Fictitious Forces and Simulated Magnetic Fields in Rotating Reference Frames

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We show that the Wigner-Bargmann program of grounding nonrelativistic quantum mechanics in the unitary projective representations of the Galilei group can be extended to include all noninertial reference frames. The key concept is the Galilean line group, the group of transformations that ties together all accelerating reference frames, and its representations. These representations are constructed under the natural constraint that they reduce to the well-known unitary, projective representations of the Galilei group when the transformations are restricted to inertial reference frames. This constraint can be accommodated only for a class of representations with a sufficiently rich cocycle structure. Unlike the projective representations of the Galilei group, these cocycle representations of the Galilean line group do not correspond to central extensions of the group. Rather, they correspond to a class of nonassociative extensions, known as loop prolongations, that are determined by three-cocycles. As an application, we show that the phase shifts due to the rotation of Earth that have been observed in neutron interferometry experiments and the rotational effects that lead to simulated magnetic fields in optical lattices can be rigorously derived from the representations of the loop prolongations of the Galilean line group.

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Introduction.—Following the very interesting experiment of Werner, Staudenmann, and Colella [1] that measured an interference effect in the neutron wave function due to the rotation of Earth, there have been several attempts to derive the noninertial effects of rotating reference frames in quantum mechanics. The first among these attempts was by Sakurai [2] who used the similarity between the Lorentz force law $\boldsymbol{F} = e\boldsymbol{v} \times \boldsymbol{B}$ and the Coriolis force $\boldsymbol{F} = 2m\boldsymbol{v} \times \boldsymbol{\omega}$ to calculate the phase shift in the neutron interferometry experiment of [1]. Later, Mashhoon [3] used what he called a "simple, yet tentative, extension of the hypothesis of locality" to obtain the same phase shift. Anandan critiqued this reasoning [4] and, subsequently, Anandan and Suzuki [5] studied the problem by drawing on the analogy between the Galilean transformation properties of the Schrödinger equation and the minimal coupling of a $U(1)$-gauge connection. All of these approaches are heuristic, essentially relying on inspired appeal to analogy rather than being grounded on first principles. The purpose of this Letter is to present a formulation of quantum mechanics that holds in all noninertial reference frames and show that the noninertial effects of rotating reference frames leading to the observed phase shift naturally follow from this formulation. Moreover, our formulation predicts a new contribution to the phase shift in reference frames with time dependent angular velocity, which must be tested with new experiments.

On the experimental side, the analogy between the Lorentz and Coriolis forces continues to be explored. The key observation is that the effect of the Coriolis force on a massive particle, be it charged or neutral, is identical to the effect of an equivalent magnetic field on a charged particle. In this regard, a suitable rotating frame can be used to overcome limitations that arise from the charge neutrality of atoms in cold atom experiments. In fact, several remarkable experiments [6–10] have shown the appearance of vortices in rotating atomic gases and Bose-Einstein condensates, a property generally attributed to superfluids and superconductors in magnetic fields. Simulated magnetic fields generated by the Coriolis effect have also been demonstrated by the behavior of rotating atomic systems in the fractional Hall effect [11]. The formalism presented here provides a theoretical framework for understanding these varied experimental results.

Galilean line group.—Consider spacetime transformations

$$\boldsymbol{x}' = R(t)\boldsymbol{x} + \boldsymbol{a}(t), \qquad t' = t + b, \tag{1}$$

where rotations R and space translations $\boldsymbol{a}$ are functions of time. Together, they define the Euclidean line group $\mathcal{E}(3)$, the infinite dimensional group of functions on the real line taking values in the three-dimensional Euclidean group [12]. When acting on a spacetime point $(\boldsymbol{x}, t)$, R and $\boldsymbol{a}$ get evaluated at t, leading to (1). It follows from (1) that the transformations $(R, \boldsymbol{a}, b)$ form a group

$$(R_2, \boldsymbol{a}_2, b_2)(R_1, \boldsymbol{a}_1, b_1) = ((\Lambda_{b_1}R_2)R_1, \Lambda_{b_1}\boldsymbol{a}_2 + (\Lambda_{b_1}R_2)\boldsymbol{a}_1, b_2 + b_1), \tag{2}$$

where Λ_b is the shift operator $(\Lambda_b f)(t) = f(t + b)$. It accounts for the fact that in a successive application of (1), the group elements R_2 and $\boldsymbol{a}_2$ get evaluated at $t + b_1$, whereas R_1 and $\boldsymbol{a}_1$ get evaluated at t. We call the group defined by (2) the Galilean line group and denote it by $\mathcal{G}$. Properties of $\mathcal{G}$ have been studied in [13,14].

