Doubly Transient Chaos: Generic Form of Chaos in Autonomous Dissipative Systems

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Chaos is an inherently dynamical phenomenon traditionally studied for trajectories that are either permanently erratic or transiently influenced by permanently erratic ones lying on a set of measure zero. The latter gives rise to the final state sensitivity observed in connection with fractal basin boundaries in *conservative* scattering systems and *driven* dissipative systems. Here we focus on the most prevalent case of *undriven dissipative* systems, whose transient dynamics fall outside the scope of previous studies since no time-dependent solutions can exist for asymptotically long times. We show that such systems can exhibit positive finite-time Lyapunov exponents and fractal-like basin boundaries which nevertheless have codimension one. In sharp contrast to its driven and conservative counterparts, the settling rate to the (fixed-point) attractors grows exponentially in time, meaning that the fraction of trajectories away from the attractors decays superexponentially. While no invariant chaotic sets exist in such cases, the irregular behavior is governed by *transient* interactions with *transient* chaotic saddles, which act as effective, time-varying chaotic sets.

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As popularized by Gleick [1], ‘‘chaos is a science of process rather than state, of becoming rather than being.’’ But the final state depends on the process and this has been widely explored in previous studies of transient chaos, where the object of analysis is not the (possibly simple) final behavior but instead the necessarily complicated transient dynamics leading to that outcome. A canonical example is a periodically forced damped pendulum with two periodic attractors and a fractal basin boundary separating them [2]. A phenomenon of continued interest [3–9], transient chaos is determined by the presence of an invariant set that, like in other manifestations of deterministic chaos, is formed by an uncountable number of aperiodic orbits that never settle down to periodic behavior and a dense set of unstable periodic ones [10–12]. This invariant set is nonattracting and represents a zero-measure subset of the phase space whose stable manifolds form the fractal boundaries between regions converging to different final states. It is thus the temporary approach to this chaotic saddle that gives typical orbits transiently irregular dynamics, which in turn limits our ability to predict the final state.

However important, these systems exclude a large and broadly significant class of other systems that cannot have such an invariant set of time-dependent solutions. They are the dissipative but undriven (hence autonomous) systems that underlie numerous physical processes [13], including approach to thermodynamic equilibrium and various forms of self-organization and structure or pattern formation. Moreover, undriven dissipative systems exhibiting complex dynamics are common not only in general physics, where a damped autonomous double pendulum is a prototypic example, but also in areas as diverse as chemistry, fluid dynamics, and astrophysics.

In undriven physical systems subject to nonvanishing dissipation, the energy can only decrease and the long-time behavior is necessarily very simple: each trajectory converges to one (out of possibly many) fixed point(s) in the case of the closed systems considered here. More important, this behavior is guaranteed for all orbits, not only for typical ones, indicating that typical orbits cannot experience the temporary influence of permanently chaotic ones. Yet, the dynamics can be very complex for a transient period of time and the basin boundaries can be very intricate—properties that have often been associated with the concepts of transient chaos and fractals [12,14]. These are in fact related to the properties that give rise to the randomlike behavior of coin tossing and die throwing [15,16]. Figure 1 shows the example of a magnetic pendulum with three fixed-point attractors, where the different colors mark the initial conditions associated with the different attractors. Magnifications seem to reveal intermingled structures at smaller and smaller scales, which is suggestive of fractal basin boundaries and sensitive dependence on initial conditions. But can the boundaries be fractal and the dynamics be transiently chaotic even though *all* motion eventually ceases?

In this Letter, we investigate the nature of the transient dynamics in undriven dissipative dynamical systems. We show that, due to the lack of long-time motion, the behavior is of a completely different type compared to the one

