Nuclear Polarization Corrections to the μ^4He$^+$ Lamb Shift

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Stimulated by the proton radius conundrum, measurements of the Lamb shift in various light muonic atoms are planned at PSI. The aim is to extract the rms charge radius with high precision, limited by the uncertainty in the nuclear polarization corrections. We present an *ab initio* calculation of the nuclear polarization for μ^4He$^+$ leading to an energy correction in the $2S$-$2P$ transitions of $\delta^A_{\rm pol} = -2.47$ meV $\pm 6\%$. We use two different state-of-the-art nuclear Hamiltonians and utilize the Lorentz integral transform with hyperspherical harmonics expansion as few-body methods. We take into account the leading multipole contributions, plus Coulomb, relativistic, and finite-nucleon-size corrections. Our main source of uncertainty is the nuclear Hamiltonian, which currently limits the attainable accuracy. Our predictions considerably reduce the uncertainty with respect to previous estimates and should be instrumental to the μ^4He$^+$ experiment planned for 2013.

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Introduction.—Recent laser spectroscopy measurements of the muonic hydrogen Lamb shift [1] ($2S$-$2P$ transition) at PSI have tremendously improved the accuracy in determining the proton charge radius $\langle r_p^2\rangle^{1/2}$. Besides experimental precision, the accurate deduction of $\langle r_p^2\rangle^{1/2}$ heavily relies on theory. Theoretical estimates of quantum electrodynamics (QED), recoil, and nuclear structure corrections are needed. The proton radius extracted at PSI [2,3] is 10 times more accurate than the value determined from electron hydrogen, i.e., CODATA-2010 [4], but also deviates from it by 7σ. This discrepancy, coined the "proton radius puzzle," is challenging the understanding of experimental systematic errors and of theoretical calculations based on the standard model. Alternative explanations involving physics beyond the standard model (e.g., lepton flavor universality violations) have also been proposed (see Ref. [5] for a review). To understand this puzzle, one possible strategy is to investigate atoms with other nuclear charges Z or mass numbers A and track the persistence or variation of this discrepancy [6]. Extending the Lamb shift measurements to other muonic atoms, e.g., μD, μ^3He$^+$, and μ^4He$^+$, must be complemented by corresponding theoretical calculations. Lamb shifts in light muonic atoms are very sensitive to nuclear structure effects since a muon is 206 times heavier than an electron and thus interacts more closely with the nucleus [7,8]. The $2S$-$2P$ energy difference can be generally related to the nuclear charge radius $\langle R_c^2\rangle^{1/2}$ (in $\hbar = c = 1$ units) by [9]

$$\Delta E \equiv \delta_{\rm QED} + \delta_{\rm pol} + \delta_{\rm Zem} + m_r^3 (Z\alpha)^4 \langle R_c^2\rangle/12, \qquad (1)$$

in an expansion of $Z\alpha$ up to fifth order. Here, α is the fine-structure constant and $m_r = m_\mu M_A/(m_\mu + M_A)$ is the reduced mass related to the nuclear mass M_A and the muon mass m_μ. $\delta_{\rm Zem}$ is the third Zemach moment [10] defined via the nuclear charge density $\rho_0(\boldsymbol{R})$ as

$$\delta_{\rm Zem} = -\frac{m_r^4}{24}(Z\alpha)^5 \iint d\boldsymbol{R}\, d\boldsymbol{R}' |\boldsymbol{R} - \boldsymbol{R}'|^3 \rho_0(\boldsymbol{R}) \rho_0(\boldsymbol{R}'). \qquad (2)$$

Contributions to $\delta_{\rm QED}$ in Eq. (1) are from vacuum polarization, muon self-energy, and relativistic recoil, while $\delta_{\rm pol} = \delta^A_{\rm pol} + \delta^N_{\rm pol}$ is the sum of the nuclear polarization $\delta^A_{\rm pol}$ and the intrinsic nucleon polarizability $\delta^N_{\rm pol}$. Since calculations of $\delta_{\rm QED}$ and spectroscopy measurements of ΔE have both achieved high accuracy, the current bottleneck in accurately extracting $\langle R_c^2\rangle^{1/2}$ from Eq. (1) lies in the polarization uncertainty. In muonic helium, to determine the nuclear radii with a relative accuracy of 3×10^{-4}, $\delta_{\rm pol}$ needs to be known at the $\sim 5\%$ level [6]. Here, we focus on the nuclear polarization $\delta^A_{\rm pol}$. $\delta^N_{\rm pol}$ depends on the internal nucleon structure and can be evaluated separately [11–13] apart from nuclear dynamics.