For R constant and $\boldsymbol{a}(t)$ of the form $\boldsymbol{a}(t) = \boldsymbol{a}^{(0)} + \boldsymbol{v}t$, (1) reduces to the usual Galilean transformations and the group law (2) becomes the composition rule for the Galilei group. Hence, the Galilean line group contains the Galilei group as a subgroup. More generally, R and $\boldsymbol{a}$ are arbitrary functions of time and, as such, they implement transformations among all rotationally and linearly accelerating reference frames. For instance, $\boldsymbol{a}(t) = \boldsymbol{a}^{(0)} + \boldsymbol{v}t + (1/2)\boldsymbol{a}^{(2)}t^2$ furnishes transformation to a uniformly accelerating reference frame. Therefore, $\mathcal{G}$ is the group of transformations that ties together all reference frames of a Galilean spacetime. Extending the Wigner-Bargmann program [15,16], which asserts that quantum theory is a linear theory whose structure is largely determined by the unitary representations of the relevant spacetime symmetry group, our principal claim is that a Galilean quantum theory that holds in noninertial reference frames should be grounded in unitary representations of $\mathcal{G}$. We will show that the quantum mechanical effects of rotational reference frames discussed in the introduction, in particular the simulation of magnetic effects, can be obtained from these representations.

Loop prolongations.—If a representation of the Galilean line group is to describe a particle, it must contain as a subrepresentation a unitary, irreducible, projective representation of the Galilei group because it is these representations that describe particles in Galilean quantum mechanics of inertial reference frames [16–18]. Recall that a projective representation is one in which the homomorphism property holds only up to a phase

$$\hat{U}(g_2)\hat{U}(g_1) = e^{i\xi(g_2,g_1)}\hat{U}(g_2g_1). \tag{3}$$

The associativity requirement for $\hat{U}(g)$ leads to a constraint, called the two-cocycle condition, on $\xi(g_2, g_1)$

$$(\delta\xi)(g_3, g_2, g_1) := \xi(g_3, g_2g_1) + \xi(g_2, g_1) - \xi(g_3g_2, g_1) - \xi(g_3, g_2) = 0. \tag{4}$$

For this reason, the function $\xi(g_2, g_1)$ that defines the phase in (3) is also called a two-cocycle.

The unitary irreducible projective representations of the Galilei group can be constructed by the Wigner-Mackey method of induced representations [16,17]. For the spin-zero case, to which we limit ourselves in order to avoid inessential complications, the representation can be defined by the transformation formula

$$\hat{U}(g)|\boldsymbol{q}\rangle = e^{im(\boldsymbol{q}'\cdot\boldsymbol{a}^{(0)} - (1/2)\boldsymbol{v}\cdot\boldsymbol{a}^{(0)} + (1/2)\boldsymbol{q}'^2 b)}|\boldsymbol{q}'\rangle, \tag{5}$$

where $\boldsymbol{q}' = R\boldsymbol{q} + \boldsymbol{v}$ is the Galilean transformation formula for velocity. (We label states by velocity rather than momentum.) Substituting (5) in (3), we deduce a two-cocycle for the Galilei group

$$\xi(g_2, g_1) = \frac{m}{2}(\boldsymbol{a}_2^{(0)}\cdot R_2^{(0)}\boldsymbol{v}_1 - \boldsymbol{v}_2\cdot R_2^{(0)}\boldsymbol{a}_1^{(0)} + b_1\boldsymbol{v}_2\cdot R_2^{(0)}\boldsymbol{v}_1). \tag{6}$$

In (5) and (6), as well as below, the superscript (0) indicates the time independent nature of rotations and spatial translations of the Galilei group (zeroth order Taylor coefficients of R and $\boldsymbol{a}$ of $\mathcal{G}$). In passing, we note that a projective representation of a group is equivalent to a true representation of its central extension [16,17].