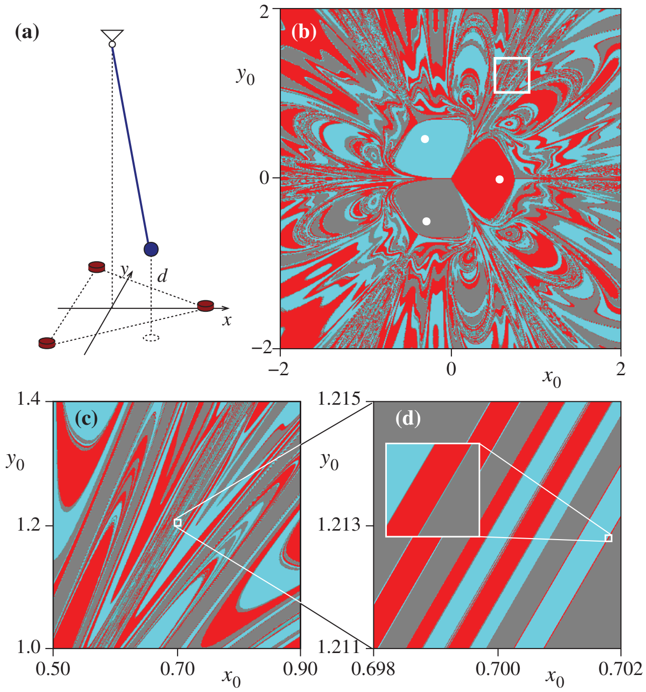

FIG. 1 (color). (a) Autonomous magnetic pendulum as described in the text. (b) Color-coded basins of attraction of the three fixed-point attractors of the system (white dots) for trajectories initiated with zero velocity. (c)–(d) Successive magnifications of the attraction basins shown in (b).

previously established for driven systems. Our principal results are that in undriven systems: (i) the measured dimension of the basin boundaries can be noninteger and the finite-time Lyapunov exponents can be positive over all finite scales but neither holds true asymptotically; (ii) the basin boundaries have (asymptotic) fractal codimension one; (iii) the survival probability away from the attractors decays superexponentially, as

$$P(t) \sim e^{-(\kappa_0/\gamma)e^{\gamma t}}, \tag{1}$$

leading to a settling rate $\kappa(t) = \kappa_0 e^{\gamma t}$, which grows exponentially in time; (iv) while no invariant chaotic set exists on which long-time averages required for chaos characteristics can be defined, the transient behavior is governed by a transient chaotic saddle that is prominent over a specific energy interval. We refer to this phenomenon as *doubly transient chaos*.

For concreteness, we focus on the magnetic pendulum as a model system, which captures the generic properties of interest. The system consists of three identical magnets at the corners of a regular horizontal triangle of unit edge length and the pendulum itself, formed by an iron bob suspended from above the center of the triangle through a massless rod [Fig. 1(a)]. The bob is subject to the influence of gravity, attractive magnetic forces, and drag due to air friction. For simplicity, we further assume that the length of the pendulum rod is long compared to the distance between the magnets, which allows us to describe the dynamics using a small-angle approximation. Following Refs. [12,14], we assume an inverse-square law interaction between the bob and the magnets as if they were point magnetic charges. The resulting dimensionless equations of motion are

$$\ddot{x} = -\omega_0^2 x - \alpha\dot{x} + \sum_{i=1}^{3} \frac{\tilde{x}_i - x}{D_i(\tilde{x}_i, \tilde{y}_i)^3}, \tag{2}$$

$$\ddot{y} = -\omega_0^2 y - \alpha\dot{y} + \sum_{i=1}^{3} \frac{\tilde{y}_i - y}{D_i(\tilde{x}_i, \tilde{y}_i)^3}, \tag{3}$$

where $(\tilde{x}_i, \tilde{y}_i)$ are the coordinates of the ith magnet, ω_0 is the natural frequency, and α is the damping coefficient; here $D_i(\tilde{x}_i, \tilde{y}_i) = \sqrt{(\tilde{x}_i - x)^2 + (\tilde{y}_i - y)^2 + d^2}$ and d are the distances from the pendulum bob to the ith magnet and to the magnets' plane, respectively. The coordinates of the magnets are $(\tilde{x}_1, \tilde{y}_1) = (\frac{1}{\sqrt{3}}, 0)$, $(\tilde{x}_2, \tilde{y}_2) = (-\frac{1}{2\sqrt{3}}, -\frac{1}{2})$, and $(\tilde{x}_3, \tilde{y}_3) = (-\frac{1}{2\sqrt{3}}, \frac{1}{2})$. In our simulations we set $\omega_0 = 0.5$, $\alpha = 0.2$, and $d = 0.3$ (except when stated otherwise), which is representative of all cases for which the fixed point at the origin is unstable. The magnetic pendulum then has three stable fixed points, and hence three attractors, as shown in Fig. 1(b) for the bob released from positions (x_0, y_0) with zero initial velocity.

