Constraints on Conformal Field Theories in Diverse Dimensions from the Bootstrap Mechanism

Ferdinando Gliozzi

School of Computing and Mathematics and Centre for Mathematical Science, Plymouth University, Plymouth PL4 8AA, United Kingdom

INFN, Sezione di Torino, via Pietro Giuria, 1, I-10125 Torino, Italy

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Recently an efficient numerical method has been developed to implement the constraints of crossing symmetry and unitarity on the operator dimensions and operator product expansion coefficients of conformal field theories in diverse space-time dimensions. It appears that the calculations can be done only for theories lying at the boundary of the allowed parameter space. Here it is pointed out that a similar method can be applied to a larger class of conformal field theories, whether unitary or not, and no free parameter remains, provided we know the fusion algebra of the low lying primary operators. As an example we calculate using first principles, with no phenomenological input, the lowest scaling dimensions of the local operators associated with the Yang-Lee edge singularity in three and four space dimensions. The edge exponents compare favorably with the latest numerical estimates. A consistency check of this approach on the 3D critical Ising model is also made.

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One of the manifold expressions of the bootstrap dream is the conformal bootstrap, i.e., the idea that the crossing symmetry of the four-point functions of a conformal field theory (CFT) is so constraining [1–3] that, in some cases, it could uniquely fix the spectrum of the allowed scaling dimensions of the theory. In the case of two-dimensional rational CFTs, i.e., those with a finite number of Virasoro primary fields, there is an almost perfect implementation of this idea encoded in the Vafa equations [4]. These are Diophantine equations built by combining the crossing symmetry with the surprising modular properties of the fusion algebra [5]. As a result, the spectrum of Virasoro primary fields turns out to be discrete and all the scaling dimensions are determined modulo an integer.

The aim of this Letter is to reformulate the numerical method recently developed to implement the constraints of crossing symmetry in CFTs in diverse space-time dimensions [6–8] so that it resembles, to a certain degree, Vafa equations, provided rational CFTs in two dimensions are replaced with truncable CFTs in D dimensions, i.e., theories whose four-point functions can be well approximated by a finite sum of conformal blocks (in a way to be specified later); the Diophantine equations are replaced by transcendental ones whose solutions yield approximate values of the scaling dimensions of the primary operators of the theory. The level of the approximation is controlled by the number of conformal blocks considered.

As in Vafa equations, unitarity plays no role in this reformulation. While unitarity should be invoked for a sensible quantum field theory, many interesting critical systems do not correspond to unitary theories; thus, it is important to extend the method to these systems.

The starting point of this kind of analysis is a suitable parameterization of the four-point function of a scalar field $\varphi(x)$ in a D-dimensional CFT. The $SO(D+1,1)$ conformal invariance makes it possible to write

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\varphi}|x_{34}|^{2\Delta_\varphi}}, \qquad (1)$$

where Δ_φ is the scaling dimension of φ, x_{ij}^2 is the square of the distance between x_i and x_j, $g(u,v)$ is an arbitrary function of the cross ratios $u = x_{12}^2x_{34}^2/x_{13}^2x_{24}^2$ and $v = x_{14}^2x_{23}^2/x_{13}^2x_{24}^2$. The function g can be expanded in terms of the conformal blocks $G_{\Delta,L}(u,v)$, i.e., the eigenfunctions of the Casimir operator of $SO(D+1,1)$:

$$g(u,v) = 1 + \sum_{\Delta,L} \mathsf{p}_{\Delta,L} G_{\Delta,L}(u,v). \qquad (2)$$

The coefficients $\mathsf{p}_{\Delta,L}$ determine, up to a sign, the operator product expansion (OPE) of $\varphi(x_1)\varphi(x_2)$. Namely, if $\mathsf{p}_{\Delta,L} \neq 0$, there is in this OPE a primary operator $\mathcal{O}$ of scaling dimension Δ and spin L with an OPE coefficient $\lambda_{\varphi\varphi\mathcal{O}}$ with $\lambda^2_{\varphi\varphi\mathcal{O}} = \mathsf{p}_{\Delta,L}$. In the following it is not necessary to know the detailed form of the OPE but simply its fusion rule that, in the case at hand, we write as

$$[\Delta_\varphi] \times [\Delta_\varphi] = \sum_i N_i [\Delta_i, L_i], \qquad (3)$$

where the integer N_i denotes the number of different primary operators of quantum numbers Δ_i and L_i and we set $[\Delta] \equiv [\Delta, 0]$.