Our task is to construct representations of the Galilean line group under the constraint that they reduce to (5). This constraint means that physically relevant representations of $\mathcal{G}$ must be of the same general form as (3), with a suitable two-cocycle $\xi(g_2, g_1)$, g_2, $g_1 \in \mathcal{G}$, that reduces to (6) when restricted to ordinary Galilei transformations. Now, since (6) involves velocities, a two-cocycle of $\mathcal{G}$ reducing to (6) must contain the derivatives $\dot{\boldsymbol{a}}$ of spatial translations. This leads to an additional complication, namely that under time dependent rotations, $\boldsymbol{a}$ and $\dot{\boldsymbol{a}}$ do not transform the same way owing to the inhomogeneous $\dot{R}$ term in $(d/dt)(R\boldsymbol{a}) = R\dot{\boldsymbol{a}} + \dot{R}\boldsymbol{a}$. This difficulty is a familiar one from gauge field theories. In the present case, the trouble is algebraic: any function $\xi(g_2, g_1)$ of the Galilean line group that reduces to (6) fails to fulfill the two-cocycle condition [the generalization of (4)]

$$(\delta\xi)(g_3, g_2, g_1) := \Lambda_{b_1}\xi(g_3, g_2) + \xi(g_3g_2, g_1) - \xi(g_2, g_1) - \xi(g_3, g_2g_1) = 0. \tag{7}$$

Mathematically, since (7) is the condition that ensures associativity, its failure means that there exist no group extensions of the Galilean line group that contain a central extension of the Galilei group. However, there exist non-associative extensions of $\mathcal{G}$ that fit very nicely into the theory of loop prolongations developed by Eilenberg and MacLane [19]. A loop is a set with a binary operation that fulfills the axioms of a group, except for associativity and therewith, also the existence of a two-sided inverse (left and right inverses may be distinct). Also, given three elements a, b, c of a loop L, there always exists a unique element $A(a, b, c) \in L$, called an associator, such that

$$a(bc) = A(a, b, c)[(ab)c]. \tag{8}$$

Associators measure deviations from associativity, much like commutators measure the lack of commutativity. We will not review the general theory of [19] here, but only mention that the construction of loop prolongations runs parallel to that of group extensions, with a little additional care to handle the complications resulting from the failure of (7).

Since the reduction to the ordinary Galilean case is the key requirement, the construction of a loop prolongation of $\mathcal{G}$ must start with a function $\xi(g_2, g_1)$ on $\mathcal{G}\times\mathcal{G}$ that reduces to (6). The simplest choice is:

$$\xi(g_2, g_1) = (m/2)((\Lambda_{b_1} \boldsymbol{a}_2) \cdot (\Lambda_{b_1} R_2)\dot{\boldsymbol{a}}_1 - (\Lambda_{b_1} \dot{\boldsymbol{a}}_2) \cdot (\Lambda_{b_1} R_2)\boldsymbol{a}_1). \quad (9)$$

Substituting (9) in (7),

$$(\delta\xi)(g_3, g_2, g_1) = \frac{m}{2}\Lambda_{b_1}\boldsymbol{\omega}_2 \cdot (\Lambda_{b_1} R_2 \boldsymbol{a}_1 \times \Lambda_{b_2+b_1}(R_3^T \boldsymbol{a}_3)) - \frac{m}{2}\Lambda_{b_2+b_1}\boldsymbol{\omega}_3 \cdot (\Lambda_{b_1}\boldsymbol{a}_2 \times \Lambda_{b_1} R_2 \boldsymbol{a}_1) \neq 0, \quad (10)$$

where the angular velocity vector $\boldsymbol{\omega}$ is related to the time derivative of the rotation matrix by the usual formula $\boldsymbol{\omega} \times \boldsymbol{a} = \dot{R}R^T \boldsymbol{a}$. As mentioned above, we see here that the time dependence of rotations violates the cocycle condition (7). In fact, if restricted to constant rotations, (9) does become a two-cocycle and leads to group extensions of the linear acceleration subgroup of the Galilean line group, albeit these extensions are noncentral [13,20].