First consider the average rate κ_E of energy dissipation due to damping. The energy decays on average as $E(t) \sim \exp(-\kappa_E t)$ with $\kappa_E \approx 0.16$ for random initial conditions in the $O(10^{-4})$ vicinity of the basin boundaries in Fig. 1(b) [Supplemental Material [17], Fig. S1(a)]. Two nearby trajectories in different basins tend to separate from each other over a relatively short period of time but they do so exponentially fast [Supplemental Material [17], Fig. S1(c)]. During the period of exponential separation, a small initial distance δ diverges as $\delta\exp(\bar{\lambda}t)$, where $\bar{\lambda} \approx 0.68$ is the average finite-time (largest) Lyapunov exponent, which is approximately constant over a relatively long time for the aggregate of trajectories close to the basin boundaries [Supplemental Material [17], Fig. S1(b)]. The average energies of the trajectories during exponential separation fall within a narrower range than the initial energies [Supplemental Material [17], Fig. S1(d)], indicating that initially close trajectories tend to move together when the energy is high and are already in the vicinity of their attractors when the energy is low. The deviation of the average dissipation rate κ_E from α during the period of exponential separation indicates that fast separation takes place when the speed of the pendulum is low, as it would be expected when an orbit approaches an unstable fixed point such as those often embedded in chaotic saddles.

Our system does not have a chaotic saddle; it has in fact only a handful of unstable periodic orbits and all of them are fixed points. They include the fixed point at the origin and three others along the symmetry axes connecting the attractors to the origin. The basin boundary points are

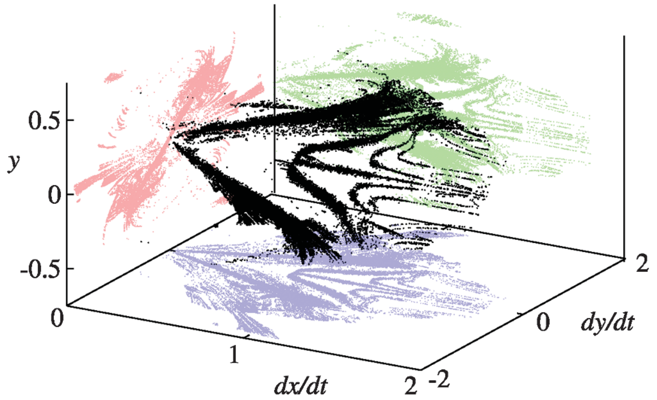

FIG. 2 (color). Transient chaotic saddle of the magnetic pendulum (black), represented through the Poincaré section defined by $x = 0$ and $\dot{x} > 0$. The colored shades correspond to projections of the set on the different coordinate planes.

expected to belong typically to stable manifolds of unstable fixed points that are locally stable along three directions in the 4D phase space. The unstable fixed point at the origin does not satisfy this condition since it has only two stable directions for the parameters we consider. The three unstable fixed points along the symmetry axes, however, have three eigenvalues with negative real parts. We argue, nevertheless, that this description alone does not capture the complexity of the observed dynamical behavior and propose that, during the period of rapid separation, the trajectories wander erratically in the vicinity of a set that plays the role of a chaotic saddle. This set can be estimated from the positions where the trajectories separate exponentially from each other. The result is shown in Fig. 2 and is strikingly similar to the usual chaotic saddles governing transient chaos. However, this set consists of only pieces of trajectories in the phase space and as such is not an invariant set of orbits. Moreover, this set manifests itself only during the period of exponential separation, which motivates us to refer to it as a *transient* chaotic saddle.

A central aspect of dissipative systems concerns the time the trajectories take to reach (a predefined neighborhood of) any of the attractors, which is referred to as the *settling time*. Figure 3(a) shows the settling time for trajectories of our system initiated on a straight line with zero initial velocity. This function exhibits a set of infinitely high peaks determined by the intersections of the initial line with the stable manifold of the nonattracting invariant set (which are typically basin boundary points). In a driven hyperbolic chaotic transient, these singular points would form a Cantor set that is statistically self-similar. In our case the settling time is fundamentally different, exhibiting no self-similar structures. The singular points still form a set that resembles those of driven systems over several decades, but subsequent magnifications indicate that this set (and hence the basin boundaries) become increasingly sparse at sufficiently small scales (see Supplemental Material [17], Fig. S3). We now quantify this systematic scale dependence.

