Tensile Strength and the Mining of Black Holes

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There are a number of important thought experiments that involve raising and lowering boxes full of radiation in the vicinity of black hole horizons. This Letter looks at the limitations placed on these thought experiments by the null energy condition, which imposes a fundamental bound on the tensile-strength-to-weight ratio of the materials involved, makes it impossible to build a box near the horizon that is wider than a single wavelength of the Hawking quanta, and puts a severe constraint on the operation of "space elevators" near black holes. In particular, it is shown that proposals for mining black holes by lowering boxes near the horizon, collecting some Hawking radiation, and dragging it out to infinity cannot proceed nearly as rapidly as has previously been claimed. As a consequence of this limitation, the boxes and all the moving parts are superfluous and black holes can be destroyed equally rapidly by threading the horizon with strings.

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Introduction.—Classical black holes live forever. The area theorem shows that not only can black holes not be destroyed, their horizon area cannot decrease at all [1]. Though Penrose-style processes can extract energy stored in electric, magnetic, or gravitokinetic fields outside the horizon of charged or spinning black holes [2,3], no energy can be extracted from the hole itself, and once the charge or spin is gone we are left with a Schwarzschild black hole that is both inert and eternal.

Quantum black holes, however, disintegrate into Hawking radiation. But black hole evaporation is slow. A $(3+1)$-dimensional Schwarzschild black hole of mass M self-destructs in a time [4]

$$\text{lifetime} = \frac{5120\pi b G^2}{\hbar c^4} M^3, \tag{1}$$

which for a solar mass black hole comes to 10^{57} times the age of the Universe. (The $O(1)$ constant b depends on the number and nature of the massless species, and on the graybody corrections to the Stefan-Boltzmann law [5]. Henceforth we use Planck units so $G = \hbar = c = 1$.) Can the hole be made to relinquish its energy sooner?

Unruh and Wald have argued that it can [6]. They have argued that by lowering a box down close to the horizon, filling it with Hawking radiation, and raising the box back out to infinity, that the black hole can be stripped of its thermal atmosphere and destroyed in a time that scales like the Schwarzschild time M [7]. I will show that this prescription will result in the black hole horizon swelling and consuming the box, so that rather than using a box to rob the black hole of its radiation, the black hole instead robs us of our box.

The strongest constraints are going to come from limitations the energy conditions place on the properties of the mining apparatus. The mining of black holes is meant to be a quasistatic process, so we can use versions of these conditions averaged over semiclassical distances. It seems doubtful that there is any notion of black hole thermodynamics that could be salvaged if we allow our equipment to systemically violate the macroscopically averaged null energy condition, which demands that the tension T of a static rope cannot exceed μ, its mass per unit length, $\mu \geq T$. A rope that is tense must also be dense. [This fundamental limit, $T/\mu = c^2 = 9 \times 10^{16}\ \mathrm{N/(kg/m)}$ in SI units, far exceeds the breaking point of any material that derives its strength from interatomic forces (e.g., defect-free carbon nanotubes can sustain no more than $T/\mu = 10^{-8}$). "Ropes" that saturate the condition have no longitudinal rest frame; examples include electric field lines, flux tubes, and cosmic and fundamental strings.] Subject to a greater force, the rope must stretch or the rope must break; what the rope cannot do is resist.

Tensile strength.—A general static spherically symmetric spacetime has metric

$$ds^2 = -\chi(r)^2 dt^2 + \frac{dr^2}{f(r)^2} + r^2 d\Omega_2^2. \tag{2}$$

For a Schwarzschild black hole, $\chi = f = (1 - 2M/r)^{1/2}$. A general static spherically symmetric matter distribution has stress energy

$$T^\mu_\nu = \mathrm{diag}\{-\rho, p_r, p_\theta, p_\theta\} = \frac{1}{4\pi r^2}\mathrm{diag}\{-\mu, -T, S, S\}, \tag{3}$$

where $\mu(r)$ is the mass per unit radial length, $T(r)$ is the radial tension, and $S(r)$ is the angular compression stress. The condition for equilibrium of a static distribution is $\nabla_\mu T^\mu_r = 0$, or

$$\frac{dT}{dr} + \frac{1}{\chi}\frac{d\chi}{dr}T + \frac{2}{r}S = \frac{1}{\chi}\frac{d\chi}{dr}\mu. \tag{4}$$

The weight of the material (the rhs) must be supported by increasing the radial tension (the first term), holding the radial tension fixed while the tension below redshifts away (the second term), or by angular compression stress (the third term).