It can be shown that the collection of elements $\bar{\mathcal{G}} := \{\bar{g} \equiv (\varphi, g)\}$, where $g \in \mathcal{G}$ and φ is an arbitrary scalar function of time, fulfill all axioms of a loop prolongation [14,19]. The composition rule for $\bar{\mathcal{G}}$ is $\bar{g}_2\bar{g}_1 = (\Lambda_{b_1}\varphi_2 + \varphi_1 + (1/m)\xi(g_2, g_1), g_2 g_1)$ [14]. When $g \in \mathcal{G}$ are restricted to Galilei transformations, $\bar{\mathcal{G}}$ reduces to a central extension of the Galilei group, our crucial embedding requirement.

A direct calculation using (8) shows that the associators of $\bar{\mathcal{G}}$ are of the form

$$A(\bar{g}_3, \bar{g}_2, \bar{g}_1) = (\delta\xi(g_3, g_2, g_1), e), \quad (11)$$

where $(\delta\xi)(g_3, g_2, g_1)$ is defined by (10) and $e = (I, \mathbf{0}, 0)$ is the identity element of $\mathcal{G}$. This means associators do not act on the spacetime points and this situation is precisely as in central extensions of the Galilei group. Further, (10) that defines the associators of $\bar{\mathcal{G}}$ is a three-cocycle. We call the loop prolongation $\bar{\mathcal{G}}$ defined by the three-cocycle (10) the Galilean line loop. Though not common in the physics literature, three-cocycles have been looked at in connection with magnetic monopoles [21,22]. However, the content of the present case appears to be quite different from these previous cases.

Unitary representations of the Galilean line loop.—The main thesis that we advocate in this Letter is that quantum mechanics in noninertial reference frames should be grounded in unitary, possibly cocycle, representations of the Galilean line loop $\bar{\mathcal{G}}$. In particular, a particle should be described by a unitary, irreducible representation (irrep) of $\bar{\mathcal{G}}$ and multiparticle systems by tensor products of such irreducible representations.

The irreps of $\bar{\mathcal{G}}$ can be constructed by the method of induced representations. Operators $\hat{U}(\bar{g})$ furnishing the representation can be defined by their action on the velocity eigenvectors $|\boldsymbol{q}\rangle$:

$$\hat{U}(\bar{g})|\boldsymbol{q}\rangle = e^{i\xi(\bar{g};\boldsymbol{q})}|\Lambda_{-b}\boldsymbol{q}'\rangle, \quad (12)$$

where

$$\xi(\bar{g};\boldsymbol{q}) = m\left(\varphi + \boldsymbol{q}' \cdot \boldsymbol{a} - \frac{1}{2}\boldsymbol{a} \cdot \dot{\boldsymbol{a}} + \frac{1}{2}(\Lambda_{-b} - 1)\boldsymbol{q}' \cdot \boldsymbol{a}_{\boldsymbol{q}'}\right),$$
$$\boldsymbol{a}_{\boldsymbol{q}} = \int dt\boldsymbol{q}, \quad \boldsymbol{q} = \frac{d}{dt}\boldsymbol{a}_{\boldsymbol{q}}, \quad (13)$$
$$\boldsymbol{q}' = R\boldsymbol{q} + \dot{R}\boldsymbol{a}_{\boldsymbol{q}} + \dot{\boldsymbol{a}} = \frac{d}{dt}(R\boldsymbol{a}_{\boldsymbol{q}} + \boldsymbol{a}) = \frac{d}{dt}\boldsymbol{a}_{\boldsymbol{q}'}.$$

It follows from (13) that $\boldsymbol{a}_{\boldsymbol{q}}$ is our standard boost in that the operator $\hat{U}(g_{\boldsymbol{q}})$, $g_{\boldsymbol{q}} = (0, I, \boldsymbol{a}_{\boldsymbol{q}}, 0)$, transforms the rest vector $|\mathbf{0}\rangle$ to $|\boldsymbol{q}\rangle$. In evaluating the integral $\int dt\boldsymbol{q}$ to determine $\boldsymbol{a}_{\boldsymbol{q}}$, we choose the boundary condition that the constant of integration be set to zero.

