

# Optimal Quantum-Enhanced Interferometry Using a Laser Power Source

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We consider an interferometer powered by laser light (a coherent state) into one input port and ask the following question: what is the best state to inject into the second input port, given a constraint on the mean number of photons this state can carry, in order to optimize the interferometer's phase sensitivity? This question is the practical question for high-sensitivity interferometry. We answer the question by considering the quantum Cramér-Rao bound for such a setup. The answer is squeezed vacuum.

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The discovery that squeezed vacuum, injected into the normally unused port of an interferometer, provides phase sensitivity below the shot-noise limit [1] led to 30 years of technology development, beginning with initial proof-of-principle experiments [2,3] and culminating recently in the use of squeezed light to beat the shot-noise limit in the GEO 600 gravitational-wave detector [4] and the Hanford LIGO detector [5].

In the last decade much work has been devoted to exploring ultimate quantum limits on estimating the differential phase shift between two optical paths and to finding the states that achieve these limits. Given exactly  $N$  photons, the optimal state, in the absence of photon loss, is a  $N00N$  state,  $(|N, 0\rangle + |0, N\rangle)/\sqrt{2}$  [6–9], i.e., a superposition of all photons proceeding down one path with all photons proceeding down the other path. The  $N00N$  state is the optical analogue of the cat state that is optimal for atomic (Ramsey) interferometry [10]. Since the  $N00N$  state is extremely sensitive to photon loss, considerable effort has gone into determining optimal  $N$ -photon input states and corresponding sensitivities in the presence of photon loss [11–14].

While these states indeed deliver optimal or near-optimal performance, given a fixed input energy, we argue that they are not of practical relevance because they are very hard to produce with current technology and are therefore only available with quite low photon numbers. Consequently, the phase resolution obtained from using these optimal states cannot compete with the resolution obtained from a classical interferometer operating at or near the shot-noise limit with a strong, commercially available laser.

This does not mean, however, that nonclassical states are useless for metrology. The use of squeezed states to enhance the sensitivity of the GEO 600 and LIGO interferometers is testimony to the efficacy of squeezed light

in a situation where the lasers powering the interferometer have been made as powerful as design constraints allow. This Letter turns the focus away from states that have only been created with very small numbers of photons and instead investigates a particular, practical question: when an interferometer is powered by a laser producing coherent-state light, what is the best state to put into an interferometer's secondary input port? The answer is not surprising: squeezed vacuum.

The setting for our analysis is depicted in Fig. 1. Specifically, we consider a situation where laser light, described by a coherent state  $|\alpha\rangle = D(a_1, \alpha)|0\rangle$  of a mode  $a_1$ , is fed into the primary input port of a 50:50 beam splitter. The secondary input port is illuminated by

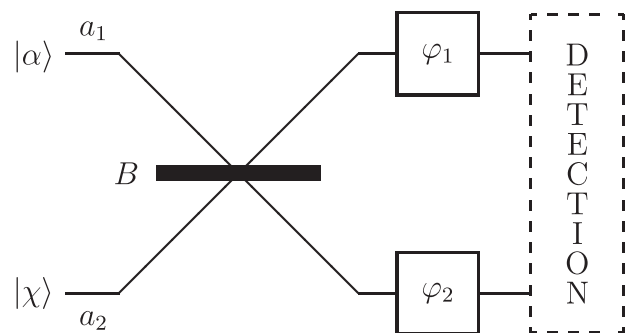


FIG. 1. Measurement of a differential phase shift. An (upper) mode  $a_1$  in a coherent state  $|\alpha\rangle$  and a (lower) mode  $a_2$  in an arbitrary pure state  $|\chi\rangle$  are incident on a 50:50 beam splitter, which performs the unitary transformation  $B$  of Eq. (1). After the beam splitter, phase shifts  $\varphi_1$  and  $\varphi_2$  are imposed in the two arms; the action of the phase shifters is contained in the unitary operator  $U$  of Eq. (2). Finally, a measurement is made to detect the phase shifts. When the measurement is pushed beyond a second 50:50 beam splitter, the result is a Mach-Zehnder interferometer, which is sensitive only to the differential phase shift  $\phi_d = \varphi_1 - \varphi_2$ .

mode  $a_2$ , which is in an arbitrary pure state  $|\chi\rangle$ . The beam splitter performs the unitary transformation

$$B = e^{-iJ\pi/4}, \quad J \equiv a_1^\dagger a_2 + a_2^\dagger a_1. \quad (1)$$

