Time-Dependent Hartree-Fock Solution of Gross-Neveu Models: Twisted-Kink Constituents of Baryons and Breathers

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We find the general solution to the time-dependent Hartree-Fock problem for the Gross-Neveu models, with both discrete (GN_2) and continuous [Nambu-Jona-Lasinio (NJL_2)] chiral symmetry. We find new multibaryon, multibreather, and twisted breather solutions, and show that all GN_2 baryons and breathers are composed of constituent twisted kinks of the NJL_2 model.

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Self-interacting fermion systems describe a wide range of physical phenomena in particle, condensed matter, and atomic physics [1–10]. Applications include solitons, excitons, polaritons, breathers, and inhomogeneous phases in superconductors, conducting polymers, liquid crystals, particle physics, and cold atomic gases, and also illustrate the widespread phenomenon of induced fermion number [11]. The Gross-Neveu models [GN_2 and Nambu-Jona-Lasinio (NJL_2)] in $(1+1)$-dimensional quantum field theory describe N species of massless, self-interacting Dirac fermions [12]:

$$\mathcal{L}_{\rm GN} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}(\bar{\psi}\psi)^2, \tag{1}$$

$$\mathcal{L}_{\rm NJL} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \psi)^2]. \tag{2}$$

These are soluble paradigms of symmetry breaking phenomena in strong interaction particle physics and condensed matter physics [1,13]. In the 't Hooft limit $N \to \infty$, $Ng^2 = \text{const}$, semiclassical methods become exact, as pioneered in this context by Dashen, Hasslacher, and Neveu (DHN) [14,15]. Classically, the GN_2 model has a discrete chiral symmetry, while the NJL_2 model has a continuous chiral symmetry. At finite temperature and density, and at large N, these models exhibit inhomogeneous phases with crystalline condensates, directly associated with chiral symmetry breaking [16]. The basic physics of these GN phases is the Peierls effect of condensed matter physics [3,17–19]. This analysis of equilibrium thermodynamics is based on exact spatially inhomogeneous solutions to the gap equation, or equivalently the Hartree-Fock problem, which solves the Dirac equation subject to constraints on the scalar and pseudoscalar condensates [16,20]. Here, we extend these results to the complete exact solution of the *time-dependent* Hartree-Fock (TDHF) problem, relevant for scattering processes, transport phenomena, and nonequilibrium physics:

$$\begin{gathered}[i\partial\!\!\!/ - S(x,t)]\psi_\alpha = 0; \\ S = -g^2 \sum_\beta^{\rm occ} \bar{\psi}_\beta \psi_\beta \quad \text{for } GN_2,\end{gathered} \tag{3}$$

$$\begin{gathered}[i\partial\!\!\!/ - S(x,t) - i\gamma_5 P(x,t)]\psi_\alpha = 0 \quad \text{for } NJL_2, \\ S = -g^2 \sum_\beta^{\rm occ} \bar{\psi}_\beta \psi_\beta, \qquad P = -g^2 \sum_\beta^{\rm occ} \bar{\psi}_\beta i\gamma_5 \psi_\beta.\end{gathered} \tag{4}$$

We solve these TDHF problems in full generality, describing the dynamics, including scattering, of nontrivial topological objects such as kinks, baryons, and breathers. Surprisingly, we found that the most efficient strategy is to solve the (apparently more complicated) NJL_2 model first, and then obtain GN_2 solutions by imposing further constraints on these solutions. This reveals, for example, that the GN_2 baryons and breathers found by Dashen, Hasslacher, and Neveu [14] are in fact bound objects of twisted NJL_2 kinks, and that the scattering of GN_2 baryons and breathers can be deduced from the scattering of twisted kinks. This includes new breather and multibreather solutions in NJL_2, as well as new multibaryon and multibreather solutions for GN_2.