The lhs of (1) is invariant under any permutation of the four coordinates x_i, while the rhs is not, unless $g(u,v)$ fulfils two functional equations which express the crossing symmetry constraints. The $x_1 \leftrightarrow x_2$ interchange gives $g(u,v) = g(u/v, 1/v)$ which is followed only by the conformal blocks of even spin in (2), so only they contribute

to (2). The $x_1 \leftrightarrow x_3$ interchange yields a functional equation that we write in the form used in [6]

$$\sum_{\Delta,L} \mathsf{p}_{\Delta,L} \frac{v^{\Delta_\varphi} G_{\Delta,L}(u,v) - u^{\Delta_\varphi} G_{\Delta,L}(v,u)}{u^{\Delta_\varphi} - v^{\Delta_\varphi}} = 1. \quad (4)$$

Following [6] we put $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. In Euclidean space $\bar{z}$ is the complex conjugate of z while in Minkowski space they can be treated as independent real variables. The conformal blocks are smooth functions in the region $0 \leq z, \bar{z} < 1$. The central idea of [6] is to Taylor expand (4) about the symmetric point $z = \bar{z} = 1/2$ and transform the functional equation into an infinite set of linear equations in the infinite unknowns $\mathsf{p}_{\Delta,L}$, made with the derivatives of (4) of any order. To be more specific, following [7] we make the change of variables $z = (a + \sqrt{b})/2$, $\bar{z} = (a - \sqrt{b})/2$ and Taylor expand around $a = 1$ and $b = 0$. It is easy to see that this expansion will contain only even powers of $(a-1)$ and integer powers of b. The crossing symmetry constraint (4) can then be rewritten as one inhomogeneous equation

$$\sum_{\Delta,L} \mathsf{p}_{\Delta,L} \mathsf{f}^{(0,0)}_{\Delta_\varphi,\Delta_L} = 1, \quad (5)$$

and an infinite number of homogeneous equations

$$\sum_{\Delta,L} \mathsf{p}_{\Delta,L} \mathsf{f}^{(2m,n)}_{\Delta_\varphi,\Delta_L} = 0, \ (m, n \in \mathbb{N}, m+n \neq 0), \quad (6)$$

with

$$\mathsf{f}^{(m,n)}_{\alpha,\beta} = \left(\partial_a^m \partial_b^n \frac{v^\alpha G_\beta(u,v) - u^\alpha G_\beta(v,u)}{u^\alpha - v^\alpha} \right)_{a,b=1,0}. \quad (7)$$

At this point our analysis differs from that of [6]. Instead of looking for a model-independent unitary bound as a consequence of these equations, we apply the above considerations to some specific CFT, like, for instance, a critical φ^4 theory or a massless free field theory, where we assume the fusion algebra (3) is known, at least for the low-lying primary operators.

We say that this theory is truncable at level N if the partial sum of the first N conformal blocks of the infinite series (2) gives an exact solution of a set of $M \geq N$ equations of the homogeneous system (6). Now a system of M linear homogeneous equations with N unknowns admits a nonidentically vanishing solution only if all the minors of order N are vanishing [9]. This gives rise to $\kappa \leq \binom{M}{N}$ independent relations among the scaling dimensions $\{\Delta\}_N \equiv [\Delta_\varphi, \Delta, \Delta', \ldots]$ of the first $N+1$ primary operators

$$\mathsf{d}_i(\{\Delta\}_N) \equiv \det[\mathsf{f}^{(2m_i,n_i)}_{\Delta_\varphi,\Delta\in\{\Delta\}_N}] = 0, \ (i = 1, 2 \ldots \kappa), \quad (8)$$

where m_i, n_i indicate the rows belonging to the minor i.