The representation (12) reduces precisely to the Galilean group representation (5). By way of this property, the m in (13) acquires interpretation as inertial mass. Further, composing two elements,

$$\hat{U}(\bar{g}_2)\hat{U}(\bar{g}_1) \mid \boldsymbol{q}\rangle = e^{i\xi_2(\bar{g}_2,\bar{g}_1;\boldsymbol{q})}\hat{U}(\bar{g}_2\bar{g}_1) \mid \boldsymbol{q}\rangle, \quad (14)$$

where

$$\xi_2(\bar{g}_2, \bar{g}_1; \boldsymbol{q}) := \xi(\bar{g}_1; \boldsymbol{q}) + \xi(\bar{g}_2; \Lambda_{-b_1}\boldsymbol{q}') - \xi(\bar{g}_2\bar{g}_1; \boldsymbol{q})$$
$$= m(\Lambda_{b_1}\boldsymbol{\omega}_2) \cdot (R_1 \boldsymbol{a}_{\boldsymbol{q}} \times \boldsymbol{a}_1 - (\Lambda_{b_1} R_2)\boldsymbol{a}_1 \times \Lambda_{b_1}\boldsymbol{a}_2) + m(1 - \Lambda_{b_1})\varphi_2 + m(\Lambda_{-b_1} - 1)\xi(g_2, g_1 g_{\boldsymbol{q}}). \quad (15)$$

It is clear from (15) that the representation structure of the loop $\bar{\mathcal{G}}$ is more intricate than that of projective representations of groups. In the latter case, once the central extension has been taken, the equivalent representation is a true representation. In contrast, a phase factor appears in (14), showing that the representation of the loop prolongation is itself a cocycle representation. A noteworthy consequence of the phase structure of the representation is that the Bargmann argument that mass is super selected applies in this general case, too.

The repeated application of (14) and (15) gives

$$(\hat{U}(\bar{g}_3)\hat{U}(\bar{g}_2))\hat{U}(\bar{g}_1) = \hat{U}(\bar{g}_3)(\hat{U}(\bar{g}_2)\hat{U}(\bar{g}_1)). \quad (16)$$

This shows that even though the Galilean line loop $\bar{\mathcal{G}}$ is nonassociative, the operators $\hat{U}(\bar{g})$ furnishing its representation do associate, a necessary requirement for linear operators in a vector space. This is a direct outcome of the structure of the associators (11) and composition (14) in that the difference between the phases on the two sides of (16) exactly cancels the operator $\hat{U}(A(\bar{g}_3, \bar{g}_2, \bar{g}_1))$ representing the corresponding associator. The details of this calculation as well as the construction of the representation (12) can be found in [14]. In sharp contrast, the operators representing symmetry transformations do not associate, and thus, are not even defined in the three-cocycle analysis of [21].

Even though $\bar{\mathcal{G}}$ is a loop, all of the physical transformations that correspond to the usual observables form one parameter subgroups. Hence, the observables for a free particle in a noninertial reference frame can be obtained from (12) in the usual way. In particular, from the definition that the Hamiltonian is the generator of time translations, $\hat{H} := i(d\hat{U}(0, I, 0, b)/db)|_{b=0}$, we obtain from (12),

$$\hat{H} = \frac{\hat{\boldsymbol{P}}^2}{2m} + m\dot{\boldsymbol{q}} \cdot \left(\hat{\boldsymbol{X}} + \frac{1}{2}\boldsymbol{a}_q\right). \tag{17}$$

Here, $\hat{\boldsymbol{P}}$ is the momentum operator, defined as the generator of constant spatial translations $\boldsymbol{a}^{(0)}$, and $\hat{\boldsymbol{X}}$ is the position operator canonically conjugated to $\hat{\boldsymbol{P}}$, $\hat{\boldsymbol{X}} := (i/m)\nabla_q$. The first term of (17) is the usual kinetic energy term. It results from the usual transformation $\hat{H}' := \hat{U}(\bar{g})\hat{H}_I\hat{U}(\bar{g})^\dagger$ of the inertial frame Hamiltonian $\hat{H}_I$ under the line group. The second term $\hat{H}_{\rm fic} \equiv m\dot{\boldsymbol{q}} \cdot (\hat{\boldsymbol{X}} + \frac{1}{2}\boldsymbol{a}_q)$ carries the fictitious force effects of the noninertial reference frame. Not surprisingly, it is proportional to the inertial mass. Therefore, we may write the Schrödinger equation in a noninertial reference frame as

$$i\frac{\partial \psi'}{\partial t} = (\hat{H}' + \hat{H}_{\rm fic})\psi', \tag{18}$$

where $\psi' = \hat{U}(\bar{g})\psi$ is the transformed wave function.