We stress that while it is well known that the classical equations for the GN_2 and NJL_2 models are closely related to integrable models [21], this fact is only directly useful for the solution of the TDHF problem for the simplest case of kink scattering in the GN_2 model, which reduces to the integrable sinh Gordon equation. The more general self-consistent TDHF solutions to Eqs. (3) and (4) involving twisted kinks, baryons, and breathers do not satisfy the sinh Gordon equation; instead, we find a general "master equation" [see Eq. (8)], whose solution reduces to a finite algebraic problem solvable in terms of determinants.

We also stress that these more general solutions require a self-consistency condition relating the filling fraction of valence fermion states to the parameters of the condensate solution, as for the static GN_2 baryon [14], the static twisted kink [22], and the GN_2 breather [14]. For our time-dependent solutions, this important fact means that during scattering processes, there is nontrivial backreaction between fermions and their associated condensates and densities [23]. Kink scattering in the GN_2 model, described by sinh Gordon solitons [24,25], is much simpler, as there is no self-consistency condition or backreaction.

With Dirac matrices $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$, and $\gamma_5 = -\sigma_3$ and light-cone coordinates (note $\bar{z}$ is *not* the complex conjugate of z) $z = x - t$, $\bar{z} = x + t$, $\partial_0 = \bar{\partial} - \partial$, and $\partial_1 = \bar{\partial} + \partial$, the Dirac equation in Eq. (4) is

$$2i\bar{\partial}\psi_2 = \Delta\psi_1, \qquad 2i\partial\psi_1 = -\Delta^*\psi_2, \qquad \Delta \equiv S - iP. \tag{5}$$

Write the complex potential Δ and continuum spinor ψ_ζ:

$$\Delta = \frac{\mathcal{N}}{\mathcal{D}}, \qquad \psi_\zeta = \frac{e^{i(\zeta\bar{z} - z/\zeta)/2}}{\mathcal{D}\sqrt{1+\zeta^2}}\begin{pmatrix}\zeta\mathcal{N}_1\\ -\mathcal{N}_2\end{pmatrix}, \tag{6}$$

where $\mathcal{D}$ is real, and the complex light-cone spectral parameter ζ is related to the energy E and momentum k as $k = (1/2)(\zeta - (1/\zeta))$, $E = -(1/2)(\zeta + (1/\zeta))$, in units of m, the dynamically generated fermion mass. The ansatz for ψ_ζ anticipates the fact that the potential Δ is transparent.

We solve Eq. (5), and associated TDHF consistency conditions, using an ansatz method, positing a decomposition with a finite number n of simple poles:

$$\mathcal{N}_{1,2}(\zeta) = \mathcal{N}_{1,2}^{(0)} + \sum_{i=1}^{n}\frac{1}{\zeta - \zeta_i}\mathcal{N}_{1,2}^{(i)}. \tag{7}$$

Inserting Eqs. (6) and (7) into the Dirac equation (5), and matching powers of ζ, we learn that the residues $\mathcal{N}_{1,2}^{(i)}$ must satisfy various sum rules, the self-consistency of which further requires $\mathcal{D}$ and $\mathcal{N}$ to satisfy the master equation [26] (see Supplemental Material [27]):

$$4\partial\bar{\partial}\ln\mathcal{D} = 1 - |\Delta|^2. \tag{8}$$

Furthermore, the following equations must hold for all $i = 1, \ldots, n$,

$$\begin{aligned}&2i(\mathcal{D}\bar{\partial} - \bar{\partial}\mathcal{D})\mathcal{N}_2^{(i)} - \zeta_i(\mathcal{D}\mathcal{N}_2^{(i)} - \mathcal{N}\mathcal{N}_1^{(i)}) = 0,\\ &2i\zeta_i(\mathcal{D}\partial - \partial\mathcal{D})\mathcal{N}_1^{(i)} + \mathcal{D}\mathcal{N}_1^{(i)} - \mathcal{N}^*\mathcal{N}_2^{(i)} = 0.\end{aligned} \tag{9}$$

