Ballistic Spreading of Entanglement in a Diffusive Nonintegrable System

Hyungwon Kim and David A. Huse
Physics Department, Princeton University, Princeton, New Jersey 08544, USA
(Received 24 June 2013; published 20 September 2013)

We study the time evolution of the entanglement entropy of a one-dimensional nonintegrable spin chain, starting from random nonentangled initial pure states. We use exact diagonalization of a nonintegrable quantum Ising chain with transverse and longitudinal fields to obtain the exact quantum dynamics. We show that the entanglement entropy increases linearly with time before finite-size saturation begins, demonstrating a ballistic spreading of the entanglement, while the energy transport in the same system is diffusive. Thus, we explicitly demonstrate that the spreading of entanglement is much faster than the energy diffusion in this nonintegrable system.

DOI: 10.1103/PhysRevLett.111.127205 PACS numbers: 75.10.Pq, 03.65.Ud

Entanglement is one of the unique features of quantum mechanics that does not exist in classical physics. Originally, quantum entanglement was viewed with some skepticism [1,2], but recently, the study of entanglement has become a central part of many-body quantum physics and quantum information science. Despite impressive recent progress in understanding entanglement from various viewpoints, many of its aspects remain to be further explored.

One natural question about entanglement is its quantum dynamics under unitary time evolution. If one starts an isolated quantum system in a nonentangled initial product pure state, how does the entanglement grow with time? Entanglement is not a conserved quantity like energy, that is transported. Instead, it is more like an infection or epidemic [3] that multiplies and spreads. An initial state that is a product state has the information about the initial state of each local degree of freedom (spins in our model below) initially localized on that degree of freedom. Under the system's unitary time evolution, quantum information about each spin's initial state can spread with time to other spins, due to the spin-spin interactions. This can make those spins that share this information entangled.

In real physical systems, information and entanglement can spread as fast as the speed of light (or sound). For a lattice spin model, which lacks propagating light or sound, an upper limit on the speed of any information spread is given by the Lieb-Robinson bound, which is set by the spin-spin interactions [4] (for recent reviews, see Refs. [5,6]). For integrable one-dimensional models, the entanglement does indeed spread ballistically [7–9], which is to be expected since such systems have ballistically propagating quasiparticles that can serve as carriers of the information. For various localized models, on the other hand, the entanglement has been shown to spread much more slowly, only logarithmically with time [8,10–16]. In the present Letter, we consider an intermediate case, a quantum Ising spin chain that is neither integrable nor localized, whose energy transport is diffusive.

Here, we investigate the spread of entanglement in a diffusive nonintegrable system, at high temperature where there are no ballistically propagating quasiparticles and the only conserved quantity is the energy which moves diffusively [17]. Diagonalizing the entire Hamiltonian matrix, we numerically study the time evolution of the entanglement and the diffusive dynamics of energy transport for highly excited thermal states of the system. We show that the entanglement spreads ballistically, while the energy moves only diffusively and thus slowly. Although we choose a specific model Hamiltonian to study the quantum dynamics, this result should be valid generally for nonlocalized and nonintegrable systems that do not have ballistically propagating quasiparticles or long-wavelength propagating modes such as light or acoustic sound.

As a simple nonintegrable model Hamiltonian, we choose a spin-1/2 Ising chain with both transverse and longitudinal fields. The model is translationally invariant, except at the ends of the chain, which we leave open. Leaving the ends open allows the longest distance within the chain to be its full length, so we can study energy transport over that distance and the spread of bipartite entanglement from the center of the chain to its ends. If we had used periodic boundary conditions instead, the longest distances that we could study would be cut in half. Given the limited lengths that one can study with exact diagonalization, this factor of 2 is quite important. Our Hamiltonian is

$$H = \sum_{i=1}^{L} g\sigma_i^x + \sum_{i=2}^{L-1} h\sigma_i^z + (h-J)(\sigma_1^z + \sigma_L^z) + \sum_{i=1}^{L-1} J\sigma_i^z\sigma_{i+1}^z. \tag{1}$$