The nuclear polarization is induced by a two-photon exchange process (Fig. 1), where the nucleus in an atom is virtually excited by its Coulomb interaction with the lepton. Effects on the leptonic spectrum are evaluated in second-order perturbation theory with inputs from nuclear structure functions, also called response functions. In early calculations, structure functions were either calculated using simple nuclear potentials (e.g., μD [14] and μ^{12}C [15]) or extracted from measurements of photoabsorption cross sections (e.g., μ^4He$^+$ [16–18]). However, these approaches lack the desired accuracy. For example,

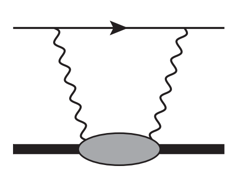

FIG. 1. The lepton-nucleus two-photon exchange.

Refs. [16–18] yielded $\delta^A_{\rm pol} = -3.1$ meV $\pm 20\%$ for $\mu^4{\rm He}^+$. Evaluations of the polarization effect in μD using state-of-the-art potentials have significantly improved the accuracy [19,20]. The purpose of this Letter is to extend these calculations to $\mu^4{\rm He}^+$. We present the first *ab initio* calculation of the nuclear polarization effects in $\mu^4{\rm He}^+$ using modern nuclear potentials. We consider systematically all terms contributing to order $(Z\alpha)^5$ and estimate the theoretical error.

Polarization contributions.—Following works on μD by Pachucki [20] and Friar [21], we separate contributions to the $\mu^4{\rm He}^+$ polarization into nonrelativistic, relativistic, Coulomb-distortion, and nucleon-size effects. The $\mu^4{\rm He}^+$ system is described as a muon interacting with the ^{4}He nucleus containing four pointlike nucleons by

$$H = H_{\rm nucl} + H_\mu - \Delta H, \tag{3}$$

where $H_{\rm nucl}$ denotes the nuclear Hamiltonian and H_μ is the nonrelativistic Hamiltonian of a muon in the Coulomb potential of a pointlike nucleus

$$H_\mu = p^2/2m_r - Z\alpha/r. \tag{4}$$

Here, $p = |\boldsymbol{p}|$ $(r = |\boldsymbol{r}|)$ is the relative momentum (distance) of the muon from the center of mass (c.m.) of the nucleus. The last term in Eq. (3)

$$\Delta H = \sum_a^Z \Delta V(\boldsymbol{r}, \boldsymbol{R}_a) \equiv \sum_a^Z \alpha\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{R}_a|} - \frac{1}{r}\right) \tag{5}$$

represents the difference between the muon interaction with the nucleus and the sum of Coulomb interactions between the muon and each proton, located at a distance $\boldsymbol{R}_a$ from the c.m. Polarization effects are evaluated as corrections due to ΔH in second-order perturbation theory. Utilizing the point-nucleon charge density operator

$$\hat{\rho}(\boldsymbol{R}) \equiv \frac{1}{Z}\sum_a^Z \delta(\boldsymbol{R} - \boldsymbol{R}_a), \tag{6}$$

the nuclear polarization correction assumes the form

$$\delta^A_{\rm pol} = -\sum\!\!\!\!\!\!\int_{N\neq N_0} \iint d\boldsymbol{R} d\boldsymbol{R}' \rho_N^*(\boldsymbol{R}) P(\boldsymbol{R}, \boldsymbol{R}', \omega_N) \rho_N(\boldsymbol{R}'), \tag{7}$$

where $\rho_N(\boldsymbol{R}) = \langle N|\hat{\rho}(\boldsymbol{R})|N_0\rangle$ is the charge density transition matrix element and

$$P(\boldsymbol{R}, \boldsymbol{R}', \omega_N) = -Z^2 \int d\boldsymbol{r} d\boldsymbol{r}' \Delta V(\boldsymbol{r}, \boldsymbol{R}) \langle \mu_0|\boldsymbol{r}\rangle \times \langle \boldsymbol{r}|\frac{1}{H_\mu + \omega_N - \epsilon_{\mu_0}}|\boldsymbol{r}'\rangle\langle \boldsymbol{r}'|\mu_0\rangle \Delta V(\boldsymbol{r}', \boldsymbol{R}') \tag{8}$$