The null energy condition (NEC) requires that $T^{\mu}_{\nu} k_{\mu} k^{\nu} \geq 0$ for every null vector k^{μ}. Choosing the radial null vector reproduces $\mu > T$. Choosing the angular null vector requires that $\mu > -S$. Let us investigate the implications of this condition as it pertains to ropes and boxes near black holes.

Ropes must be heavy: A hanging rope has radial tension ($T > 0$) but no angular stress ($S = 0$). If a rope of constant μ is suspended from infinity down towards the black hole horizon, what tension in the rope is required to keep it static? Equation (4) tells us that the required tension is independent of r and independent of M

$$T(\mu, r, M) = \mu. \tag{5}$$

Closer to the horizon the gravitational field $g \equiv d\log\chi/ds$ is stronger, but there is less rope to support below. (This solution acts as an explicit constructive counterexample to the claim of Ref. [8] that there can be no such solution. Compare my Eq. (4) to the paper's erroneous Eq. (5), which disregards the changing radial component of t^{a}; see also Ref. [9].) The ability of a constant tension rope to support itself is a uniquely relativistic effect; in Newtonian mechanics the tension must always increase with height to compensate for the increased weight, but in curved spacetime the weight of the rope redshifts away.

Thus for a constant-μ rope to support itself, it must saturate the NEC bound. And even a constant-μ rope that saturates the NEC bound must expend all its tensile strength supporting its own weight, if it stretches all the way down to the horizon, leaving none over to support a box. By getting rid of the rope below a certain height, we can free up tensile strength, but only enough to support a box no heavier than the weight of the excised rope. For a thin box of proper mass $\mathfrak{m} \ll M$ at a radius $r = R$ in Schwarzschild spacetime, integrating Eq. (4) gives the required tension and therefore the required density in the rope as $\mathfrak{m} g|_{r=R}$, or

$$\mu \geq T = \frac{1}{\chi}\frac{M\mathfrak{m}}{r^2}\bigg|_{r=R}. \tag{6}$$

It is sometimes said that the force required at infinity to hold a box fixed near a black hole remains bounded even for masses arbitrarily close to the horizon: though the gravitational field gets ever stronger, the redshifted gravitational force remains finite. This is technically true, but misleading. The NEC demands that the rope be heavy, which means that by the time you are far from the hole very little of the tension is devoted to supporting the weight of the box, and almost all is devoted to supporting the weight of the rope. The force at infinity required to suspend both box and supporting NEC-obedient rope does diverge as the horizon is approached.

Boxes must be narrow: A second consequence of the NEC is that a single box hanging from a single rope near a black-hole horizon can be no wider than the local Hawking wavelength. To see this, let us first see what constraints the NEC places on ropes suspended not from one point, as in the previous discussion, but from two.

In Newtonian mechanics, the profile adopted by a constant density rope suspended from two points in a constant gravitational field is the catenary $y = y_b - a + a\cosh(x - x_b)/a$, where a, x_b, and y_b are determined by the boundary conditions. The tension at the bottom is $\mu g a$. This Newtonian intuition suggests that the NEC will bound the radius of curvature at the bottom a to be less than about g^{-1}.

