Quantum Self-Correction in the 3D Cubic Code Model

Sergey Bravyi[1] and Jeongwan Haah[2]

[1]IBM Watson Research Center, Yorktown Heights, New York 10598, USA

[2]Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA

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A big open question in the quantum information theory concerns the feasibility of a self-correcting quantum memory. A quantum state recorded in such memory can be stored reliably for a macroscopic time without need for active error correction, if the memory is in contact with a cold enough thermal bath. Here we report analytic and numerical evidence for self-correcting behavior in the quantum spin lattice model known as the 3D cubic code. We prove that its memory time is at least $L^{c\beta}$, where L is the lattice size, β is the inverse temperature of the bath, and $c > 0$ is a constant coefficient. However, this bound applies only if the lattice size L does not exceed a critical value which grows exponentially with β. In that sense, the model can be called a partially self-correcting memory. We also report a Monte Carlo simulation indicating that our analytic bounds on the memory time are tight up to constant coefficients. To model the readout step we introduce a new decoding algorithm, which can be implemented efficiently for any topological stabilizer code. A longer version of this work can be found in Bravyi and Haah, arXiv:1112.3252.

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Any practical memory device must function reliably in the presence of small hardware imperfections and protect the recorded data against thermal noise. Building a memory capable of storing quantum information is particularly challenging since a quantum state must be protected against both bit-flip and phase-flip errors.

Ground states of topologically ordered many-body systems, such as fractional quantum Hall liquids [1,2] and unpaired Majorana fermions in nanowires and 2D heterostructures [3–7], were proposed as a natural quantum data repository insensitive to small hardware imperfections. A qubit encoded into the ground subspace of a topologically ordered system is almost perfectly decoupled from any local perturbation due to the local indistinguishability of the ground states [8–11].

To undo the effect of noise, a user of any memory, either classical or quantum, must invoke some form of error correction. It was shown by Dennis *et al.* [12] that topological memory based on Kitaev's 2D toric code model [8] can tolerate stochastic local noise provided that error correction is performed frequently enough to prevent errors from accumulating. An intriguing open question raised in [12,13] is whether topological memories can be *self-correcting*, that is, whether active error correction can be imitated by the natural dynamics of the memory system coupled to a thermal bath. Ideally, the lifetime of such memory can be made arbitrarily large by increasing the system size. The physical mechanism behind self-correction envisioned in [13] relies on "energy barriers" separating distinct ground states and energy dissipation. Unfortunately, examples of self-correcting memories are only known to exist in the 4D geometry [14].

One reason that self-correction is hard, if not impossible, to achieve in 2D or 3D is the presence of pointlike excitations carrying a topological charge [12,15,16]. Anyons in the 2D toric code [8] provide a paradigmatic example of such excitations. Even though creation of anyon pairs from the ground state is suppressed by a constant energy gap, anyons can diffuse over large distances at no extra energy cost. If the average distance traveled by anyons due to the diffusion is comparable with the typical anyon separation, information recorded in the memory is lost [17]. The lack of self-correction in the 2D toric code model was rigorously confirmed by Alicki *et al.* [18] who showed that its memory time is a constant independent of the lattice size.

Here we assess self-correcting properties of a new class of topological memories where the thermal diffusion of topological excitations is suppressed by energy barriers. The first model in this class, known as the 3D cubic code, has been recently discovered by Haah [19]. The model describes quantum spins-1/2, or qubits, that live at sites of the regular 3D cubic lattice of linear size L with periodic boundary conditions. Each site of the lattice is occupied by two qubits. The cubic code Hamiltonian introduced in [19] has the form

$$H = -J\sum_c G_c^X + G_c^Z,$$

where the sum runs over all L^3 elementary cubes c and the operators G_c^X, G_c^Z act on the qubits of c as shown in Fig. 1. The positive coupling constant J will be set to $J = 1/2$ for simplicity.