is the muonic matrix element. Here, $\omega_N = E_N - E_{N_0}$, and E_{N_0}, E_N, $|N_0\rangle$, and $|N\rangle$ are the nuclear ground- and excited-state energies and wave functions, respectively. The symbol $\sum\!\!\!\!\!\!\int$ indicates a sum over discrete plus an integration over continuum states. ϵ_{μ_0} and $|\mu_0\rangle$ are the unperturbed atomic energy and wave function in either the $2S$ or $2P$ state. In Eq. (7), the nucleus is excited into all possible intermediate states, which represents the inelastic part of the two-photon exchange, while the elastic part is known as a finite-size effect [9].

The leading contribution to $\delta^A_{\rm pol}$ is obtained in the nonrelativistic limit, neglecting in Eq. (8) the Coulomb-potential part of H_μ. Only contributions to the $2S$ state are considered, as $2P$-state effects enter only at order $(Z\alpha)^6$. In this limit, we have

$$P(\boldsymbol{R}, \boldsymbol{R}', \omega) = -Z^2\phi^2(0) \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left(\frac{4\pi\alpha}{q^2}\right)^2 (1 - e^{i\boldsymbol{q}\cdot\boldsymbol{R}}) \times \frac{1}{q^2/2m_r + \omega}(1 - e^{-i\boldsymbol{q}\cdot\boldsymbol{R}'}), \tag{9}$$

where $\phi^2(0) = (m_r Z\alpha)^3/8\pi$ is the normalization coefficient of the muon $2S$ state. After integrating over $\boldsymbol{q}$ in Eq. (9), terms not depending on both $\boldsymbol{R}$ and $\boldsymbol{R}'$ drop out, due to the orthogonality of the nuclear eigenstates. The resulting muonic matrix element P is then a function of $\xi\sqrt{2m_r\omega}$, with $\xi \equiv |\boldsymbol{R} - \boldsymbol{R}'|$. Expanding P in powers of $\xi\sqrt{2m_r\omega}$ up to fourth order yields

$$P(\xi, \omega) \simeq \frac{m_r^3(Z\alpha)^5}{12}\sqrt{\frac{2m_r}{\omega}}\left[\xi^2 - \frac{\sqrt{2m_r\omega}}{4}\xi^3 + \frac{m_r\omega}{10}\xi^4\right]. \tag{10}$$

ξ indicates the "virtual" distance a proton travels during the two-photon exchange. According to the uncertainty principle, it is related to ω by $\xi \sim 1/\sqrt{2M_A\omega}$. Therefore, the expansion parameter $\xi\sqrt{2m_r\omega}$ in Eq. (10) is of order $\sqrt{m_r/M_A} \approx 0.17$.

In the following, we will relate the different $\delta^A_{\rm pol}$ terms coming from Eq. (10) to structure functions. Details will be given in a forthcoming paper. The structure functions are defined as

$$S_O(\omega) \equiv \frac{1}{2J_0 + 1} \sum\!\!\!\!\!\!\int_{N\neq N_0, J} |\langle N_0 J_0 \| \hat{O} \| N J\rangle|^2 \delta(\omega - \omega_N), \tag{11}$$

where $\hat{O}$ is a general operator. Here, we use the reduced matrix elements by employing the Wigner-Eckart theorem [22]. J_0 (J) is the total angular momentum of the ground (excited) state of ^{4}He.

The leading contribution to the nuclear polarization, denoted by the superscript (0), is the electric-dipole correction, which originates from the ξ^2 term in Eq. (10)

$$\delta^{(0)}_{D1} = -\frac{2\pi m_r^3}{9}(Z\alpha)^5 \int_{\omega_{\rm th}}^\infty d\omega \sqrt{\frac{2m_r}{\omega}} S_{D_1}(\omega), \tag{12}$$

where $\hat{D}_1 = (1/Z)\sum_a^Z R_a Y_1(\hat{R}_a)$, Y_1 is the rank-one spherical harmonics, and $\omega_{\rm th}$ indicates the threshold excitation energy of ^{4}He, i.e., $\omega_{\rm th} = 19.8$ MeV.

