(-x)$, replacing $p \rightarrow -p$ in the incoming kink profile leads to an identical leading-order phase shift for a kink-antikink collision.

Comparison with simulations.—To check our expression, we numerically evolve the full equation of motion. From the numerical result $\tilde{\phi}(x, t)$, we can define

$$\tilde{h}(x, t) = \tilde{\phi}(x, t) - \phi_K(x) - \phi_K(\gamma(x + v\tau)) \quad (21)$$

and extract the phase shift from \tilde{h} using Eq. (10). We use a finite integration range in Eq. (10) such that the exponential tail of $\phi'_K(x)$ outside is negligible. This range is smaller than the box size where we evolve the full equation of motion, and we make the extraction only after the fast-moving kink is sufficiently far from the stationary one.

Note that while the analytical result of Δx is time independent, the numerical result need not be. There are a number of sources of time dependence. First, the orthogonality between $\phi'_K(x) \propto f_0(x)$ and other $\{f_a(x)\}$ is not exact, given a finite x integral. Fortunately, this additional time dependence is usually periodic and can easily be removed by an average. Another source is a higher-order dependence

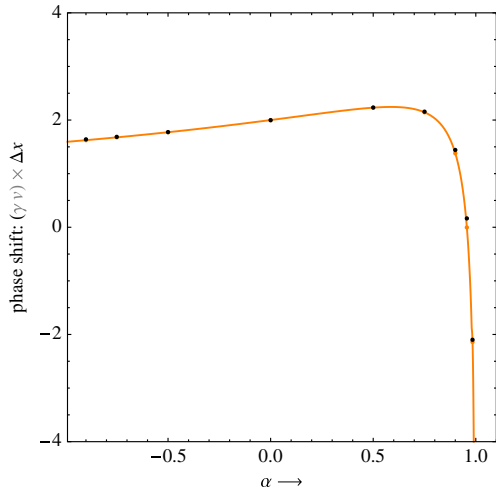


FIG. 4 (color online). The phase shift [multiplied by (γv)] undergone by a stationary kink colliding with an incoming kink as a function of the α parameter in the potential $V(\phi) = (1 - \cos\phi)(1 - \alpha\sin^2\phi)$. For this plot, $(\gamma v) = 100$. The orange curve (and orange dots) is the theoretical prediction at leading order in $(\gamma v)^{-1}$, and the black dots are the simulation results.

in time—the kink’s velocity. It is not surprising that the kink changes velocity after a collision. Our analytical calculation shows that the leading-order velocity change is zero, but the next order is generally not. Our numerical data indeed show that as (γv) increases, the linear dependence drops faster than the constant piece. We fit the slope of this linear dependence and remove it from the constant piece. It is this constant piece that is compared with Eq. (4).

We applied both the analytical and the numerical methods to the collision of kinks in the model defined by $V(\phi) = (1 - \cos\phi)(1 - \alpha\sin^2\phi)$, $-1 < \alpha < 1$. (22)

We carried out a large number of detailed numerical simulations with $-1 < \alpha < 1$ and $3 \leq (\gamma v) \leq 100$. The parameter range we cover is clearly beyond small deformation from the sine Gordon model ($\alpha = 0$) [12]. While we have chosen a potential that is symmetric around each minimum in the detailed numerical example, we hasten to add that Eq. (4) does not rely on this symmetry or this particular form.

In Fig. 3, we plot Δx as a function of (γv) and in Fig. 4 as a function of α . Both figures demonstrate the excellent agreement between analytical predictions and numerical results. Note that when $\alpha \geq 0.9564$, the phase shift becomes negative.

Conclusion.—Understanding soliton interactions has been an active area of research for more than 50 years. Many interesting physical phenomena involve solitons such as fluxons in Josephson junctions [13], nonlinear optical solitons [3], reheating after inflation [9], and domain wall collisions in cosmology [5]. Apart from

numerical simulations, there are two well-known analytical methods: perturbative analysis around integrable systems and dynamical-systems analysis of collective-coordinate ordinary differential equations (ODEs) [10,17,18]. Here, and in our related paper [16], we demonstrate a novel third method—a kinematics based scattering theory at relativistic velocities. Our method is not limited to small deformations from special models. It is also complementary to numerical techniques which become unwieldy for ultrarelativistic solitary wave collisions.

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