Simulated magnetic fields from rotating reference frames.—As a special case, consider the transformation from an inertial reference frame to a rotating reference frame. Then, from (12) and (13),

$$|\boldsymbol{q}\rangle = \hat{U}(0, R, \boldsymbol{0}, 0)|\boldsymbol{q}^{(0)}\rangle, \tag{19}$$

$$\boldsymbol{q} = R\boldsymbol{q}^{(0)} + \boldsymbol{\omega} \times \boldsymbol{a}_q, \tag{20}$$

where the superscript (0) indicates the time independence of velocity in an inertial frame. Differentiating (20), we obtain

$$\dot{\boldsymbol{q}} \equiv \ddot{\boldsymbol{a}}_q = \dot{\boldsymbol{\omega}} \times \boldsymbol{a}_q + 2\boldsymbol{\omega} \times \boldsymbol{q} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{a}_q), \tag{21}$$

where the Euler, Coriolis, and centrifugal terms have emerged naturally. The substitution of (21) in (17) gives

$$\hat{H} = \hat{A}_0 + \frac{1}{2m}(\hat{\boldsymbol{P}} - \hat{\boldsymbol{A}})^2, \tag{22}$$

where

$$\hat{A}_0 = -2m\left(\boldsymbol{\omega} \times \left(\hat{\boldsymbol{X}} + \frac{1}{2}\boldsymbol{a}_q\right)\right) \cdot (\boldsymbol{\omega} \times \hat{\boldsymbol{X}}) - m\boldsymbol{a}_q \cdot (\dot{\boldsymbol{\omega}} \times \hat{\boldsymbol{X}}),$$
$$\hat{\boldsymbol{A}} = 2m\boldsymbol{\omega} \times \left(\hat{\boldsymbol{X}} + \frac{1}{2}\boldsymbol{a}_q\right). \tag{23}$$

Hence, we have shown that the gauge connection that appears in a rotational reference frame can be rigorously derived from the representations of the Galilean line loop. The $(1/2)\boldsymbol{a}_q$ in $\hat{\boldsymbol{X}} + (1/2)\boldsymbol{a}_q$ is not significant as it results from the choice of the two-cocycle (6) and may be removed by a suitable phase. Even then, $\hat{\boldsymbol{A}}$ and the first term of $\hat{A}_0$ differ by factors of 2 and 4 from what has been obtained previously by the analogy between the Coriolis and Lorentz forces [2,5]. [See (25) below.] Further, besides being rigorous, our result holds for nonconstant $\boldsymbol{\omega}$ as well and in that case there is another term, $m\boldsymbol{a}_q \cdot (\dot{\boldsymbol{\omega}} \times \hat{\boldsymbol{X}})$, which would be missed if only rotations were considered, rather than the full Galilean line group. This suggests a more intricate gauge structure that should be tested by new experiments with time dependent $\boldsymbol{\omega}$.

As a possible setup for such experiments, consider a frame of reference that oscillates about the z axis so that the azimuthal angle is a sinusoidal function of time, $\phi(t) = \phi_0 \sin\omega t$, where ϕ_0 and ω are constants. The corresponding rotation, angular velocity and acceleration matrices are

$$R(t) = \begin{pmatrix} \cos\phi(t) & -\sin\phi(t) & 0 \\ \sin\phi(t) & \cos\phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Omega(t) = \dot{R}R^T = \omega\phi_0\cos\omega t\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\dot{\Omega}(t) = -\omega^2\phi_0\sin\omega t\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Hamiltonian may then be explicitly obtained by substituting these expressions into (22) and (23). In particular, the Euler term

$$m\boldsymbol{a}_q \cdot (\dot{\boldsymbol{\omega}} \times \hat{\boldsymbol{X}}) = m\omega^2 t\phi(t)[(\cos\phi(t)\hat{X}_2 + \sin\phi(t)\hat{X}_1)q_1^{(0)} + (\cos\phi(t)\hat{X}_1 - \sin\phi(t)\hat{X}_2)q_2^{(0)}]. \tag{24}$$

Since thermal neutrons are usually used in interferometry [3], the contribution of (24) to the phase shift may be determined by integrating the gauge potential from the source to the detector along two suitable paths leading to interference. Or, we may consider a simple system such as a particle in a box in an oscillating reference frame and its energy level shifts due to (24). To that end, we may use the relation $\psi' = \hat{U}(R(t))\psi$ to solve (18).