Let us consider the region just outside a fixed Schwarzschild horizon. Using $f = \chi = (1 - 2M/r)^{1/2}$ and moving to the near horizon limit ($\chi \ll 1$) of Eq. (2) gives Rindler coordinates

$$ds^2 = -\chi^2 dt^2 + 16M^2 d\chi^2 + 4M^2(d\theta^2 + \cos^2\theta d\phi^2). \tag{7}$$

The horizon lies at $\chi = 0$, where the local gravitational field $g = 1/(4\chi M)$ diverges. The action of a static constant-ϕ constant-μ NEC-saturating string hanging with shape $\chi(\theta)$ is proportional to

$$S \sim \mu \int dt d\theta \cos\theta \chi \sqrt{1 + 4\chi'(\theta)^2}. \tag{8}$$

If we consider strings that are not only close to the horizon but also take up a small angular scale we can treat the $\cos\theta$ as essentially fixed so that we are treating the horizon as a plane. Then the action provides the same function to be extremized as the potential energy in the Newtonian case, and the solution is

$$\chi(\theta) = \frac{1}{2}\chi_0 \cosh\frac{\theta}{\chi_0}. \tag{9}$$

Remarkably, the shape of a constant-μ, constant-T string in the nonconstant gravitational field of a black hole has the same functional form as the shape of a constant-μ, nonconstant-T rope in a constant Newtonian potential [10–13]. The difference is that in this case we have specified the tension so if we fix the point of closest approach to the horizon, we also fix the radius of curvature there. Since we can make T less than μ, we can make the rope hang steeper than Eq. (9). Since we cannot make T greater than μ, we cannot make the rope hang shallower than Eq. (9).

The implication of this is that two points at the same χ can be connected by a NEC-satisfying string only if they are sufficiently close. If they are separated by a distance $2M\Delta\theta$, then a string can be hung between them only if there is some solution to the above equation. Minimizing $\chi(\theta)$ with respect to χ_0 gives the bound $\Delta\theta < (1.3254\ldots)\chi$.

We can repeat this analysis for the rotationally symmetric box formed by hanging a NEC-saturating sheet from a circular support. In the near-horizon, small-angular-size, fixed-Schwarzschild limit the action is proportional to

$$S \sim \int d\theta \chi \theta \sqrt{1 + 4\chi'(\theta)^2}, \tag{10}$$

so that the shape must satisfy

$$\frac{2\chi''}{1 + 4\chi'^2} = \frac{1}{2\chi} - \frac{2\chi'}{\theta}. \tag{11}$$

The dilation symmetry ensures that if $\chi(\theta)$ is a solution, so is $\lambda\chi(\theta/\lambda)$, just as for the case of the NEC-saturating string. The shape equation can be numerically integrated to reveal that a sheet can be hung from a circle without breaking only if $\Delta\theta < (2.1754\ldots)\chi$.

At a height $4M\chi$ above the horizon the wavelength of a Hawking photon is approximately $M\Delta\theta \sim 4M\chi$, so the NEC permits us to build boxes only just wide enough to fit a single wavelength of Hawking radiation, no matter whether we use ropes or sheets. If we employ many supporting strings, then we can hang many boxes that collectively cover a larger area, but no single box with a single point of support can have a width that exceeds the local Hawking wavelength. For every Hawking wavelength we wish to enclose, we need another supporting rope.

Backreaction and melting: If the rope is too heavy, it will itself undergo gravitational collapse. The $G_t^t = 8\pi T_t^t$ component of Einstein's equation implies that a string produces a deficit angle of $8\pi\mu$ [14,15]; once the deficit angle reaches 4π gravitational collapse is inevitable, so the string must be lighter than half a Planck mass per Planck length

$$\mu < \frac{1}{2}. \tag{12}$$

The rope cannot be too light, though. If the rope is so slight, or the temperature so great, that the weight of even a single Hawking photon exceeds the rope's carrying capacity, Eq. (6), then the rope cannot prevent the photon falling into the hole and carrying at least part of the rope with it. Since at a redshift χ a typical Hawking quantum has an energy $\mathfrak{m} \sim (\chi M)^{-1}$, the string can bear the photon only when

$$\mu > (\chi M)^{-2}. \tag{13}$$

For a fundamental string, this corresponds to the redshift at which the Hawking radiation has reached the melting point of the string, the Hagedorn temperature [16,17].