The subleading ξ^3 term is independent of ω. Replacing $\sum\!\!\!\!\!\!\int_{N\neq N_0} |N\rangle\langle N|$ with $1 - |N_0\rangle\langle N_0|$, the contribution of this term, denoted by the superscript (1), is

$$\delta^{(1)} = -\frac{m_r^4}{24}(Z\alpha)^5 \iint d\boldsymbol{R}d\boldsymbol{R}'|\boldsymbol{R}-\boldsymbol{R}'|^3[\langle N_0|\hat{\rho}^\dagger(\boldsymbol{R})\hat{\rho}(\boldsymbol{R}') \times|N_0\rangle - \rho_0(\boldsymbol{R})\rho_0(\boldsymbol{R}')], \tag{13}$$

where $\rho_0(\boldsymbol{R}) \equiv \rho_{N_0}(\boldsymbol{R}) = \langle N_0|\hat{\rho}(\boldsymbol{R})|N_0\rangle$ is the charge density, satisfying $\int d\boldsymbol{R}\rho_0(\boldsymbol{R}) = 1$. It is convenient to write Eq. (13) as $\delta^{(1)} = \delta^{(1)}_{R3pp} + \delta^{(1)}_{Z3}$. The first term $\delta^{(1)}_{R3pp}$ is the ground-state expectation value of the proton-proton distance cubed. The second term $\delta^{(1)}_{Z3}$ cancels exactly the third Zemach moment $\delta_{\rm Zem}$ that appears in the finite-size corrections to the Lamb shift (1). This cancellation was also found by Pachucki [20] and Friar [21,23] in μD. Here, we retain this term and calculate $\delta^{(1)}_{R3pp}$ and $\delta^{(1)}_{Z3}$ first in the point-nucleon limit and then add finite-nucleon-size corrections.

Contributions from the sub-subleading ξ^4 term, denoted with the superscript (2), are

$$\delta^{(2)} = \frac{m_r^5}{18}(Z\alpha)^5\int_{\omega_{\rm th}}^{\infty} d\omega\sqrt{\frac{\omega}{2m_r}}\Big[S_{R^2}(\omega) + \frac{16\pi}{25}S_Q(\omega) + \frac{16\pi}{5}\mathcal{S}_{D_1D_3}(\omega)\Big], \tag{14}$$

where S_{R^2} and S_Q are the respective structure functions of the monopole $\hat{R}^2 = (1/Z)\sum_a^Z R_a^2$ and quadrupole $\hat{Q} = (1/Z)\sum_a^Z R_a^2 Y_2(\hat{R}_a)$ operators. $\mathcal{S}_{D_1D_3}$ indicates the interference between two multipolarity-one operators $\hat{D}_1$ and $\hat{D}_3 = (1/Z)\sum_a^Z R_a^3 Y_1(\hat{R}_a)$ and is calculated as

$$\mathcal{S}_{D_1D_3}(\omega) = [S_{D_1+D_3}(\omega) - S_{D_1}(\omega) - S_{D_3}(\omega)]/2. \tag{15}$$

Effects from $\hat{R}^2$, Q, and the interference term in Eq. (14) are defined, respectively, as $\delta^{(2)}_{R2}$, $\delta^{(2)}_{Q}$, and $\delta^{(2)}_{D1D3}$.

Since the electric-dipole contribution $\delta^{(0)}_{D1}$ dominates in the nonrelativistic approximation, we add relativistic corrections solely to this term. These corrections can be obtained from the longitudinal (L) and transverse (T) parts of the two-photon exchange amplitude [15,16]. Replacing the nonrelativistic Green's function with relativistic expressions, we obtain relativistic corrections as

$$\delta^{(0)}_{L(T)} = \frac{2m_r^3}{9}(Z\alpha)^5\int_{\omega_{\rm th}}^{\infty} d\omega\,\Theta_{L(T)}\Big(\frac{\omega}{m_r}\Big)S_{D_1}(\omega), \tag{16}$$

where the two energy-dependent weights are

$$\Theta_L(\lambda) = \pi\sqrt{2/\lambda} + 2\mathcal{F}(\lambda), \tag{17}$$

$$\Theta_T(\lambda) = \lambda + \lambda\ln(2\lambda) + \lambda^2\mathcal{F}(\lambda), \tag{18}$$

with

$$\mathcal{F}(\lambda) = \sqrt{(\lambda-2)/\lambda}\,{\rm arctanh}[\sqrt{(\lambda-2)/\lambda}] - \sqrt{(\lambda+2)/\lambda}\,{\rm arctanh}[\sqrt{\lambda/(\lambda+2)}], \tag{19}$$

and $\lambda \equiv \omega/m_r$ ranges from ~ 0.2 to infinity. Expressions similar to Eqs. (17) and (18) are also derived by Martorell *et al.* [24], whose transverse form is, however, valid only for $\lambda \ge 2$.