σ_i^x and σ_i^z are the Pauli matrices of the spin at site i. After searching a bit in the space of parameters to see where we have both fast entanglement spread and slow energy diffusion and none of the terms singly dominates the energy spectrum, we chose the longitudinal field $h = (\sqrt{5}+1)/4 = 0.8090\ldots$ and the transverse field

$g = (\sqrt{5}+5)/8 = 0.9045\ldots$ and set the interaction $J = 1$ (and also set the Planck constant $\hbar$ to one); all results reported here are for these values. Our qualitative results and conclusions do not depend on these parameter choices as long as g, h, and J are all of similar magnitude to each other to keep the system robustly nonintegrable [18]. Note that the magnitude of the energy "cost" to flip a spin in the bulk, from the applied longitudinal field and its interactions with its neighbors, is $2h$ or $4J \pm 2h$. To keep the end sites similar in this respect to the bulk, we reduce the strength of the longitudinal field on the end spins by J. This is to avoid having some slow low-energy modes near the ends that introduced small additional finite-size effects when we also applied the same magnitude of longitudinal field to the end spins.

This Hamiltonian has one symmetry, namely, inverting the chain about its center. We always work with even L, so the center of the chain is on the bond between sites $L/2$ and $(L/2)+1$. This symmetry allows us to separate the system's state space into sectors that are even and odd under this parity symmetry and diagonalize within each sector separately. Any mixed parity state can be obtained from a linear combination of even and odd parity states. The statistics of energy-level spacings within each parity sector of this nonintegrable Hamiltonian should follow Gaussian orthogonal ensemble statistics [19]. There are 32 896 energy levels in the even sector for $L = 16$, the largest system we have diagonalized. Their level spacing statistics is in excellent agreement with the "r test" introduced in Ref. [20] and the Wigner-like surmise described in Ref. [21], as expected, indicating that this is indeed a robustly nonintegrable model with no extra symmetries (see the Supplemental Material [18]).

First, we consider the time evolution of the bipartite entanglement across the central bond between the two halves of the chain. We quantify the entanglement entropy in bits using the von Neumann entropy $S(t) = -\mathrm{tr}[\rho_A(t)\log_2\rho_A(t)] = -\mathrm{tr}[\rho_B(t)\log_2\rho_B(t)]$ of the probability operators (known as reduced density matrices) at time t of either the left half (A) or the right half (B) of the chain. As initial states, we consider random product states (with thus zero initial entanglement) $|\psi(0)\rangle = |\mathbf{s}_1\rangle|\mathbf{s}_2\rangle\ldots|\mathbf{s}_L\rangle$, where each spin at site i initially points in a random direction on its Bloch sphere

$$|\mathbf{s}_i\rangle = \cos\left(\frac{\theta_i}{2}\right)|\uparrow_i\rangle + e^{i\phi_i}\sin\left(\frac{\theta_i}{2}\right)|\downarrow_i\rangle, \tag{2}$$

where $\theta_i \in [0, \pi)$ and $\phi_i \in [0, 2\pi)$. Such an initial state is in general neither even nor odd and thus explores the entire Hilbert space of the pure states as it evolves with unitary Hamiltonian dynamics. This ensemble of initial states maximizes the thermodynamic entropy and thus corresponds to infinite temperature. For each time t, we generate 200 random initial product states, let them evolve to time t, compute $S(t)$ for each initial state, and then average.