The twin constraints of backreaction and melting limit the usefulness of ropes less tense than the NEC bound. We have already seen that only a $T = \mu$ rope can support itself if it is to have constant mass per unit length, so a $T < \mu$ rope must be tapered, with a higher tension and linear density at the top than at the bottom. If we input $T(r) = -w\mu(r)$ into Eq. (4) then the solution is

$$\mu(\chi) = \mu_\infty \chi^{-((1+w)/w)}. \tag{14}$$

The string tapers from finite linear density μ_∞ at infinity to zero linear density at the horizon. On the one hand, the rope must not be so linearly dense at infinity as to gravitationally collapse, so $\mu_\infty < 1/2$. On the other hand, the rope must not be so light at the black hole end that it melts and loses control of the box. A rope that is as dense at infinity as is consistent with backreaction melts at a redshift $\chi \sim M^{(2w/(1-w))}$. But most thought experiments take place down at a redshift that scales as $\chi \sim M^{-1}$ so that the temperature stays fixed even as the black hole is taken very large and semiclassical. It is only at these redshifts that the Generalized Second Law may be in jeopardy [2,18–25], only at these redshifts that buoyancy produced by the Hawking atmosphere becomes significant [6,26,27], and only at these redshifts that we could hope to mine enough energy that the lifetime scales as the light-crossing time M [7]. So only a NEC-saturating ($w = -1$) string will do. Not twine, not steel, not nanotubes—to reach interesting redshifts requires our rope to have the maximum possible tensile strength permitted by the laws of nature.

The destruction of black holes.—Unaided Hawking radiation releases approximately one quantum per light-crossing time M. If in the same time N such quanta could be liberated, the lifetime would fall to

$$\text{lifetime} \sim \frac{M^3}{N}. \tag{15}$$

Near the horizon of a black hole the metric is given by Eq. (7); the area remains fixed at M^2 but every other length scale is given by χM: the distance to the horizon, the wavelength of Hawking quanta, and the time spacing with which they arrive. Every locally measured time χM a photon of wavelength χM passes through each cell of area $(\chi M)^2$. Due to the gravitational time dilation near the horizon, a locally measured time of χM corresponds to an asymptotically measured time of M. Thus through a given angular area element near the horizon, the number of photons passing per asymptotically measured light-crossing time is

$$N = \frac{\text{area}}{\chi^2 M^2}. \tag{16}$$

Taking the area to be a whole sphere surrounding the hole, this number of photons is $N \sim 1/\chi^2$. What this means is that almost all photons that make it past a sphere of redshift $\chi \ll 1$ do not make it out to infinity. This effect can be understood already in the geometric optics approximation (marginally applicable here because the wavelength and redshift-doubling length coincide). As a matter of geometry, null rays near the horizon must be aimed within an angle $\Delta\psi \sim \chi$ of the vertical in order to escape the hole—any greater deviation and they loop round and are recaptured by the hole. Angular momentum makes escaping the hole more difficult.

The mining proposal [6,7,28] is that we reach in with a box and help these photons over the angular momentum barrier. No matter whether it expends its own energy climbing out of the gravitational potential, or we have to expend energy dragging it out, the net energy we recover from a photon is χ times the proper energy it had when we captured it, which is to say $\chi(1/\chi M) = (1/M)$. It is not that we recover more energy per photon, it is rather that we recover more photons. (We adopt the perspective of a quasistatic observer at a fixed value of χ, for whom the black hole is surrounded by a thermal atmosphere of mineable Hawking radiation. Even though they see no thermal atmosphere, mining can also be understood from the perspective of an infalling inertial observer; see Sec. IV of Ref. [6].) How many can be liberated per time M is going to be determined, through Eq. (16), by how deep we can mine. We turn to that question now.

If the mass per unit length of an individual NEC-saturating rope is μ_s, and the total number of ropes we wish to deploy is N_s, then we have the following constraints.

Constraint 1: melting. If we are to scoop from a redshift χ, then the strings cannot have melted at that depth. Equation (13) then implies

$$\frac{1}{\chi^2 M^2} < \mu_s. \tag{17}$$

Constraint 2: gravitational backreaction. The collective mass of the quasistatic strings must not induce gravitational collapse. Equation (12) then implies

$$\mu_s N_s < 1/2. \tag{18}$$

Constraint 3: box width. We have seen that it is impossible to construct a box that is wider than the local wavelength of the radiation, so it is impossible for a single rope to lift parametrically more than a single Hawking quantum per light-crossing time