We shall refer to operators G_c^X, G_c^Z as stabilizer generators, or simply stabilizers. Note that each stabilizer acts

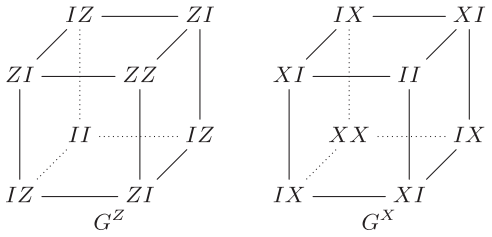

FIG. 1. Stabilizer generators of the 3D cubic code. Here $X \equiv \sigma^x$ and $Z \equiv \sigma^x$ represent single-qubit Pauli operators, while I is the identity operator. Double-letter indices represent two-qubit Pauli operators, for example, $IZ \equiv I \otimes Z$, $ZZ \equiv Z \otimes Z$, $II \equiv I \otimes I$, etc.

nontrivially only on 8 qubits. Let us recall some basic properties of the cubic code [19]. First, one can easily check that the stabilizers $G_c^X, G_{c'}^Z$ commute with each other for all c, c'. A ground state of H is a common $+1$ eigenstate of all stabilizers. The degeneracy of the ground states is $2^{k(L)}$ for some integer $2 \le k(L) \le 4L$. The ground subspace of H has topological order; that is, different ground states cannot be distinguished locally on any subset of qubits with linear size $<L$. Excited states of H can be described by configurations of *defects*, that is, stabilizers whose eigenvalue is -1. Each defect costs one unit of energy.

The key property of the model is that no local operator can create a single defect from the ground state or move a defect farther than a certain constant distance away without creating other defects [19]. This property was used in [20] to show that any sequence of local errors mapping a ground state of H to an orthogonal ground state must traverse an energy barrier proportional to $\log(L)$. This Letter extends results of Ref. [20] by analyzing the thermal dynamics of the cubic code and calculating its memory time as a function of the lattice size and the temperature.

Following Refs. [14,17,18,21], we shall model interaction between the memory and the thermal bath using the Davies weak coupling limit [22]. It provides a Markovian master equation of the following form:

$$\dot{\rho}(t) = -i[H, \rho(t)] + \mathcal{L}(\rho(t)), \qquad t \ge 0. \tag{1}$$

Here $\rho(t)$ is the state of the memory at time t such that $\rho(0)$ is a ground state of H. The Lindblad generator $\mathcal{L}$ describes dissipation of energy and has a form

$$\mathcal{L}(\rho) = \sum_{\alpha} \sum_{\omega} h(\alpha, \omega)\left(A_{\alpha,\omega}\rho A_{\alpha,\omega}^{\dagger} - \frac{1}{2}\{\rho, A_{\alpha,\omega}^{\dagger}A_{\alpha,\omega}\}\right), \tag{2}$$

where α runs over all lattice sites. We assume there are local Hermitian operators A_α that couple the memory with the thermal bath which yields the spectral components $A_{\alpha,\omega}$ in Eq. (2) defined by $\sum_\omega e^{-i\omega t}A_{\alpha,\omega} = e^{iHt}A_\alpha e^{-iHt}$, and $h(\alpha, \omega)$ are coefficients that obey the detailed balance equation $h(\alpha, -\omega) = e^{-\beta\omega}h(\alpha, \omega)$. Thus, $h(\alpha, \omega)$ is the energy fluctuation rate. The detailed balance equation is the only part of our model that depends on the bath temperature. It guarantees that the Gibbs state $\rho_\beta \sim e^{-\beta H}$ is a fixed point of the dynamics, $\mathcal{L}(\rho_\beta) = 0$, which is furthermore unique under natural ergodicity conditions [23]. We assume that $\|A_\alpha\| \le 1$ and $h(\alpha, \omega) = O(1)$ for all α and ω.

The encoded information is retrieved from the final state $\rho(t)$ by measuring an error syndrome (a configuration of defects) and performing an error correction. The former involves a nondestructive eigenvalue measurement of all generators G_c^X, G_c^Z. The latter is specified by an algorithm that takes as input the measured syndrome s and returns a correcting Pauli operator annihilating all the defects in s. Let Φ_{ec} be the linear map describing the net action of the syndrome measurement and the error correction. We shall refer to Φ_{ec} as a decoder. Note that syndrome measurement and error correction can also be used to prepare the initial state $\rho(0)$.

Our first result is an upper bound on the storage error, that is, the trace distance between the initial encoded state and the final error corrected state.