By including a Coulomb distorted muonic wave function in the intermediate state of the two-photon exchange in Fig. 1, $\delta^{(0)}_{D1}$ is corrected in both the $2S$ and $2P$ states. We follow the derivation by Friar [18] and provide Coulomb-distortion corrections up to second order in a $Z\alpha\sqrt{2m_r/\omega}$ expansion. The Coulomb-distortion correction is given as the difference between the $2S$ and $2P$ levels,

$$\delta^{(0)}_C = -\frac{2\pi m_r^3}{9}(Z\alpha)^6\int_{\omega_{\rm th}}^{\infty} d\omega\Big[\frac{m_r}{\omega}\Big(\frac{1}{6} + \ln\frac{2m_r Z^2\alpha^2}{\omega}\Big) - \frac{17}{16}Z\alpha\Big(\frac{2m_r}{\omega}\Big)^{3/2}\Big]S_{D_1}(\omega). \tag{20}$$

Even though $\delta^{(0)}_C$ is of order $(Z\alpha)^6$, its contribution is significantly enhanced by the $2S$-logarithmic term in Eq. (20).

Considering the finite size of the nucleons, the proton position in Eq. (5) should be replaced by a convolution over the proton charge density, and a similar term should be added for the neutron. For the proton and neutron form factors [25], we use low-momentum approximations: $G_p^E(q^2) \simeq 1 - 2q^2/\beta^2$ and $G_n^E(q^2) \simeq \lambda q^2$. Following Ref. [21], we choose $\beta = 4.120\ {\rm fm}^{-1}$ and $\lambda = 0.019\,35\ {\rm fm}^2$, which reproduce $\langle r_p^2\rangle^{1/2} = 0.8409$ fm [3] and $\langle r_n^2\rangle = -0.1161\ {\rm fm}^2$ [26]. Since corrections to $\delta^{(0)}$ vanish, the leading nucleon-size (NS) correction enters in $\delta^{(1)}$ as

$$\delta^{(1)}_{\rm NS} = -m_r^4(Z\alpha)^5\Big[\frac{2}{\beta^2} - \lambda\Big]\iint d\boldsymbol{R}d\boldsymbol{R}'|\boldsymbol{R}-\boldsymbol{R}'|[\langle N_0|\hat{\rho}^\dagger \times(\boldsymbol{R})\hat{\rho}(\boldsymbol{R}')|N_0\rangle - \rho_0(\boldsymbol{R})\rho_0(\boldsymbol{R}')], \tag{21}$$

where we have used the isospin symmetry of the ^{4}He ground state [27,28]. The prefactors $2/\beta^2$ and $-\lambda$ account for respective contributions from proton-proton and neutron-proton correlations, whereas neutron-neutron correlations are neglected. Similarly to $\delta^{(1)}$, contributions to $\delta^{(1)}_{\rm NS}$ from the two integrands in Eq. (21) are denoted as $\delta^{(1)}_{\rm NS} = \delta^{(1)}_{R1pp} + \delta^{(1)}_{Z1}$.

The subleading nucleon-size correction enters in $\delta^{(2)}$ as

$$\delta^{(2)}_{\rm NS} = -\frac{16\pi}{9}m_r^5(Z\alpha)^5\Big[\frac{2}{\beta^2} - \lambda\Big]\int_{\omega_{\rm th}}^{\infty} d\omega\sqrt{\frac{\omega}{2m_r}}S_{D_1}(\omega). \tag{22}$$

Summing up the nuclear polarization corrections to the Lamb shift, we have

$$\begin{aligned}\delta^A_{\rm pol} &= \delta^{(0)} + \delta^{(1)} + \delta^{(2)} + \delta_{\rm NS}\\ &= [\delta^{(0)}_{D1} + \delta^{(0)}_L + \delta^{(0)}_T + \delta^{(0)}_C] + [\delta^{(1)}_{R3pp} + \delta^{(1)}_{Z3}]\\ &\quad + [\delta^{(2)}_{R2} + \delta^{(2)}_{Q2} + \delta^{(2)}_{D1D3}] + [\delta^{(1)}_{R1pp} + \delta^{(1)}_{Z1} + \delta^{(2)}_{\rm NS}].\end{aligned} \tag{23}$$