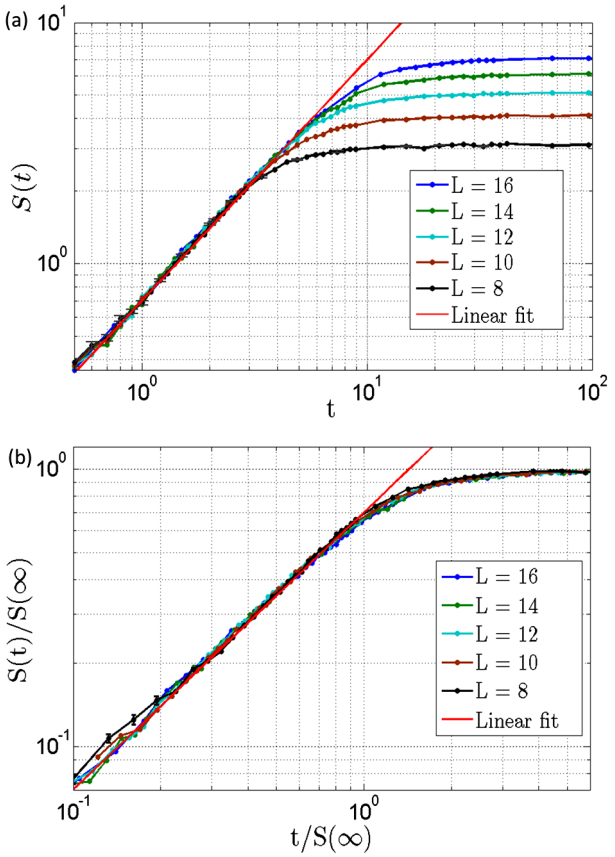

FIG. 1 (color online). (a) Spreading of entanglement entropy $S(t)$ for chains of length L. Initially, the entanglement grows linearly with time for all cases, with the same speed $v \cong 0.70$. Then, the entanglement saturates at long time. This saturation begins earlier for smaller L, as expected. The linear fit function is $f(t) = 0.70t$. Standard error is less than 0.04 for all points, and thus the error bars are only visible at early times. (b) Same data scaled by the infinite-time entropy for each L. Note that we use logarithmic scales both here and in Fig. 2.

By doing so, the standard error at each time is uncorrelated. The results are shown in Fig. 1. Ballistic linear growth of $S(t)$ at early time is clearly seen, and the growth rate before the saturation is independent of L. (Note that there is an even earlier time regime at $t \ll 1$ where the entanglement initially grows as $\sim t^2|\log t|$; this regime is just the initial development of some entanglement between the two spins immediately adjacent to the central bond.)

In the long-time limit, the time evolved state, on average, should behave like a random pure state (a random linear combination of product states). In Ref. [22], it is shown that the average of the entanglement entropy of random pure states is

$$S^R = \log_2 m - \frac{m}{2n\ln 2} - \mathcal{O}\left(\frac{1}{mn}\right), \tag{3}$$

where m and n are the dimension of the Hilbert space in each subsystem, with $m \le n$. Since $m = n = 2^{L/2}$ in our

case, $S^R \simeq L/2$ in the large L limit. This limiting value indicates that the entanglement spreads over the entire subsystem of length $L/2$. Therefore, before saturation begins, we can interpret $S(t)$ (in bits) as a measure of the distance over which entanglement has spread and its growth rate thus as the speed of the ballistic entanglement spreading. It is clear from Fig. 1(a) that at long time ($t > 20 \sim 100$ depending on the system size), $S(t)$ saturates close to S^R. We find that the deviation of the saturation value from $L/2 - 1/(2\ln 2)$ [Eq. (3)] is small (~ 0.19 for $L = 8$ and ~ 0.11 for $L = 16$; see Fig. 7 of the Supplemental Material [18]). Since the entanglement entropy saturates because of the finite length L, this deviation from S^R is a correction to the leading finite-size effect, which should be negligible in the thermodynamic limit.

This behavior suggests the finite-size and finite-time scaling form for the entanglement entropy:

$$S(t) \approx S_L(\infty) F(t/S_L(\infty)), \tag{4}$$

where $S_L(\infty)$ is the infinite-time average value of the entanglement entropy [23] for chain length L, the scaling function $F(x) \sim vx$ for $x \to 0$ (v is the spreading rate), and $F(x) \to 1$ for $x \to \infty$. Figure 1(b) confirms that this scaling works well.