At this perturbative order these equations encode the whole amount of information extracted from crossing symmetry, in the sense that the first M homogeneous equations are exactly solved if and only if all these minors are vanishing (the inhomogeneous Eq. (5) is simply a normalization condition). What can they tell us about the physical properties of this truncable CFT? The scaling dimension of the energy-momentum tensor is fixed to be $\Delta_T = D$, while we assume initially that the other N Δ's are free parameters. They are progressively constrained by increasing the number κ of equations. The maximum allowed value of κ in a generic case is $\kappa = N$, of course, when the system (8) has a discrete number of solutions $\{\Delta_{\mathsf{a}}\}_N$ $(\mathsf{a} = 1, 2, \ldots)$, or even no solution. The latter possibility is expected when one blindly includes in the fusion rule terms which should not be there. On the contrary, if the CFT we are studying truly exists, and we can reasonably infer its fusion algebra, we expect that $\mathsf{a} \neq 0$ and that the exact spectrum of the first $N+1$ primary operators $\{\Delta^\star\}_N$ is close to one of those discrete solutions (we shall illustrate it with some examples).

Choosing a (partially) different set of M homogeneous equations, the discrete solutions $\{\Delta_{\mathsf{a}}\}_N$ slightly shift. The extent of this displacement gives a measure of the error made in truncating the expansion (2) at N conformal blocks. It is also easy to see that adding a new term in the conformal block expansion does not spoil the discrete solution $\{\Delta_{\mathsf{a}}\}_N$, but simply induces a small correction on it (we leave the pleasure of proving it to the interested reader). Thus, if a CFT is truncable at level N, it is also truncable in general at level $N+1$ and so on. This fact leads us to conjecture that the truncable CFTs could coincide with those with a finite number of primary operators of conformal dimension $\Delta < K$ for any positive K. This subclass of CFTs contains many physically interesting examples.

As a first application of the present method let us consider a massless free field theory in D space-time dimensions. In this case the fusion rule is

$$[\Delta_\varphi] \times [\Delta_\varphi] = 1 + [\Delta_{\varphi^2}] + [D, 2] + [\Delta_{\varphi^2} + 4, 4] + \ldots, \quad (9)$$

where $\Delta_{\varphi^2} = 2\Delta_\varphi = D - 2$, but we treat Δ_{φ^2} and Δ_φ as free parameters and truncate (9) at $L = 4$, resulting in three unknowns $\mathsf{p}_{\Delta,L}$. The vanishing of each 3×3 minor of the homogeneous system gives a relation $\mathsf{d}_i(\Delta_\varphi, \Delta_{\varphi^2}) = 0$ between these two parameters. Figure 1 shows four such relations in the $D = 3$ case. We see that their mutual intersections accumulate around the expected exact value. Adding now the inhomogeneous equation (5) we obtain the OPE coefficients. They accumulate near the exact values given, at $D = 3$, by the conformal block expansion

$$g(u,v) - 1 \equiv \sqrt{u} + \sqrt{\frac{u}{v}} = 2G_{1,0} + \frac{1}{4} G_{3,2} + \frac{1}{64} G_{5,4} + \ldots \quad (10)$$

(We used, for the rhs, the normalizations of [7]).