Computational tools.—The ^{4}He structure functions involve a sum over all the spectrum, including energies beyond the three-body disintegration threshold. Thus, we calculate them using the Lorentz integral transform (LIT) method [29,30], which allows exact calculations in this energy range. We use the effective interaction hyperspherical harmonics [31] few-body technique to solve the ^{4}He ground state and the LIT equations. The same methods were used, e.g., for the first realistic calculation of the ^{4}He dipole structure function in Ref. [32].

For the nuclear Hamiltonian, we use two state-of-the-art potential models that include three-nucleon ($3N$) forces: (i) the Argonne v_{18} [33] nucleon-nucleon (NN) force supplemented by the Urbana IX [34] $3N$ force, denoted by AV18 + UIX, and (ii) a chiral effective field theory potential [35,36], denoted by χEFT, where the NN and $3N$ forces are at next-to-next-to-next-to-leading order and next-to-next-to-leading order in the chiral expansion, respectively. For the chiral $3N$ force, we use the parametrization of the low-energy constants obtained in Ref. [37] ($c_D = 1$ and $c_E = -0.029$). The calculated ^{4}He binding energy, point-proton radius, and electric-dipole polarizability α_E are, respectively, 28.422 MeV, 1.432 fm, and 0.0651 fm^3 for the AV18 + UIX potential. The corresponding numbers for the χEFT force are 28.343 MeV, 1.475 fm, and 0.0694 fm^3. These numbers are in good agreement with previous calculations [28,38,39]. The theoretical AV18 + UIX (χEFT) binding energy and radius are, respectively, within 0.3% (0.1%) and 3% (0.3%) of the experimental values. The uncertainty of α_E, spanned by these two potentials, agrees with one in a recent study from variations of the χEFT low-energy constants [39] and is much smaller than the experimental error.

Results.—We first check the formalism by comparing our μD results with Pachucki [20]. In Table I, we present all corrections related to the dipole structure function $S_{D_1}(\omega)$ obtained from the AV18 potential [40]. We find a good agreement for $\delta^{(0)}_{D1}$ and $\delta^{(0)}_{C}$. A difference in the relativistic corrections appears because in Ref. [20], $\delta^{(0)}_{L}$ includes only the leading term of Θ_L [Eq. (17)] in an ω/m_r expansion and neglects Θ_T [Eq. (18)], since it is one order higher in ω/m_r. These higher-order terms, which we include, provide additional relativistic corrections to the Lamb shift in μD. Consequently, the -1.680 meV result of Ref. [20] changes to -1.698 meV, where in this case, the cancellation of the third Zemach moment is implemented as in Ref. [20].

Now, we turn to μ^4He$^+$ and discuss the first *ab initio* calculations for $\delta^A_{\rm pol}$. Numerical results for the AV18 + UIX and χEFT potentials are presented in Table II, leading to an average value of $\delta^A_{\rm pol} = -2.475$ meV. In the point-nucleon treatment, we observe that the leading contribution $\delta^{(0)}$, amounting to -3.743 meV with AV18 + UIX and -3.981 meV with χEFT, strongly dominates in $\delta^A_{\rm pol}$.

Regarding the subleading terms, each individual term in $\delta^{(1)}$ (or $\delta^{(2)}$) is not necessarily small. Only their complete combination at each order fulfills the expansion in $\xi\sqrt{2m_r\omega} \sim \sqrt{m_r/M_A}$ as a consequence of the uncertainty principle, and yields $\delta^{(1)} = 0.775$ meV and $\delta^{(2)} = 0.089$ meV when averaging AV18 + UIX and χEFT calculations. As expected, $\delta^{(1)}$ and $\delta^{(2)}$ are, respectively, one and two orders smaller in $\sqrt{m_r/M_A}$ than $\delta^{(0)}$. The nucleon-size correction contributes an additional $\delta_{\rm NS} = 0.523$ meV on average. The latter depends on the value of $\langle r_p^2\rangle^{1/2}$: using 0.8775 fm [4] will increase $\delta_{\rm NS}$ to 0.579 meV.