The two optical paths after the beam splitter experience phase shifts  $\varphi_1$  and  $\varphi_2$ ; the phase-shift unitary operator is

$$U = e^{i(\varphi_1 a_1^\dagger a_1 + \varphi_2 a_2^\dagger a_2)} = e^{iN_s \phi_s/2} e^{iN_d \phi_d/2}. \quad (2)$$

In the second form we introduce the sum and difference phase shifts  $\phi_s = \varphi_1 + \varphi_2$  and  $\phi_d = \varphi_1 - \varphi_2$ , and the corresponding sum and difference number operators;  $N_s = a_1^\dagger a_1 + a_2^\dagger a_2$  is the total number operator for the two modes, and  $N_d = a_1^\dagger a_1 - a_2^\dagger a_2$  is the number-difference operator. We assume that there are no losses in this configuration. The two-mode state after the phase shifters is

$$|\psi\rangle = UB|\psi_{\text{in}}\rangle, \quad (3)$$

where  $|\psi_{\text{in}}\rangle$  is the state before the beam splitter.

We use the quantum Fisher information to investigate the optimal resolution for estimating the phase shifts  $\phi_s$  and  $\phi_d$ . The advantage of the quantum Fisher information is that it gives a bound on phase resolution, called the quantum Cramér-Rao bound (QCRB), that applies to all quantum measurements on the two optical paths and to all procedures for estimating the phase shifts from the measurement results. In particular, let  $\phi_s^{\text{est}}$  and  $\phi_d^{\text{est}}$  denote estimators of the sum and difference phase shifts, and introduce the covariance matrix of the estimators

$$\Sigma = \begin{pmatrix} \langle(\Delta\phi_s^{\text{est}})^2\rangle & \langle\Delta\phi_s^{\text{est}}\Delta\phi_d^{\text{est}}\rangle \\ \langle\Delta\phi_d^{\text{est}}\Delta\phi_s^{\text{est}}\rangle & \langle(\Delta\phi_d^{\text{est}})^2\rangle \end{pmatrix}, \quad (4)$$

where here and throughout  $\Delta O \equiv O - \langle O \rangle$  denotes the deviation of a quantity from its mean. The QCRB is the matrix inequality

$$\Sigma \geq \mathcal{F}^{-1}, \quad (5)$$

where  $\mathcal{F}$  is the (real, symmetric) Fisher-information matrix [15,16]. The matrix QCRB implies that  $\text{tr}\Sigma \geq \text{tr}\mathcal{F}^{-1}$  and  $\det\Sigma \geq \det\mathcal{F}^{-1}$ ; for more than one parameter, the matrix QCRB cannot generally be saturated [17].

For pure states, the Fisher-information matrix is given by [18]

$$\mathcal{F}_{jk} = 4(\langle\partial_j\psi|\partial_k\psi\rangle - \langle\partial_j\psi|\psi\rangle\langle\psi|\partial_k\psi\rangle), \quad (6)$$

where  $j$  and  $k$  take on the values  $s$  and  $d$  and thus the derivatives are with respect to  $\phi_s$  and  $\phi_d$ . We retain both  $\phi_s$  and  $\phi_d$  in our analysis for the present, but eventually specialize to estimation of the differential phase shift alone. This would be the case if the final measurement were moved behind a second 50:50 beam splitter, giving a standard (Mach-Zehnder) interferometric configuration.

There are important practical reasons for considering the configuration of Fig. 1. The first is that in a typical phase

measurement, the easiest way to improve sensitivity is to buy more photons. The cheapest coherent source being a laser, the relevant model is that of a laser producing an input coherent state with the largest possible amplitude. To avoid the phase noise of the laser, either intrinsic or excess, one splits the laser light at a 50:50 beam splitter. Phase shifts are imposed in the two arms, and then in a Mach-Zehnder configuration the light in the two arms is recombined at a second 50:50 beam splitter, after which differenced photodetection or differenced homodyne detection is used to detect the differential phase shift. This interferometric technique is insensitive to the common-mode phase shift  $\phi_s$  in the two arms, which is just another way of saying that it is insensitive to the laser noise. Yet another way of putting this is that each arm serves as a phase reference for the other.