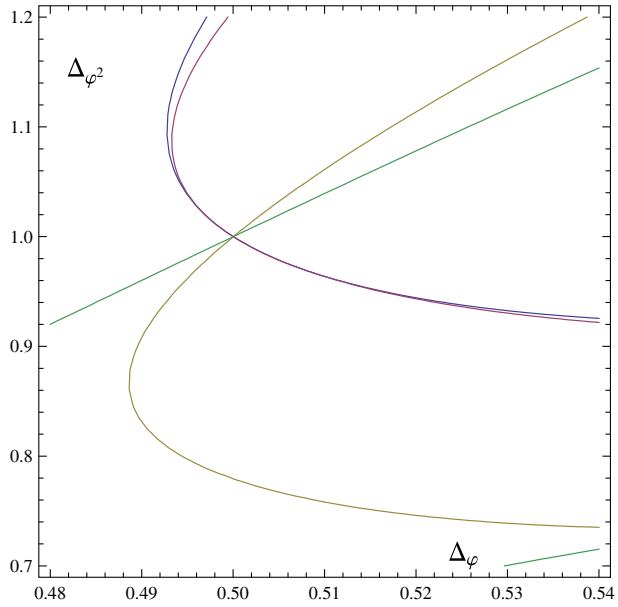

FIG. 1 (color online). Each curve represents the locus of vanishing of a 3×3 minor of the homogeneous system (6) in the Δ_φ, Δ_{φ^2} plane in the free scalar massless theory in $D = 3$ dimensions.

Next, we switch on the interaction by adding to the action a φ^3 term with imaginary coupling. Precisely, we put

$$S = \int d^D x \left[\frac{1}{2} (\partial \varphi)^2 + i(h - h_c)\varphi + ig\varphi^3 \right]. \quad (11)$$

This nonunitary theory is known to describe in the infrared the universality class of the Yang-Lee edge singularity [10]. Such a singularity occurs in any ferromagnetic D-dimensional Ising model above its critical temperature $T > T_c$. The zeros of the partition function in the complex plane of the magnetic field h are located on the imaginary ih axis above a critical value $ih_c(T)$. In the thermodynamic limit the density of these zeros behaves near h_c like $(h - h_c)^\sigma$, where the critical exponent σ is related to the scaling dimension of the field φ by [10]

$$\sigma = \frac{\Delta_\varphi}{D - \Delta_\varphi}. \quad (12)$$

This edge exponent is exactly known in $D = 2$ and $D = 6$. Our purpose now is to evaluate it in $D = 3$ and $D = 4$ using the present method and to compare it with most recent numerical results. Meanwhile we also check the method in $D = 2$, where the complete spectrum of primary operators is known as well as the OPE coefficients [11].

The φ^3 interaction tells us that the upper critical dimensionality of this model is $D_u = 6$, above which the classical mean-field value $\sigma = 1/2$ applies. In $6 - \epsilon$ there are apparently two relevant operators, φ and φ^2; however the latter is, in fact, a redundant operator, as at the nontrivial φ^3 fixed point it is proportional to $\partial^2 \varphi$ by the equation of motion. Thus φ^2 and its derivatives become descendant operators of the only relevant primary operator φ of this universality class. Actually, this is the only difference between the operator content of the Gaussian fixed point of the free-field theory and the Yang-Lee edge universality class, as long as the approximate renormalization group analysis applies. As a result, the fusion rule (9) becomes even simpler

$$[\Delta_\varphi] \times [\Delta_\varphi] = 1 + [\Delta_\varphi] + [D, 2] + [\Delta_4, 4] + \ldots \quad (13)$$

It characterizes the universality class of the Yang-Lee edge singularity in any space dimension.

Before inserting such a fusion rule in Eqs. (8), we try to simplify the notation a bit. Each equation of the homogeneous system (6) is labeled by the pair of integers (m, n). We enumerate these equations using the following arbitrary dictionary

$$1, 2, 3, 4, 5, 6, \ldots \leftrightarrow (1, 0)(2, 0)(0, 1)(0, 2)(1, 1)(0, 3)\ldots \quad (14)$$

Let us begin with the $D = 2$ case. A 2×2 minor of the truncation of (13) at $N = 2$ can be written explicitly in this case as

$$\mathsf{d}_{12}(\Delta_\varphi) = \det \begin{pmatrix} \partial_a^2 G_{\Delta_\varphi,0} & \partial_a^2 G_{2,2} \\ \partial_a^4 G_{\Delta_\varphi,0} & \partial_a^4 G_{2,2} \end{pmatrix}. \quad (15)$$

It has a zero at $\Delta_\varphi \simeq -0.422$. Similarly $\mathsf{d}_{13}(\Delta_\varphi)$ vanishes at $\Delta_\varphi \simeq -0.362$. The exact value is at $\Delta_\varphi = -(2/5)$.