$$N < N_s. \tag{19}$$

The principal lower bound on the lifetime comes from combining Eqs. (18) and (19). If we have at our disposal a NEC-saturating rope with a given fixed μ_s, then gravitational backreaction limits the number of ropes we may deploy, so the black hole can be destroyed in a time no shorter than

$$\text{lifetime} \geq \mu_s M^3. \tag{20}$$

This is a factor of $G\mu_s$ shorter than the unaided evaporation time. If we have a number of different weights of suitable NEC-saturating rope at our disposal, this lower bound indicates we should choose the one with the smallest μ_s. But only up to a point. If the string is too light then it melts before it can get deep enough. If we wish to collect many photons then we must reach deep, but deep means hot. Equation (17) implies that if we have complete freedom to pick μ_s, then the optimal tradeoff between backreaction and melting is given by picking $\mu_s \sim M^{-1}$. This gives a second lower bound on the lifetime

$$\text{lifetime} \geq M^2. \tag{21}$$

For intermediate mass black holes, this may be more restrictive than Eq. (20). [In $n+1$ dimensions, the Schwarzschild radius is $r_S \sim M^{1/(n-2)}$ and the evaporation lifetime is $M r_S^2 \sim r_S^n$. The analogue of Eq. (20) is that the lifetime cannot be made shorter than $\mu_s r_S^3$, while the analogue of Eq. (21) is that the lifetime cannot be made shorter than $r_S^{(3n-1)/(n+1)}$. In $3+1$ dimensions the lifetimes of the largest black holes scale as the same power of r_S whether or not the black holes are being mined: only the prefactor is improved. But in higher dimensions mining is much more effective, while still slower than the r_S predicted by Ref. [7].]

The lower bounds Eqs. (20) and (21) limit the rate at which black holes can be mined with boxes. But boxes are not the only ways to mine black holes [29,30]. Lawrence and Martinec [31] showed that, even without a box, a string dangled into a black hole wicks away Hawking radiation. The equation for perturbations on the string is just that of the s-wave bulk Hawking mode, and since the s-wave bulk mode carries away the majority of the energy in conventional Hawking radiation [5], so a single string carries away parametrically as much energy as the whole bulk Hawking emission [32–34]. (The string carries this energy exclusively in transverse modes, since it has neither longitudinal nor conduction modes [31].) Photons with high angular momentum cling to the string, deposit their angular momentum, and are channeled up to infinity. Frolov and Fursaev [29] argued that by employing many strings black holes can be systematically mined.

Black hole mining with strings is slow: each string can carry away just one quantum per light-crossing time. But the foregoing analysis shows that the narrow boxes demanded by the NEC can do no better. The constraints on the number of strings that can support boxes carry directly over to constraints on the number of strings that can be dangled into the horizon, so the rates of these two types of mining are parametrically identical. [Frolov and Fursaev [29] considered multiple strings sticking into black holes and achieved the same limits on mining as is captured in Eqs. (20) and (21). They derived the limit Eq. (21) by considerations of string reconnection: the strings are safe from reconnecting with one another and being expelled from the hole if kept more than one string length apart. However, it seems like there are other ways the strings could be safe from reconnection: for example, they could be oriented strings. Happily, we have seen other ways to derive the same limit. I thank Don Marolf for discussions on this point, similar discussions will appear in Ref. [35]].

When it comes to black hole mining, then, boxes are superfluous. Worse than superfluous, they are an encumbrance: the many extraneous moving parts present more failure modes and burden the mining process with unnecessary $O(1)$ overheads. Rather than scoop the Hawking radiation up with boxes, both the simplest and the fastest way to destroy black holes is to puncture the horizon with a large number of NEC-saturating strings, allowing Hawking modes even of high angular momentum to flow up and away along a soaring multitude of celestial brane drains.

There is a great trove of energy stored in the thermal atmosphere of a black hole, by some measures all the energy the hole possesses. But we see this energy only faintly, in the rare Hawking quanta that make it out, and we grasp for it at our peril. To reach close to the horizon demands that our equipment be strong, the threat of gravitational backreaction demands that our equipment be light, but the null energy condition demands that that which is strong must also be heavy.

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