Theorem 1: There exists a decoder Φ_{ec} and constants $c, c' > 0$ such that for any inverse temperature $\beta > 0$, any $L \le e^{c'\beta}$, any state $\rho(0)$ supported on the ground subspace of H, and any evolution time $t \ge 0$ one has

$$\|\rho(0) - \Phi_{ec}(\rho(t))\|_1 \le O(t)2^{k(L)}L^{3-c\beta}. \tag{3}$$

The error correction algorithm used by the decoder has running time poly(L).

In the following we shall be interested in the smallest ground state degeneracy, $k(L) = 2$. This happens for any odd $3 \le L \le 200$ such that L is not a multiple of 15 or 63 [19,24]. If one is willing to tolerate a fixed storage error ϵ, say $\epsilon = 0.01$, the memory time T_{mem} can be defined as the smallest $t \ge 0$ such that $\|\rho(0) - \Phi_{ec}(\rho(t))\|_1 \ge \epsilon$. Theorem then implies that

$$T_{\mathrm{mem}} \ge L^{c\beta-3} \quad \text{for } L \le L^\star \simeq e^{c'\beta}. \tag{4}$$

Here we neglected the overall constant coefficient. This shows that for low temperature, $\beta \gg 1$, the memory time is a growing function of the lattice size L with a power law scaling, whose exponent is proportional to β. This power law growth occurs over a broad range of lattice sizes, and, to the best of our knowledge, provides the first realistic example of a topological memory with a self-correcting behavior. The bound Eq. (4) agrees with a naive estimate of the memory time based on the Arrhenius law and the logarithmic scaling of the energy barrier separating distinct ground states [20]. For a given temperature β, the best lower bound on the memory time $T_{\mathrm{mem}}(\beta)$ is obtained by choosing $L = e^{c'\beta}$, that is,

$$T_{\mathrm{mem}}(\beta) \ge e^{cc'\beta^2} \tag{5}$$

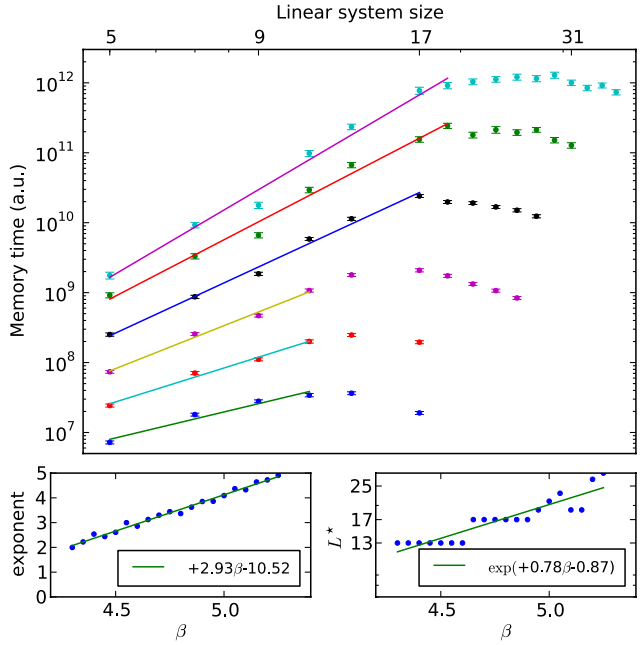

FIG. 2 (color online). The memory time T_{mem} versus the system size L. The data for $\beta = 4.3, 4.5, 4.7, 4.9, 5.1, 5.25$ are shown (starting from the bottom). The left bottom plot shows the exponent of the power law fit of T_{mem} for the first few system sizes. It indicates that $T_{\mathrm{mem}} \propto L^{2.93\beta-10.5}$ for $L < L^{\star}$, where $L^{\star}$ is the optimal system size where T_{mem} reaches maximum.