The residues of ψ_ζ at the poles $\zeta = \zeta_i$ provide normalizable bound state spinor solutions:

$$\psi^{(i)} = \frac{1}{\mathcal{D}V_i}\begin{pmatrix}\zeta_i\mathcal{N}_1^{(i)}\\ -\mathcal{N}_2^{(i)}\end{pmatrix}, \qquad V_i \equiv e^{-i(\zeta_i\bar{z} - z/\zeta_i)/2}. \tag{10}$$

An alternative set of normalizable bound state spinors comes from ψ_ζ at the complex conjugate poles ζ_i^*:

$$\phi^{(i)} = \frac{V_i^*}{\mathcal{D}}\begin{pmatrix}\zeta_i^*\mathcal{N}_1(\zeta_i^*)\\ -\mathcal{N}_2(\zeta_i^*)\end{pmatrix}. \tag{11}$$

These two sets of bound states are linearly related $\psi^{(i)} = \sum_j\Omega_{ij}\phi^{(j)}$. The condition that $\mathcal{D}$ is real and has no zeroes restricts the matrix Ω to the form

$$\Omega_{ij} = i\hat{\Omega}_{ij}/(\zeta_j^*)^2, \tag{12}$$

where $\hat{\Omega}$ is a positive definite Hermitean matrix. Together with Eq. (7), we obtain a finite dimensional algebraic system:

$$\begin{aligned}\mathcal{N}_1(\zeta_j^*) + \sum_{i,k}\frac{1}{\zeta_i(-\zeta_j^* + \zeta_i)}V_i\Omega_{ik}V_k^*\zeta_k^*\mathcal{N}_1(\zeta_k^*) = \mathcal{D},\\ \mathcal{N}_2(\zeta_j^*) + \sum_{i,k}\frac{1}{-\zeta_j^* + \zeta_i}V_i\Omega_{ik}V_k^*\mathcal{N}_2(\zeta_k^*) = \mathcal{N}.\end{aligned} \tag{13}$$

We have found a remarkably simple solution to this algebraic system, which yields a compact determinant expression for all the ansatz quantities in the TDHF solution:

$$\begin{aligned}\mathcal{D} &= \det(\omega + B), & \mathcal{N} &= \det(\omega + A),\\ \mathcal{N}_1(\zeta) &= \det(\omega + C), & \mathcal{N}_2(\zeta) &= \det(\omega + D),\end{aligned} \tag{14}$$

with matrices

$$B_{ij} = \frac{iV_i^*V_j}{\zeta_j - \zeta_i^*} = \frac{\zeta_i^*}{\zeta_j}A_{ij}, \qquad C_{ij} = \frac{\zeta - \zeta_i^*}{\zeta - \zeta_j}B_{ij} = \frac{\zeta_i^*}{\zeta_j}D_{ij}, \tag{15}$$

where ω is a positive definite Hermitean matrix: $\omega_{ij} = \zeta_i^*\hat{\Omega}_{ij}^{-1}\zeta_j$. This gives the complete solution to the Dirac equation for time-dependent transparent (complex) potential Δ. In the nonrelativistic limit, writing $|\Delta| \approx 1 + V$, with $V \ll 1$, the master equation (8) reduces to the known Kay-Moses "log det" form of the general transparent static Schrödinger potential [28]

$$V(x) = -\partial_x^2\ln\det(\mathbf{1} + A), \quad A_{ij} = \sqrt{a_i a_j}\frac{e^{(\kappa_i+\kappa_j)x}}{\kappa_i + \kappa_j} \tag{16}$$

and its time-dependent Schrödinger generalization [29]. Our solution (14) and (15) also provides a new closed-form solution to the finite algebraic problem, found recently in Ref. [20], for the *static* transparent NJL_2 Dirac equation.