Now, let us consider the diffusive dynamics of this system. As an example, we study the diffusive spreading of an initially localized energy inhomogeneity. First, we prepare the system in the maximal thermodynamic entropy mixed state (equilibrium at infinite temperature) and put a small energy perturbation on the center bond. Then, we observe how this extra local energy spreads over the system under unitary time evolution. Specifically, the initial probability operator (density matrix) is

$$\rho(0) = \frac{1}{2^L}(I + \epsilon \sigma^z_{L/2} \sigma^z_{L/2+1}), \tag{5}$$

where I is the identity operator and ϵ is a small number. Note that I commutes with H, so we only need to time evolve the perturbation. Then, we compute the local energy $\langle H_r \rangle(t)$ at each site and bond r at time t. The position index r is an integer (1 to L) for each site and a half-integer (3/2 to $L - 1/2$) for each bond. Explicitly,

$$H_r = \begin{cases} g\sigma^x_r + h\sigma^z_r & \text{sites } 2 \le r \le L-1 \\ g\sigma^x_r + (h-J)\sigma^z_r & r = 1 \text{ or } L \\ J\sigma^z_{r-1/2}\sigma^z_{r+1/2} & \text{bonds } 3/2 \le r \le L-1/2. \end{cases} \tag{6}$$

This is just a decomposition of the Hamiltonian $H = \sum_r H_r$. Trivially, $\langle H_r \rangle = \epsilon \delta_{r,(L+1)/2}$ at time $t = 0$. To quantify the energy spreading at time t, we compute an average "distance" $R(t)$ that the energy has moved away from the center bond

$$R(t) = \frac{2}{\langle H \rangle} \sum_r \left| r - \frac{L+1}{2} \right| \langle H_r \rangle(t), \tag{7}$$

where $\langle H \rangle = \epsilon$ is the conserved total energy. Figure 2(a) is the plot of $R(t)$ for $L = 8$, 10, 12, 14, and 16. If the extra energy at long time is distributed equally to all sites and bonds, $R(\infty) \to (L/2)((2L-2)/(2L-1)) \simeq L/2$ and thus close to the maximum value of entanglement spreading [the factor of 2 in Eq. (7) is to make the long-time value of $R(t)$ comparable to that of $S(t)$]. We find that the saturation value $R(\infty)$ [24] grows linearly with the system size but is always slightly smaller than $L/2$ due to the final local energy distribution not being homogeneous near the ends of the chain.

If this dynamics is diffusive, the energy spread is $R(t) \approx (4/\sqrt{\pi})\sqrt{Dt} \sim \sqrt{t}$ (one-dimensional random walk) for sufficiently large t ($t \gtrsim 1$ in our case) before finite-size saturation begins. D is the energy diffusivity, which only depends on the interaction parameters, not the system size. Figure 2(a) clearly shows that $R(t)$ is independent of