In order to obtain more accurate results, we have to add the next term of the fusion rule (13), namely the spin 4 operator $[\Delta_4, 4]$, which depends on the new "free" parameter Δ_4.

In view of the fact that in the fusion rule of any scalar operator in $D = 2$ the energy momentum tensor $T + \bar{T}$ is always accompanied by the scalar $T\bar{T}$ associated to [4, 0], we add the latter without enlarging the number of free parameters. The intersection of $\mathsf{d}_{1234}(\Delta_\varphi, \Delta_4) = 0$ with $\mathsf{d}_{1245}(\Delta_\varphi, \Delta_4) = 0$ gives $\Delta_\varphi \simeq -0.393$ and $\Delta_4 \simeq 3.666$. The exact value of the latter is $\Delta_4 = 18/5$. Solving now the inhomogeneous system we find $\mathsf{p}_{\Delta_\varphi} \simeq -3.665$ to be compared with the exact result [11]

$$\mathsf{p}_{\Delta_\varphi} = -\frac{\Gamma(\frac{6}{5})^2 \Gamma(\frac{1}{5}) \Gamma(\frac{2}{5})}{\Gamma(\frac{4}{5})^3 \Gamma(\frac{3}{5})} \simeq -3.65312. \quad (16)$$

We can extract from $\mathsf{p}_{2,2}$ the estimate $c \simeq -4.53$ of the central charge, while its exact value is $c = -(22/5)$.

The last step is now to put $D = 3$ in our formulae. Entering in the three-dimensional world we leave the golden eden of exactly solvable models and can resort solely to the internal consistency of the approach. Using the fusion rule truncated at the spin 4 operator we have only two free parameters. The intersection of the vanishing

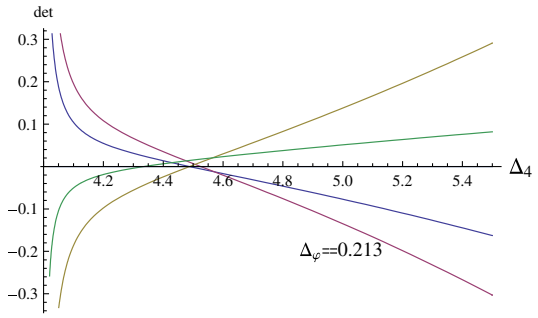

FIG. 2 (color online). Plot of some 3×3 minors around the solution (17) as functions of Δ_4. Their convergence to zero near this solution supports our estimate of the critical exponent σ for the three-dimensional Yang-Lee edge singularity.

loci of the 3×3 minors $\mathsf{d}_{123}(\Delta_\varphi, \Delta_4)$ and $\mathsf{d}_{234}(\Delta_\varphi, \Delta_4)$ and the subsequent solution of the inhomogeneous system yield

$$\Delta_\varphi \simeq 0.213, \quad \Delta_4 \simeq 4.49, \quad \mathsf{p}_{\Delta_\varphi} \simeq -3.91, \quad \mathsf{p}_{3,2} \simeq 0.006 \tag{17}$$

where $\mathsf{p}_{\Delta_\varphi} = \lambda^2_{\varphi\varphi\varphi}$ and $\mathsf{p}_{3,2} = \lambda^2_{\varphi\varphi T}$. Internal consistency requires that all the 3×3 minors made with the same equations (i.e., with the same indices $i = 1, 2, 3, 4$) should converge to zero or to small values when approaching this solution. This is illustrated in Fig. 2.