The Mach-Zehnder interferometric configuration gives shot-noise-limited sensitivity when the secondary port is illuminated by vacuum. To go beyond the shot-noise limit, one must replace the vacuum coming into the secondary port with some other, nonclassical quantum state of light; this inevitably makes the light in the two arms of the interferometer entangled, this modal entanglement having been made by the input beam splitter (see the Supplemental Material [19]). A major advantage of the setting in Fig. 1 is that the main power production is separated from the generation of nonclassical light, which only has to get a phase reference from the laser. Many analyses of phase sensitivity start by asking what entangled state in the two arms gives the best sensitivity, but this approach generally requires an entangled state that cannot be made by beam splitting a product state and thus gives up the practical advantage of separating the main power production from the production of nonclassical light.

In accordance with this discussion, the intended mode of operation of our interferometer is to have the coherent state carry many more photons than the light input to the secondary port. Since it does not hinder our analysis, however, we allow for the opposite possibility and all intermediate ones in our analysis.

An analysis similar in spirit to ours has investigated the best performance of an interferometer, given a constraint on the total mean number of photons, when the primary input port is illuminated with many more photons than the secondary input port [20]. The results show that a coherent state input to the primary port and squeezed vacuum into the other port comes very close to achieving a bound on the Fisher information that applies to all input states, both product and nonproduct states. This result holds when the photon loss exceeds a certain level, given in terms of the total mean number of photons, and thus is complementary to our result.

In our setting, the Fisher matrix for an arbitrary input state  $|\psi_{\text{in}}\rangle$  becomes

$$\mathcal{F}_{ss} = \langle\psi_{\text{in}}|B^\dagger N_s^2 B|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|B^\dagger N_s B|\psi_{\text{in}}\rangle^2, \quad (7)$$

$$\mathcal{F}_{dd} = \langle \psi_{\text{in}} | B^\dagger N_d^2 B | \psi_{\text{in}} \rangle - \langle \psi_{\text{in}} | B^\dagger N_d B | \psi_{\text{in}} \rangle^2, \quad (8)$$

$$\begin{aligned} \mathcal{F}_{sd} &= \mathcal{F}_{ds} \\ &= \langle \psi_{\text{in}} | B^\dagger N_s N_d B | \psi_{\text{in}} \rangle - \langle \psi_{\text{in}} | B^\dagger N_s B | \psi_{\text{in}} \rangle \\ &\quad \times \langle \psi_{\text{in}} | B^\dagger N_d B | \psi_{\text{in}} \rangle. \end{aligned} \quad (9)$$

We can use  $B^\dagger a_1 B = (a_1 - ia_2)/\sqrt{2}$  and  $B^\dagger a_2 B = (a_2 - ia_1)/\sqrt{2}$  to get  $B^\dagger N_s B = N_s$  and

$$B^\dagger N_d B = -i(a_1^\dagger a_2 - a_2^\dagger a_1) \equiv K. \quad (10)$$

The Fisher matrix is thus the covariance matrix of  $N_s$  and  $K$  with respect to the initial state. Notice that  $J_z = N_d/2$ ,  $J_x = J/2$ , and  $J_y = K/2$  make up the three components of an angular momentum and provide a convenient way of analyzing interferometry [21].

For the product input that is our main concern

$$|\psi_{\text{in}}\rangle = |\alpha\rangle \otimes |\chi\rangle, \quad (11)$$

the Fisher matrix becomes, with  $N_2 = a_2^\dagger a_2$ ,

$$\mathcal{F}_{ss} = |\alpha|^2 + \langle \chi | (\Delta N_2)^2 | \chi \rangle, \quad (12)$$

$$\begin{aligned} \mathcal{F}_{dd} &= |\alpha|^2 \langle \chi | (\Delta a_2 \Delta a_2^\dagger + \Delta a_2^\dagger \Delta a_2) | \chi \rangle \\ &\quad - \alpha^* \langle \chi | (\Delta a_2)^2 | \chi \rangle - \alpha^2 \langle \chi | (\Delta a_2^\dagger)^2 | \chi \rangle \\ &\quad + \langle \chi | N_2 | \chi \rangle, \end{aligned} \quad (13)$$

$$\begin{aligned} \mathcal{F}_{sd} &= \mathcal{F}_{ds} \\ &= -i\alpha^* \langle \chi | N_2 (\Delta a_2) | \chi \rangle + i\alpha \langle \chi | (\Delta a_2^\dagger) N_2 | \chi \rangle \\ &\quad - i\alpha^* \langle \chi | a_2 | \chi \rangle. \end{aligned} \quad (14)$$

Partly because the matrix QCRB cannot generally be saturated [17], but chiefly because we are mainly interested in measurements of the differential phase shift, we specialize now to single-parameter estimation of  $\phi_d$ , for which the QCRB reduces to

$$\langle (\Delta \phi_d^{\text{est}})^2 \rangle = \Sigma_{dd} \geq \frac{1}{\mathcal{F}_{dd}}. \quad (15)$$

It is known that there is a quantum measurement that achieves the single-parameter QCRB [22], i.e., has the required Fisher information, and it is also known that the resulting QCRB can be attained asymptotically in many trials by maximum-likelihood estimation.