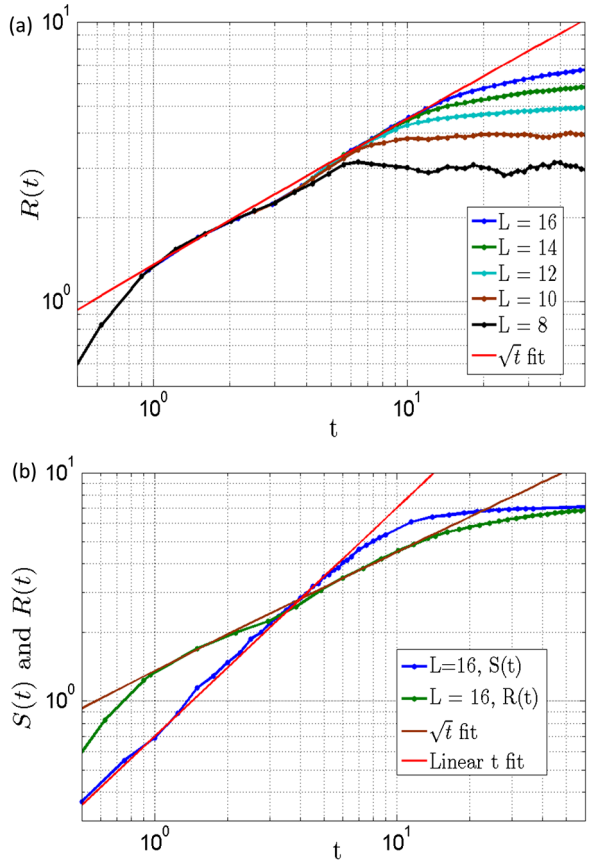

FIG. 2 (color online). (a) The average energy spreading $R(t)$ (defined in the main text) vs time. Before saturation, its behavior does not depend on the system size. As we increase the system size, diffusive $\sqrt{t}$ behavior becomes more apparent. (b) Direct comparison of $S(t)$ and $R(t)$ for $L = 16$. It is clear that the entanglement spreads faster than energy diffuses in the scaling regime before saturation.

system size at early stages, and it grows as $\sim\sqrt{t}$ before saturation begins. For $L = 8$, the frequency scale of the many-body level spacing is of order 0.1, and thus $R(t)$ begins oscillating around $t \sim 10$. Although the system sizes that we can diagonalize are not large enough to show a wide range of time scales, they do show that the speed of entanglement spreading becomes faster than the rate of diffusive energy spreading by direct comparison of $S(t)$ and $R(t)$. Figure 2(b) is the plot of $S(t)$ and $R(t)$ for $L = 16$. In the very beginning ($t \leq 1$), $R(t)$ grows faster than $S(t)$ due to microscopic details of the dynamics, but soon the linearly growing $S(t)$ overtakes $R(t)$ and approaches its saturation while $R(t)$ is growing only as $\sim\sqrt{t}$. Therefore, this is a direct demonstration of the contrast between ballistic entanglement spreading and diffusive energy transport.

In conclusion, we have demonstrated that quantum entanglement spreads ballistically in a nonintegrable diffusive system. Since there are no ballistically traveling quasiparticles, the mechanism of entanglement spreading is different from what happens in integrable systems, where these quasiparticles can carry both energy and information. At high enough temperature, almost all states are relevant to the dynamics, and the dynamics is constrained by only a few conservation laws (in our case, only the total energy). In this regime, the concept of quasiparticles is not well defined for the system we have studied. Even so, if we do heuristically describe the dynamics of our diffusive model in terms of quasiparticles, these quasiparticles scatter strongly and frequently and thus have a short mean free path. This limits the energy transport to be diffusive. But, apparently, the quantum information needed to spread entanglement is passed along in each collision, presumably to all outgoing quasiparticles from each collision. Thus, this information spreads in a cascade or shower of collisions and the edges of this shower spread ballistically.

We conjecture that for highly excited nonintegrable systems such as those we study here, there are no *local* observables whose correlations spread more rapidly than diffusively, even though the entanglement spreads ballistically. Note that this is a strong conjecture that goes well beyond what we can test numerically.

We have used the analogy from Ref. [3] between the spreading of entanglement and the spreading of an epidemic. But, it is an unusual sort of nonlocal epidemic, where the symptoms of the "disease" cannot be detected by any local observables. In Ref. [25], they make an analogy instead to a tsunami; again, this appears to be a very gentle nonlocal tsunami, whose effects can only be detected by nonlocal observables. An interesting question that we leave for future work is, what is the simplest and most local operator that can detect this ballistically spreading entanglement? We detected it using the state of the full system, but if the entanglement has only traveled a distance ℓ in each direction from the central bond, it should be detectable by some operators that only involve the spins within that distance.