We now show that this solution also gives a self-consistent solution to the fully quantized TDHF problem (4), provided certain filling-fraction conditions are satisfied by the combined soliton-fermion system, generalizing the conditions already found by DHN, Jackiw-Rebbi, and Shei [11,14,22]. Consider first the induced fermion density in the Dirac sea. Introducing a cutoff scale Λ,

$$\rho_{\rm ind} = \int_{1/\Lambda}^{\Lambda} \frac{d\zeta}{2\pi}\frac{\zeta^2+1}{2\zeta^2}(\psi_\zeta^\dagger\psi_\zeta - 1)$$
$$= \int_{1/\Lambda}^{\Lambda} \frac{d\zeta}{2\pi}\frac{1}{2\zeta^2\mathcal{D}^2}[\zeta^2(|\mathcal{N}_1|^2-\mathcal{D}^2)+|\mathcal{N}_2|^2-\mathcal{D}^2]. \quad (17)$$

The pole ansatz (7), a partial fraction decomposition, and the known asymptotic behavior of the ansatz functions combine to show that the linear and logarithmic divergent terms cancel, leading to the finite result

$$\rho_{\rm ind} = \frac{i}{4\pi}\sum_{i,j}\frac{\zeta_i^*\zeta_j\mathcal{N}_1^{(i)*}\mathcal{N}_1^{(j)} + \mathcal{N}_2^{(i)*}\mathcal{N}_2^{(j)}}{\mathcal{D}^2 V_i^* V_j}(\hat{\Omega}^{-1})_{ij}\ln\frac{\zeta_i^*}{\zeta_j}. \quad (18)$$

For consistency with axial current conservation, this must be canceled by the contribution from the discrete bound states [26]. The physical bound state spinors are in general a (orthonormal) superposition of the basis bound states (10) $\hat{\psi}^{(i)} = \sum_j C_{ij}\psi^{(j)}$, where we find that the matrix C is directly related to the matrix $\hat{\Omega}$ as $2C\hat{\Omega}C^\dagger = \mathbf{1}$. The density from the bound states (with occupation fractions ν_k) is

$$\rho_b = \sum_{i,j}\frac{\zeta_i^*\zeta_j\mathcal{N}_1^{(i)*}\mathcal{N}_1^{(j)} + \mathcal{N}_2^{(i)*}\mathcal{N}_2^{(j)}}{\mathcal{D}^2 V_i^* V_j}\sum_k \nu_k C_{ki}^* C_{kj}. \quad (19)$$

Then, the condition $\rho_{\rm ind} + \rho_b = 0$ leads to a consistency condition for the filling fractions ν_k which can be written as

$$\nu_k = \frac{1}{2\pi}\text{ eigenvalues of }(C^{\dagger -1}M\hat{\Omega}^{-1}C^{-1} + \text{H.c.}), \quad (20)$$

where M is the diagonal matrix $M_{ij} = -i\delta_{i,j}\ln(-\zeta_j^*)$. Having found a candidate solution with vanishing fermion density, we now consider the TDHF self-consistency conditions in Eq. (4). Equations (6) and (7) imply the condensate expectation value

$$\langle\bar{\psi}\psi\rangle - i\langle\bar{\psi}i\gamma_5\psi\rangle = -\frac{\Delta}{\pi}\ln\Lambda - \frac{i}{2\pi}\sum_{i,j}\frac{\zeta_i^*\mathcal{N}_1^{(i)*}}{V_i^*\mathcal{D}}\frac{\mathcal{N}_2^{(j)}}{V_j\mathcal{D}} \times(\hat{\Omega}^{-1})_{ij}\ln\frac{\zeta_i^*}{\zeta_j}. \quad (21)$$

The sum rules satisfied by the residues $\mathcal{N}_{1,2}^{(i)}$ guarantee UV and IR convergence of the latter terms, while the first term gives self-consistency from the vacuum gap equation $(Ng^2/\pi)\ln\Lambda = 1$. The second term must be canceled against the bound state contribution, and remarkably this is satisfied, provided the previously found filling-fraction condition [Eq. (20)] holds. This proves full TDHF self-consistency for the NJL$_2$ system [Eq. (4)]. For GN$_2$, we impose reality of the condensate Δ and relax the consistency condition on the pseudoscalar condensate, as discussed below.