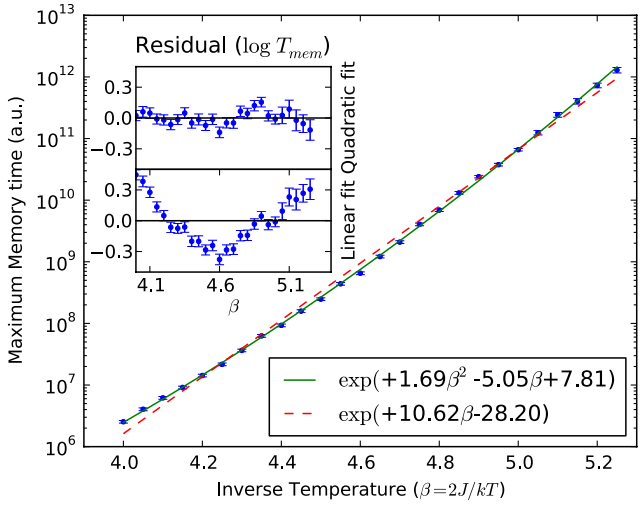

FIG. 3 (color online). The maximum memory time T_{mem} versus the inverse temperature β. The memory time is maximized with respect to the system size. The logarithm of T_{mem} clearly follows a quadratic relation with β as opposed to a linear one (inset).

for $\beta \gg 1$. For comparison, the memory time of the 2D toric code model grows only exponentially with β; see Refs. [17,18].

The memory time of the cubic code has also been computed numerically for a range of β's and L's. The numerical simulation results shown in Figs. 2 and 3 strongly suggest that our analytic bounds in Eqs. (4) and (5) are tight. In particular, we numerically found $c \approx 2.93$, which is in a good agreement with the analytic estimate $c = 2/\log 2 \approx 2.89$. The derivation of c and details of the numerical simulation can be found in the Supplemental Material [25]. We note that both theorem and our numerical simulation use the same decoder at the readout step as described below.

It should be emphasized that the cubic code offers only a partial quantum self-correction, as opposed to the truly self-correcting 4D toric code [14]. Indeed, the latter model exhibits a phase transition at nonzero temperature, detected by operators of extensive support, which we may call order parameters. They form an algebra of Pauli matrices acting on 6 logical qubits enabling one to store quantum information reliably in the thermal Gibbs state with a suitable encoding [14]. The memory time becomes infinite in the thermodynamic limit. In the case of the 3D cubic code, however, there is no order parameter that can be used to store information in the Gibbs state at nonzero temperature. Instead, information is encoded in the ground state subspace, and the self-correction is due to energy barriers that slow down the process of relaxation towards the Gibbs state ρ_β (quantum glassiness). These energy barriers, however, are not high enough for a phase transition to occur. To make this more quantitative, define the relaxation time τ to be the least $t > 0$ such that $\|\rho(t) - \rho_\beta\|_1 \leq \delta$ for a fixed small $\delta > 0$, where $\rho(0)$ is a ground state. Choosing any $0 \leq t \leq T_{\mathrm{mem}}$ and noting that $\Phi_{ec}(\rho(t)) \approx \rho(0)$, one gets $\|\rho(t) - \rho_\beta\|_1 \geq \|\rho(0) - \Phi_{ec}(\rho_\beta)\|_1 = \Omega(1)$ since $\Phi_{ec}(\rho_\beta)$ is a completely mixed state on the ground state subspace. It follows that the relaxation time τ is at least $T_{\mathrm{mem}} = \Omega(e^{cc'\beta^2})$. The numerically observed system size cutoff beyond which the memory time levels off is an interesting new phenomenon that calls for explanation.

Let us now prove theorem. The main ingredient of the proof is a new error correction algorithm which we call a renormalization group (RG) decoder. It falls into a larger family of error correction algorithms using real-space renormalization methods [26,27]. The RG decoding is a sequence of simple subroutines parametrized by integer levels $p = 0, 1, \ldots, \lfloor \log_2 L \rfloor$. At any given level p, the decoder decomposes a syndrome into disjoint connected clusters of defects, where the connectivity is defined using 2^p as a unit of length. This step can be implemented in time $O(N)$, where N is the volume of the lattice [28]. The decoder then examines each cluster individually and tries to "annihilate" it by a Pauli operator supported on a rectangular box enclosing the cluster. Clusters that cannot be annihilated in this way are passed to the next level (that is, $p + 1$). The time needed to test whether a given connected cluster can be annihilated depends on a particular code. For the cubic code we show how to perform this test in time $O(V)$, where V is the volume of the smallest rectangular box enclosing the cluster [28]. Once the

decoder reaches the highest level, $p = \lfloor \log_2 L \rfloor$, it returns the product of annihilation operators over all clusters that have been successfully annihilated.