The numerical accuracy of $\delta^A_{\rm pol}$ is also studied. The error in the effective interaction hyperspherical harmonics method is controlled by the convergence with respect to the maximum grand-angular momentum $K_{\rm max}$, which determines the size of the model space [31]. This error, obtained by taking the difference between results with $K_{\rm max} = 22\,(20)$ to those with $K_{\rm max} - 4$, is 0.4% (0.2%) for AV18 + UIX (χEFT). An additional 0.2% error is estimated by comparing the results from integrating the structure functions calculated using the LIT method with those obtained by a Lanczos sum-rule method as in Ref. [38].

The difference in $\delta^A_{\rm pol}$ that comes from using the AV18 + UIX or χEFT potential amounts to 0.134 meV and represents the uncertainty in nuclear physics. Both potentials are tuned to fit the ^{3}H binding energy, and they reproduce the

TABLE I. Nuclear polarization contributions to the $2S$-$2P$ Lamb shift ΔE (in meV) in μD, compared to Pachucki [20].

	Reference [20]	This work
$\delta^{(0)}_{D1}$	-1.910	-1.907
$\delta^{(0)}_{L}$	0.035	0.029
$\delta^{(0)}_{T}$		-0.012
$\delta^{(0)}_{C}$	0.261	0.259

TABLE II. Nuclear polarization contributions to the $2S$-$2P$ Lamb shift ΔE (in meV) in μ^4He$^+$.

		AV18 + UIX	χEFT
$\delta^{(0)}$	$\delta^{(0)}_{D1}$	-4.418	-4.701
	$\delta^{(0)}_{L}$	0.289	0.308
	$\delta^{(0)}_{T}$	-0.126	-0.134
	$\delta^{(0)}_{C}$	0.512	0.546
$\delta^{(1)}$	$\delta^{(1)}_{R3pp}$	-3.442	-3.717
	$\delta^{(1)}_{Z3}$	4.183	4.526
$\delta^{(2)}$	$\delta^{(2)}_{R2}$	0.259	0.324
	$\delta^{(2)}_{Q}$	0.484	0.561
	$\delta^{(2)}_{D1D3}$	-0.666	-0.784
$\delta_{\rm NS}$	$\delta^{(1)}_{R1pp}$	-1.036	-1.071
	$\delta^{(1)}_{Z1}$	1.753	1.811
	$\delta^{(2)}_{\rm NS}$	-0.200	-0.210
$\delta^A_{\rm pol}$		-2.408	-2.542

^{4}He binding energy to a few parts per thousand. They differ, however, in their respective predictions for the nuclear charge radius. Given the relations between the structure functions and the charge radius [38], it is plausible that the uncertainty in $\delta^A_{\rm pol}$ can be reduced using the ^{4}He charge radius to constrain the nuclear potential models. This systematic uncertainty dominates the errors in predicting $\mu^4{\rm He}^+$ polarization effects. The difference between the two models divided by $\sqrt{2}$ gives a 4% error, which can be interpreted as a 1σ deviation from the central value. The magnetic polarization is negligible in ^{4}He [41]. Terms of order $(Z\alpha)^6$, relativistic corrections to polarizations other than dipole, and higher-order nucleon-size effects will be explored in the future. The sum of all these additional corrections is expected to be a few percent. In a quadratic sum of all the errors mentioned above, we estimate the accuracy of our calculation to be $\pm6\%$. We did not include the contribution from the disputed intrinsic nucleon polarizability [11–13] because it can be estimated independently of the nuclear Hamiltonian (see, e.g., Ref. [20,21]).

Conclusions.—We perform the first *ab initio* calculation for the $\mu^4{\rm He}^+$ polarization correction, obtaining $\delta^A_{\rm pol} = -2.47$ meV $\pm6\%$. This result significantly improves the accuracy and is close to the upper bound of previous predictions $\delta^A_{\rm pol} = -3.1$ meV $\pm20\%$ [16–18]. The theoretical accuracy is limited by the uncertainty in the nuclear Hamiltonian, which is probed by using two different state-of-the-art nuclear potentials. Exploring other choices for potential parametrizations and including higher-order χEFT forces can possibly narrow this uncertainty. Our result allows a significant improvement in the precision of $\langle R_c^2\rangle$ that will be extracted from the $\mu^4{\rm He}^+$ Lamb shift measurements planned for 2013.

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