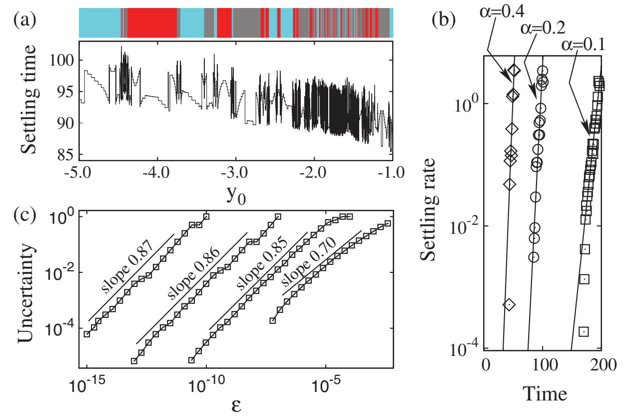

FIG. 3 (color). (a) Settling time as a function of the initial y coordinate for trajectories initiated with zero velocity on the line $x = -1$ to reach a phase-space distance 10^{-4} from any of the attractors. The top bar indicates the corresponding basins of attraction, as color coded in Fig. 1. (b) Settling rate κ for different values of the damping coefficient α, which increases exponentially as a function of time. (c) Estimation of the basin boundary (fractal) dimension using the uncertainty algorithm at successively smaller scales ε along the line considered in panel (a).

Various dynamical quantities of a chaotic set can be determined from a single generating function—the free energy function $F(\beta)$ [10,18]. This function is defined as $\beta F(\beta) = -\lim_{t\to\infty}(1/t)\log I(\beta, t)$ for $I(\beta, t) = \sum_{i=1}^{N(t)}[\ell_i(t)]^\beta$, where $N(t)$ is the number of intervals on a line of initial conditions (intersecting the stable manifold of the saddle) whose orbits have a settling time larger than t, and $\ell_i(t)$ are the lengths of these intervals. Quantities such as Lyapunov exponents, settling rates, dimensions, and entropies, which are by definition time independent and asymptotic, can all be calculated directly from this function and its derivatives. In undriven dissipative systems, the $t \to \infty$ limit is of little interest since all motion eventually ceases. But based on the settling time distribution of Fig. 3(a), we can introduce a finite-scale free energy function as $\beta F(\beta, t) = -d\log I(\beta, t)/dt$. This function is now time dependent, which means that the resulting dynamical quantities can be scale dependent.

We thus define the settling rate as the instantaneous rate $\kappa(t)$ of decay of the fraction $P(t)$ of still unsettled trajectories at time t: $dP(t)/dt = -\kappa(t)P(t)$. This corresponds to $\kappa(t) = \beta F(\beta, t)|_{\beta=1}$ when expressed using the free energy function. As shown in Fig. 3(b), the settling rate $\kappa(t)$ increases exponentially as a function of time, where the scaling exponent is $\gamma = 0.21$, 0.43, 0.56 for $\alpha = 0.1$, 0.2, 0.4, respectively. This represents a superexponential decay of $P(t)$, as summarized in Eq. (1), which becomes increasingly more pronounced as the damping coefficient α is increased. This is fundamentally different from the constant settling rate and power-law decay reported in the existing literature of hyperbolic and nonhyperbolic transient chaos, respectively [10–12]. An explanation for the superexponential decay is that (due to the exponential loss of energy) the difference between the settling times of two different trajectories scales with the difference of the logarithm of their initial energies, as $\Delta t \approx (1/\kappa_E)\Delta \ln E_0$, which causes them to reach the respective attractors after a comparable time. While we used the average dissipation rate in this simplified argument, note that the dissipation rate of individual trajectories is increasingly smaller for trajectories with the same E_0 initiated increasingly closer to the basin boundaries.