We illustrate the TDHF solution [Eqs. (14), (15), and (20)] with some examples. We write ζ_j in terms of phase and boost parameters: $\zeta_j = -e^{-i\phi_j}/\eta_j$. With just one pole, $B = e^{2x\sin\phi}$, $A = e^{-2i\phi}B$, and $\Delta = (1 + e^{-2i\phi}e^{2x\sin\phi})/(1 + e^{2x\sin\phi})$, which is Shei's twisted kink for the NJL$_2$ model [22], with filling fraction $\nu = \phi/\pi$. When $\phi = \mp\pi/2$, we get a real solution of the GN$_2$ model, the usual kink or antikink. With two poles, we obtain real Δ, for GN$_2$, either by choosing $\phi_1 = \phi_2 = \pi/2$, which gives

$$\Delta = \frac{1 - U_1 - U_2 + (\frac{\eta_1-\eta_2}{\eta_1+\eta_2})^2 U_1U_2}{1 + U_1 + U_2 + (\frac{\eta_1-\eta_2}{\eta_1+\eta_2})^2 U_1U_2}, \quad U_i \equiv \frac{\eta_i|V_i|^2}{2\sin\phi_i}, \quad (22)$$

describing scattering of two kinks, or alternatively by choosing $\zeta_1 = -\zeta_2^*$, which means $\phi_2 = \pi - \phi_1$ and $\eta_1 = \eta_2$ ($= 1$ for rest frame). Then, $V_2 = V_1^*$ and

$$B = \begin{pmatrix} U_1 & \frac{ie^{-i\phi_1}}{2}(V_1^*)^2 \\ -\frac{ie^{i\phi_1}}{2}V_1^2 & U_1 \end{pmatrix}, \quad A_{ij} = \frac{B_{ij}}{e^{i(\phi_i+\phi_j)}}. \quad (23)$$

Choosing $\omega = \mathbf{1}$, we obtain the DHN GN$_2$ baryon [14]

$$\Delta = \frac{1 + 2\cos(2\phi_1)U_1 + \cos^2(\phi_1)U_1^2}{1 + 2U_1 + \cos^2(\phi_1)U_1^2}$$
$$= 1 + y\tanh(y\bar{x} - b) - y\tanh(y\bar{x} + b), \quad (24)$$

where $y = \sin\phi_1 = \tanh(2b)$, and the x origin has been shifted. The GN$_2$ consistency condition leads to filling fractions $\nu_1 = 2\phi_1/\pi$ and $\nu_2 = 1$. This shows that the DHN GN$_2$ baryon is in fact a bound object of two twisted kinks with filling fractions (which also determine the baryon size) related to the twist angle. Furthermore, the mass of the DHN baryon is related to the masses of the constituent twisted kinks as $M = M_{\rm kink}(\phi_1) + M_{\rm kink}(\pi - \phi_1) = (2N/\pi)\sin\phi_1$. Choosing instead an off-diagonal mixing matrix

$$\omega = \begin{pmatrix} \sec\chi & \tan\chi \\ \tan\chi & \sec\chi \end{pmatrix} \quad (25)$$

leads to the DHN GN$_2$ breather [14]

$$\Delta = \frac{1 + 2\sec\chi\cos2\phi_1 U_1 - 2\tan\chi\sin\phi_1\cos(2\bar{t}\cos\phi_1)U_1 + \cos^2\phi_1 U_1^2}{1 + 2\sec\chi U_1 + 2\tan\chi\sin\phi_1\cos(2\bar{t}\cos\phi_1)U_1 + \cos^2(\phi_1)U_1^2}$$

with filling fractions $\nu_{1,2} = (1/2\pi)[(\phi_1 + \phi_2) \pm (\phi_1 - \phi_2)\sec\chi]$. Thus, the DHN GN$_2$ breather is also a bound object of two twisted kinks, with filling fractions related to the twist angles.