What we do now is to maximize  $\mathcal{F}_{dd}$  over all initial states  $|\chi\rangle$  of mode  $a_2$  subject to a constraint of fixed mean photon number  $\bar{N} = \langle \chi | N_2 | \chi \rangle$ . The optimal state turns out to be squeezed vacuum with the requisite mean photon number. We then use the results of Pezzé and Smerzi [23] to indicate how the ultimate sensitivity can be achieved in a Mach-Zehnder interferometer in which one does direct photon detection of the two outputs.

To get started on maximizing  $\mathcal{F}_{dd}$ , we assume, without loss of generality, that  $\alpha$  is real, and we write  $\mathcal{F}_{dd}$  in terms of moments of the (Hermitian) quadrature components  $x$  and  $p$  of  $a_2 = (x + ip)/\sqrt{2}$ ,

$$\mathcal{F}_{dd} = 2\alpha^2 \langle (\Delta p)^2 \rangle + \bar{N}. \quad (16)$$

Here and for the remainder of the Letter, all expectation values are taken with respect to the initial state (11). The first term in Eq. (16),  $2\alpha^2 \langle (\Delta p)^2 \rangle$ , is due to interference between the coherent state and the phase quadrature  $p$  of the light coming into the secondary port; if  $\alpha^2 \gg \bar{N}$ , this term dominates and gives the shot-noise limit when mode  $a_2$  is in vacuum and improvements beyond shot noise when  $\langle (\Delta p)^2 \rangle > 1/2$ . If  $\alpha = 0$ , the contribution from  $\bar{N}$  in Eq. (16) dominates and expresses the shot-noise limit for illumination only through the secondary port.

We now maximize the variance of  $p$ , subject to a constraint on the mean number of photons. Writing

$$\begin{aligned} 2\bar{N} + 1 &= \langle p^2 \rangle + \langle x^2 \rangle \\ &= \langle p \rangle^2 + \langle x \rangle^2 + \langle (\Delta p)^2 \rangle + \langle (\Delta x)^2 \rangle, \end{aligned} \quad (17)$$

we see that

$$\langle (\Delta p)^2 \rangle + \langle (\Delta x)^2 \rangle \leq 2\bar{N} + 1 \quad (18)$$

with equality if and only if  $\langle x \rangle = \langle p \rangle = 0$ . We also have

$$\begin{aligned} &\langle (\Delta p)^2 \rangle - \langle (\Delta x)^2 \rangle \\ &= \langle (\Delta p)^2 \rangle + \langle (\Delta x)^2 \rangle - 4\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \\ &\leq -1 + \langle (\Delta p)^2 \rangle + \langle (\Delta x)^2 \rangle \leq 4\bar{N}(\bar{N} + 1) \end{aligned} \quad (19)$$

with equality in the first inequality if and only if  $|\chi\rangle$  is a minimum-uncertainty state, i.e.,  $\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = 1/4$ . Combining Eqs. (18) and (19) bounds  $\langle (\Delta p)^2 \rangle$  and hence gives a bound on the Fisher information,

$$\mathcal{F}_{dd} \leq \alpha^2 (2\bar{N} + 2\sqrt{\bar{N}(\bar{N} + 1)} + 1) + \bar{N} \equiv \mathcal{F}_{\text{max}} \quad (20)$$

with equality if and only if  $|\chi\rangle$  is a zero-mean minimum-uncertainty state, i.e., the squeezed vacuum state  $e^{r(a^2 - a^{\dagger 2})/2}|0\rangle$ , with  $\bar{N} = \sinh^2 r$ . In terms of the squeeze parameter  $r$ , the bound on the Fisher information takes the simple form  $\mathcal{F}_{\text{max}} = \alpha^2 e^{2r} + \sinh^2 r$ .