The unbound, exponential increase of the settling rate has an important implication for the basin boundaries: their codimension is one. For an illustration, consider a single-scale Cantor set construction in which the proportion of the interval length removed at step i is λ_i. At step n, there are 2^n intervals of length $\varepsilon_n = l_n/2^n$, where $l_n = \Pi_{i=1}^{n}(1-\lambda_i)$. The box-counting dimension of the limit set then is $D_0 = \lim_{n\to\infty}(\ln 2)/[\ln 2 - (\ln l_n)/n]$. In a self-similar Cantor set, as often used to model hyperbolic chaotic systems, $\lambda_i = \lambda$ (i.e., the fraction removed is the same for all i) and hence $(\ln l_n)/n = \ln(1-\lambda)$, leading to a dimension $0 < D_0 < 1$. An example of a non-self-similar Cantor set, used to model nonhyperbolic chaotic systems [19], is the one for which $\lambda_i = 1/(i+\lambda)$ (i.e., the fraction removed decreases with i) and hence $(\ln l_n)/n = \ln[\lambda/(n+\lambda)]/n$; this leads to $D_0 = 1$ even though the Lebesgue measure is zero. The case of an exponentially increasing settling rate corresponds to $l_n = e^{-(\kappa_0/\gamma)(e^{\gamma n}-1)}$ and hence $(\ln l_n)/n = -\kappa_0(e^{\gamma n}-1)/\gamma n$. The dimension then is $D_0 = 0$ even though the set is uncountable. But since $\ln\varepsilon_n \sim -(\kappa_0/\gamma)e^{\gamma n}$ for large n, the convergence is in this case logarithmically slow with respect to the length scale, which requires going to very small scales for the accurate estimation of D_0; in this case, finite-scale calculations will always overestimate D_0. Similar results hold for any increasing settling rate such that $l_n = \lambda^{n^s}$ for $s > 1$, which includes as a particular case $l_n = (\frac{2}{3})^{n^2}$, generated by taking $\lambda_i = 1 - (\frac{2}{3})^{2n-1}$. Numerical calculation of the dimension of the basin boundaries in our system using the uncertainty algorithm [20]—which exploits the scaling ε^θ of the fraction of points within a distance ε of a basin boundary of codimension θ—shows that the estimated θ becomes increasingly close to 1 at smaller scales (corresponding to basin boundaries of dimension 3 in the full 4D phase space) [Fig. 3(c)]. This should *not* be taken as an indicator of minimal sensitive dependence on initial conditions, however, since sensitivity is minimal only when the asymptotic value of θ is approached, which, as suggested by our Cantor set construction and effectively demonstrated in Fig. 3(c), requires extremely small ε.

How general is the behavior described here? When driven by an external force, the magnetic pendulum exhibits the already known properties of driven dissipative systems (see Supplemental Material [17]). Thus the novel behavior identified here is indeed due to the undriven nature of the dynamics. As a rule of thumb, we suggest that systems that would be chaotic if the dissipation could be turned off are expected to exhibit doubly transient chaos for small but nonzero dissipation rates; the dissipation rate sets the time scale over which trajectories will get intermingled by transient chaotic saddles. In particular, this is expected for Hamiltonian systems with mixed phase space, where the addition of dissipation generally converts the local minima of the energy at the center of Kolmogorov-Arnold-Moser islands into fixed-point attractors [21].

For completeness, we contrast doubly transient chaos with other nonlinear phenomena in which signatures of chaos are observed in the absence of an invariant chaotic set. An important case concerns strange nonchaotic attractors [22] and repellers [23], which are dynamically generated fractal invariant sets whose largest Lyapunov exponents are nevertheless zero. Such properties are usually induced by quasiperiodic driving, and hence concern systems significantly different from those considered here. Another important case is stable chaos [24], which is a spatiotemporal phenomenon in which the topological entropy can be positive even though the largest Lyapunov exponent is negative. Stable chaos is usually studied in coupled map systems and the phenomenon itself is rigorously observed in the thermodynamic limit. Our characterization of undriven dissipative systems applies, however, to low-dimensional dynamics.

The doubly transient chaotic behavior analyzed here is both surprising and significant. Many authors have portrayed the dissipative magnetic pendulum and other such undriven systems in the same class as driven dissipative systems, for the excellent reason that at first glance their basin boundaries and transient dynamics do seem similar. As shown here, however, they are fundamentally different and this is reflected both quantitatively and qualitatively. A remarkable distinction is that undriven dissipative systems exhibit exponentially growing rather than constant settling rates and, consequently, fractal basin boundaries whose complexity become increasingly diluted upon magnification. These properties are expected to be common to many natural and man-made systems and, in particular, to those whose conservative counterpart is chaotic. The implications are thus rather general given the prevalence of chaos in conservative models and of undriven dissipative systems in the real world. Our characterization of doubly transient chaos is relevant, for instance, in the study of "transitional chaos" in closed chemical reaction systems evolving toward thermodynamic equilibrium [25,26], of chaotic interacting vortices when dissipation due to viscosity is accounted for [27,28], and of spinning gravitational binary systems as energy is lost due to gravitational waves [29,30].

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