Inserting this value of Δ_φ in (12) we obtain $\sigma \simeq 0.076$, to be compared with the very accurate estimate [12] $\sigma = 0.077(2)$, based on very long series expansions of dimer density in powers of the activity in a cubic lattice. Similarly, with no more effort, we can apply the same analysis to the Yang Lee edge singularity in four space dimensions, obtaining

$$\Delta_\varphi \simeq 0.823, \quad \Delta_4 \simeq 5.71, \quad \mathsf{p}_{\Delta_\varphi} \simeq -2.86, \quad \mathsf{p}_{4,2} \simeq 0.40 \tag{18}$$

and consequently $\sigma \simeq 0.259$, to be compared with $\sigma = 0.258(5)$ of [12]. Figure 3 shows this solution as the common intersection of the zero loci of 3×3 minors. In principle, the other physical parameters listed in (17) and (18) could be checked with an $\epsilon = 6 - D$ expansion.

Clearly the present method may be successfully applied to other models. Note, however, that in the fusion rules there are generally many more unknown scaling dimensions to be accounted for, which should require much larger determinants and, hence, much larger orders of vanishing derivatives.

For instance, in the 3D critical Ising model there are two more scalar primary operators of scaling dimension smaller than that of the spin 4 operator, one associated to φ^4 with $\Delta_{\varphi^4} = 3.84(4)$, the other associated to φ^6 with $\Delta_{\varphi^6} = 4.67(11)$ ($\Delta_4 = 5.0208(12)$ in this model). A consistency check of the present method is to insert these

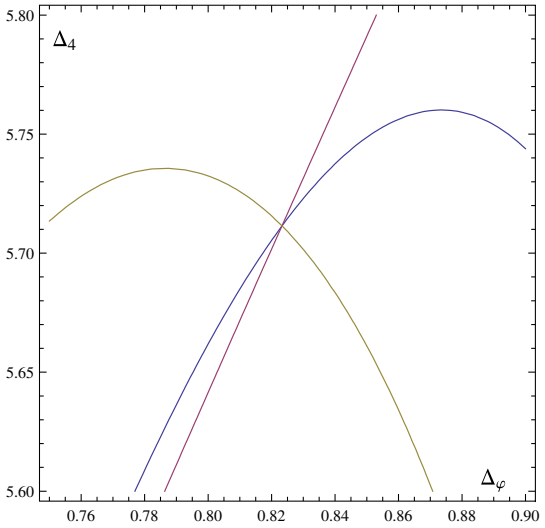

FIG. 3 (color online). Plot in the plane $(\Delta_\varphi, \Delta_4)$ of the zeros of some minors in the case of Yang-Lee edge singularity in four space dimensions.

values in the 5×5 minor associated to the first five homogeneous equations as well as to the primary operators appearing in the fusion of $[\varphi] \times [\varphi]$, namely $[\varphi^2]$, $[\varphi^4]$, $[\varphi^6]$, $[3, 2]$, $[\Delta_4, 4]$. Treating Δ_φ and Δ_{φ^2} as free parameters, the vanishing of this determinant yields the constraint $F(\Delta_\varphi, \Delta_{\varphi^2}) = 0$. This very constraint may be used to obtain a separate estimate of Δ_φ and Δ_{φ^2}, without resorting to higher derivatives. The key observation is that at this truncation level the fusion rule of $[\varphi] \times [\varphi]$ coincides with that of $[\varphi^2] \times [\varphi^2]$; therefore, in the latter case the constraint becomes $F(\Delta_{\varphi^2}, \Delta_{\varphi^2}) = 0$, which has a discrete number of solutions. One is at $\Delta_{\varphi^2} \simeq 1.447$. Inserting this value in the former constraint yields $\Delta_\varphi \simeq 0.518$. The agreement with the most precise estimates [13,14], $\Delta_\varphi = 0.5182(2)$ and $\Delta_{\varphi^2} = 1.4130(4)$, is even too good. In order to have reliable results one should check their stability against the insertion of new operators. More information is needed on primary operators of higher scaling dimensions. Perhaps the recent progress on the knowledge of the scaling properties of higher spin operators [15] could be very useful for this purpose.

In conclusion, we have reformulated the recently developed method of implementing the conformal bootstrap in diverse dimensions so that it can be applied to a larger class of conformal field theories. Its application to the Yang-Lee edge singularity in three and four space dimensions as well as to the 3D critical Ising model, gives rather good results as compared to the best numerical methods.

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