It is useful to manipulate the bound (20) in the following way:

$$\mathcal{F}_{\text{max}} = 4\alpha^2 \bar{N} + R = N_{\text{tot}}^2 - (\alpha^2 - \bar{N})^2 + R. \quad (21)$$

Here  $N_{\text{tot}} = \alpha^2 + \bar{N}$  is the total mean number of photons into both input ports, and the remainder term is given by

$$R = \bar{N} + \alpha^2 (2\sqrt{\bar{N}(\bar{N} + 1)} - 2\bar{N} + 1). \quad (22)$$

Applying  $\bar{N} \leq \sqrt{\bar{N}(\bar{N} + 1)} \leq \bar{N} + \frac{1}{2}$ , we have  $N_{\text{tot}} = \alpha^2 + \bar{N} \leq R \leq 2\alpha^2 + \bar{N} = N_{\text{tot}} + \alpha^2$ . When  $N_{\text{tot}}$  is large,

the remainder term is negligible compared to  $N_{\text{tot}}^2$ . Moreover, when  $\alpha^2 = \bar{N}$ , we have  $\mathcal{F}_{\text{max}} = N_{\text{tot}}^2 + R$ , which gives the Heisenberg limit on phase sensitivity plus a small correction that satisfies  $N_{\text{tot}} \leq R \leq 3N_{\text{tot}}/2$ . The apparent violation of the Heisenberg limit comes from not having a fixed total number of photons. That this configuration using coherent and squeezed light can achieve the Heisenberg limit was shown in Ref. [23].

The case of primary practical interest has  $\alpha^2 \gg \bar{N} = \sinh^2 r$ , in which case the maximal Fisher information reduces to  $\mathcal{F}_{\text{max}} = \alpha^2 e^{2r}$ . This corresponds to the standard picture of reduced fluctuations in the quadrature that produces differential phase fluctuations in the interferometer, and it gives the standard phase sensitivity  $1/\sqrt{\mathcal{F}_{\text{max}}} = e^{-r}/\alpha$ , for a squeezed-state interferometer. Indeed, the Fisher bound can be achieved by recombining the two optical paths at a second 50:50 beam splitter to create an interferometer and performing direct detection of the two outputs. The estimator can be taken to be the standard linear estimator that inverts the fringe pattern of the differenced photocount to estimate the differential phase shift.

Though it might be surprising, squeezed vacuum remains the optimal state into the secondary port even when the secondary port is allowed as many or more photons as the coherent-state input. We can appeal to the results of Pezzé and Smerzi [23] to show that the Mach-Zehnder configuration, with coherent-state and squeezed-vacuum inputs and direct detection at the output, can achieve the QCRB (20) for all values of the ratio  $\alpha^2/\bar{N}$ . Pezzé and Smerzi showed that for this configuration, the classical Fisher information of the probability for the output sum and difference photocounts  $P(n_s, n_d|\phi_d) = P(n_d|n_s, \phi_d)P(n_s)$  is equal to  $\mathcal{F}_{\text{max}}$  (see the Supplemental Material [24]). When  $\alpha^2 \leq \bar{N} = \sinh^2 r$ , however, the interferometer is running partially or mainly on the phase dependence of the squeezed vacuum noise, and the standard linear estimator mentioned above does not deliver optimal sensitivity [23,25]. Indeed, one can use the convexity of the Fisher information [26] to show that any estimator that uses only the differenced photocount  $n_d$ , ignoring the sum photocount  $n_s$ , does worse than keeping both [27], even though  $n_s$  is insensitive to the differential phase shift  $\phi_d$ . Instead of using an estimator to verify that the classical Fisher bound—and, hence, from our analysis, the QCRB—can be achieved, Pezzé and Smerzi simulated a Bayesian analysis that indicates the classical bound can be achieved for all ratios  $\alpha^2/\bar{N}$ .

We note that squeezed vacuum is not the state that maximizes the entanglement of the two optical paths after the input beam splitter. A number state  $|\bar{N}\rangle$  in the second mode leads to a larger value of the marginal entropy of the two paths [28].

We have analyzed a real-world scheme for measuring differential phase shifts, in which a coherent state illuminates one side of a 50:50 beam splitter and an arbitrary quantum state of light the other. We showed that given a

constraint on the total mean number of photons, the optimal state to put into the secondary input port is a squeezed vacuum state, regardless of the relative mean photon numbers of the two inputs. At least two questions beg for further attention. We do not know a simple optimal estimator when the squeezed light carries as many or more photons than the coherent input. We assumed no photon losses throughout our analysis and thus do not know the optimal state to put into the secondary port in the presence of losses, even though we suspect—and the results of Ref. [20] suggest—that it is squeezed vacuum.

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- Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.111.173601>, where we show that any nonclassical state in the secondary input port leads to modal entanglement between the two arms after the beam splitter.
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