To prove the bound Eq. (3), define an energy cutoff $m = c\log(L)$, where c is a constant to be chosen later. Let Π_- and Π_+ be the projectors onto the subspace spanned by eigenstates of H with energy $E \le m$ and $E > m$, respectively. Introduce auxiliary Lindblad generators $\mathcal{L}_-$ and $\mathcal{L}_+$ defined by Eq. (2) where $A_{\alpha,\omega}$ is replaced by $B_{\alpha,\omega} \equiv \Pi_- A_{\alpha,\omega}$ and $C_{\alpha,\omega} \equiv \Pi_+ A_{\alpha,\omega}$, respectively. Simple algebra shows that, for any state ρ diagonal in the eigenbasis of H, one has $\mathcal{L}(\rho) = \mathcal{L}_-(\rho) + \mathcal{L}_+(\rho)$ and

$$e^{\mathcal{L}t}(\rho) = e^{\mathcal{L}_- t}(\rho) + \int_0^t ds e^{\mathcal{L}_-(t-s)} \mathcal{L}_+ e^{\mathcal{L}s}(\rho). \quad (6)$$

We claim that

$$\Phi_{ec}[e^{\mathcal{L}_- t}(\rho(0))] = \rho(0) \quad (7)$$

if the readout map Φ_{ec} uses the RG decoder. Indeed, $e^{\mathcal{L}_- t}(\rho(0))$ is a mixture of states of form $|\psi\rangle = \Pi_- E_n \cdots \Pi_- E_2 \Pi_- E_1 |g\rangle$, where E_i are few-qubit Pauli operators that appear in the expansion of $A_{\alpha,\omega}$ and $|g\rangle$ is some ground state of H. Since Pauli errors map eigenvectors of H to eigenvectors of H, one has either $\psi = 0$ or $|\psi\rangle = E_n \cdots E_2 E_1 |g\rangle$. By definition of Π_-, the latter case is possible only if all intermediate states $E_j \cdots E_1 |g\rangle$ have at most m defects. We can now apply theorem 2 of Ref. [20] that characterizes excited states of the cubic code achievable from the ground state by a chain of local errors with a given energy cutoff m. The theorem asserts that the syndrome s of $|\psi\rangle$ cannot contain topologically charged clusters of defects (i.e., clusters that cannot be created from the ground state locally) separated from other defects by a distance greater than $R = \exp[\Omega(m)]$. By choosing small enough constant c, we can ensure that $R \ll L$. Simple geometric arguments then show that the syndrome s of $|\psi\rangle$ can be decomposed into well-isolated topologically neutral clusters of defects such that the RG decoder correctly identifies and annihilates each of those clusters; see lemma 4 in [28]. This proves Eq. (7).

Let $\epsilon(t) = \|\Phi_{ec}(\rho(t)) - \rho(0)\|_1$ be the storage error. From Eqs. (6) and (7) one easily gets

$$\epsilon(t) \le t \max_{0 \le s \le t} \|\mathcal{L}_+ e^{\mathcal{L}s}(\rho(0))\|_1. \quad (8)$$

Choose a constant $f = O(1)$ such that $h(\alpha, \omega) = 0$ whenever $|\omega| > f$. Let $\tilde{\Pi}_+$ be the projector onto the subspace spanned by eigenstates of H with energy $E > m - 2f$. The definition of $\mathcal{L}_+$ and the assumption $\|A_{\alpha,\omega}\| = O(1)$ imply

$$\epsilon(t) \le O(L^3 t)\mathrm{Tr}[\tilde{\Pi}_+ e^{\mathcal{L}s}(\rho(0))] = O(L^3 t)\mathrm{Tr}\tilde{\Pi}_+ e^{-\beta H}. \quad (9)$$

Here the last equality follows from the detailed balance condition and the fact that $\rho(0)$ is the ground state. The dimension of the subspace with exactly n defects is $\le 2^{k(L)}\binom{N}{n}$, where $N = 2L^3$ is the number of stabilizer generators. It follows that

$$\epsilon(t) \le O(L^3 t) 2^{k(L)} e^{-a\beta m} \sum_{n \ge m-2f} \binom{N}{n} e^{-(1-a)\beta n} \quad (10)$$

for any constant $0 < a < 1$. Substituting $m = c\log(L)$ and choosing the lattice size such that $N \le e^{(1-a)\beta}$ yields $\epsilon(t) \le O(t) 2^{k(L)} N^{1-ac\beta}$, which is equivalent to Eq. (3).

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