We thank Joel Moore and Michael Kolodrubetz for discussions. We also acknowledge Hanjun Kim, Seok Hyeong Lee, Taewook Oh, and Jonathan Sievers for fruitful discussions about numerics. The simulations presented in this work were performed on computational resources supported by the Princeton Institute for Computational Science and Engineering (PICSciE) at Princeton University. This work was supported in part by NSF under Grant No. DMR-0819860 and by funds from the DARPA Optical Lattice Emulator Program.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] E. Schrodinger, Proc. Cambridge Philos. Soc. **32**, 446 (1936).
[3] R. Omnes, arXiv:1212.0331v1.
[4] E. H. Lieb and D. W. Robinson, Commun. Math. Phys. **28**, 251 (1972).
[5] B. Nachtergaele and R. Sims, Contemp. Math. **529**, 141 (2010).
[6] M. Kliesh, C. Gogolin, and J. Eisert, arXiv:1306.0716v1.
[7] P. Calabrese and J. Cardy, J. Stat. Mech. 04 (2005) P04010.
[8] G. De Chiara, S. Montangero, P. Calabrese, and R. Fazio, J. Stat. Mech. 03 (2006) P03001.
[9] T. Hartman and J. Maldacena, arXiv:1303.1080v2.
[10] C. K. Burrell and T. J. Osborne, Phys. Rev. Lett. **99**, 167201 (2007).
[11] M. Žnidarič, T. Prosen, and P. Prelovšek, Phys. Rev. B **77**, 064426 (2008).
[12] F. Iglói, Z. Szatmári, and Y.-C. Lin, Phys. Rev. B **85**, 094417 (2012).
[13] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[14] R. Vosk and E. Altman, Phys. Rev. Lett. **110**, 067204 (2013).
[15] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013); arXiv:1305.5554v1.
[16] D. A. Huse and V. Oganesyan, arXiv:1305.4915v1.
[17] In integrable models, on the other hand, correlations of local observables can spread both diffusively and ballistically. See, e.g., J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. Lett. **103**, 216602 (2009); S. Langer, M. Heyl, I. P. McCulloch, and F. Heidrich-Meisner, Phys. Rev. B **84**, 205115 (2011).
[18] The spreading speed of the entanglement and the energy diffusivity depend on the parameters in the Hamiltonian. Some choices may produce strong initial transients, which do not scale with system size and so are irrelevant in the thermodynamic limit, or large energy diffusivity and small spreading speed of entanglement. In these cases, it is difficult to study long-time phenomena using the small-L systems we can study. Our basic result that the exponent of entanglement spreading is larger than that of

energy diffusion was reproduced for other parameter choices. See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.111.127205 for explicit demonstration of other parameter choices.

[19] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[20] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).

[21] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Phys. Rev. Lett. **110**, 084101 (2013).

[22] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993).

[23] We used two methods to estimate $S_L(\infty)$: One is computing $S(t)$ at large t such as 500, 1000, and 10 000 and seeing that $S(t)$ remains constant within the standard error at these long times. The other is expanding the state in terms of the eigenstates of H and giving each eigenstate a random phase (thus approximating infinite time) and computing the entanglement entropy from the state with random phases. We found that these two methods give the same value within the error bar (fractional difference was less than 1.5%).

[24] We estimated $R(\infty)$ as the average of a few values of well-spaced $R(t)$ near and beyond the time scale set by the inverse many-body energy-level spacing. For example, the inverse many-body level spacing for $L = 14$ is ≈ 533. Then, $R(\infty)$ is estimated by averaging $R(450)$, $R(600)$, $R(750)$, $R(900)$, and $R(1050)$. Fractional fluctuation within that time range was as small as 0.12% (this fractional fluctuation becomes smaller as we increase the system size).

[25] H. Liu and S. J. Suh, arXiv:1305.7244v2.