As a final remark, we note that our Hamiltonian, both in its general form (17) and specific form (22) for rotations, is different from the corresponding classical Hamiltonian. For rotations, the classical Hamiltonian is $H_{\rm clas} = (1/2m)(\boldsymbol{p}' - \boldsymbol{A})^2 + A_0$, where $\boldsymbol{p}'$ is the canonical momentum corresponding to $\boldsymbol{x}' = R(t)\boldsymbol{x}$ and

$$\boldsymbol{A} = m\boldsymbol{\omega} \times \boldsymbol{x}', \qquad A_0 = -\frac{m}{2}(\boldsymbol{\omega} \times \boldsymbol{x}')^2. \tag{25}$$

Notably, the Euler term is absent in the classical Hamiltonian and the remaining terms of the gauge connection differ from those of (23) by factors of 2 and 4. However, the classical Hamiltonian leads to an equation of motion that is identical to (21). These differences are clearly rooted in the structure of canonical transformations in that, in contrast to (17), accelerations appear only in the equations of motion and never in the transformed Hamiltonian. This observation leads to the inference that the usual way of "quantizing" a theory by promoting the classical Hamiltonian to an operator is at odds with the Wigner-Bargmann approach when covariance under arbitrary coordinate transformations is required. In this regard, besides its applications to simulated magnetic fields in rotating reference frames, the study presented here also helps settle two broad theoretical issues: the correct way to ground quantum mechanics in arbitrary reference frames and the physical relevance of three-cocycles.

[1] S. A. Werner, J.-L. Staudenmann, and R. Colella, Phys. Rev. Lett. **42**, 1103 (1979).
[2] J. J. Sakurai, Phys. Rev. D **21**, 2993 (1980).
[3] B. Mashhoon, Phys. Rev. Lett. **61**, 2639 (1988).
[4] J. Anandan, Phys. Rev. Lett. **68**, 3809 (1992).
[5] J. Anandan and J. Suzuki, arXiv:quant-ph/0305081v2.
[6] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, Nature (London) **435**, 1047 (2005).
[7] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, and E. A. Cornell, Phys. Rev. Lett. **92**, 040404 (2004).
[8] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).
[9] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science **292**, 476 (2001).
[10] Y.-J. Lin, R. L. Compton, K. Jiménez-García, J. V. Porto, and I. B. Spielman, arXiv:1007.0294v1.
[11] N. Gemelke, E. Sarajlic, and S. Chu, arXiv:1007.2677v1.
[12] W. H. Klink, Ann. Phys. (N.Y.) **260**, 27 (1997).
[13] B. A. MacGregor, A. E. McCoy, and S. Wickramasekara, Ann. Phys. (Amsterdam) **327**, 2310 (2012).
[14] W. H. Klink and S. Wickramasekara, Ann. Phys. (Amsterdam) **336**, 261 (2013).
[15] E. P. Wigner, Ann. Math. **40**, 149 (1939); *Group Theory and Quantum Mechanics* (Academic Press, New York, 1959); S. Weinberg, *Quantum Theory of Fields,* (Cambridge University Press, Cambridge, England, 1995), Vol. I.
[16] V. Bargmann, Ann. Math. **59**, 1 (1954).
[17] J.-M. Levy-Leblond, J. Math. Phys. (N.Y.) **4**, 776 (1963).
[18] R. E. Warren and W. H. Klink, J. Math. Phys. (N.Y.) **11**, 1155 (1970).
[19] S. Eilenberg and S. MacLane, Duke Math. J. **14**, 435 (1947).
[20] W. H. Klink and S. Wickramasekara, arXiv:1303.6273.
[21] R. Jackiw, Phys. Rev. Lett. **54**, 159 (1985).
[22] Bo-Yu Hou, Bo-Yuan Hou, and Pei Wang, Ann. Phys. (N.Y.) **171**, 172 (1986); B. Grossman, Phys. Lett. B **152**, 93 (1985).