Using the off-diagonal form (25) for ω, but not imposing the reality condition $\phi_1 + \phi_2 = \pi$, we obtain a new twisted breather solution to the NJL_2 model, shown in Fig. 1. At the three pole level, we find the scattering of three kinks if $\omega = \mathbf{1}$. These are GN_2 kinks if $\phi_1 = \phi_2 = \phi_3 = \pi/2$, and twisted NJL_2 kinks otherwise. The scattering of a GN_2 baryon and a kink is obtained by choosing $\zeta_1 = -\zeta_2^*$, and the scattering of a breather with a kink is obtained by choosing the off-diagonal form (25) for a 2×2 sub-block of ω. New breather solutions are obtained by choosing $\eta_1 = \eta_2 = \eta_3$ and a more general 3×3 off-diagonal form of ω. At the four pole level, in addition to the scattering of four (in general twisted) kinks, we can combine the spectral parameters of the kinks pairwise, e.g., as $\zeta_1 = -\zeta_2^*$ and $\zeta_3 = -\zeta_4^*$, to obtain the scattering of two baryons [23]. Further choosing the corresponding 2×2 sub-blocks of the mixing matrix ω to have the breather form (25), we obtain the scattering of two GN_2 breathers [30]. Relaxing the pairwise reality conditions $\phi_1 + \phi_2 = \pi$ and $\phi_3 + \phi_4 = \pi$, we obtain another new solution, the scattering of two twisted NJL_2 breathers, as shown in Fig. 2. Choosing equal boost parameters η_i and a more

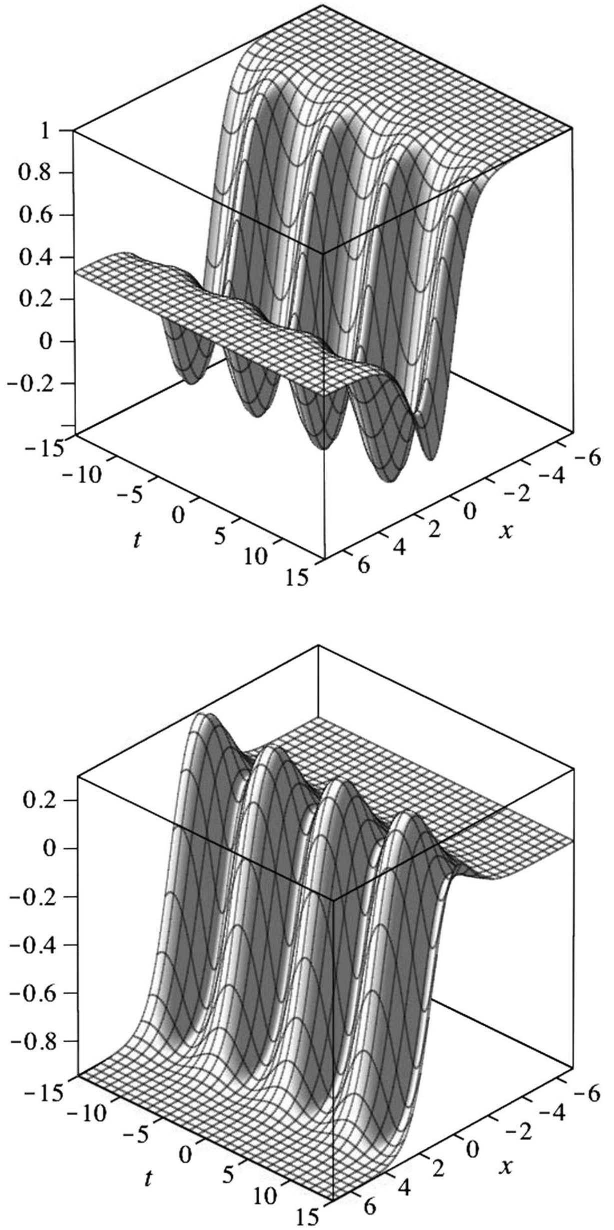

FIG. 1. The real (upper) and imaginary (lower) parts of the condensate $\Delta(x, t) = S - iP$ for the twisted NJL_2 breather. Note the periodic breathing in the time direction of both scalar and pseudoscalar components, and that $|\Delta|^2 \to 1$ asymptotically.

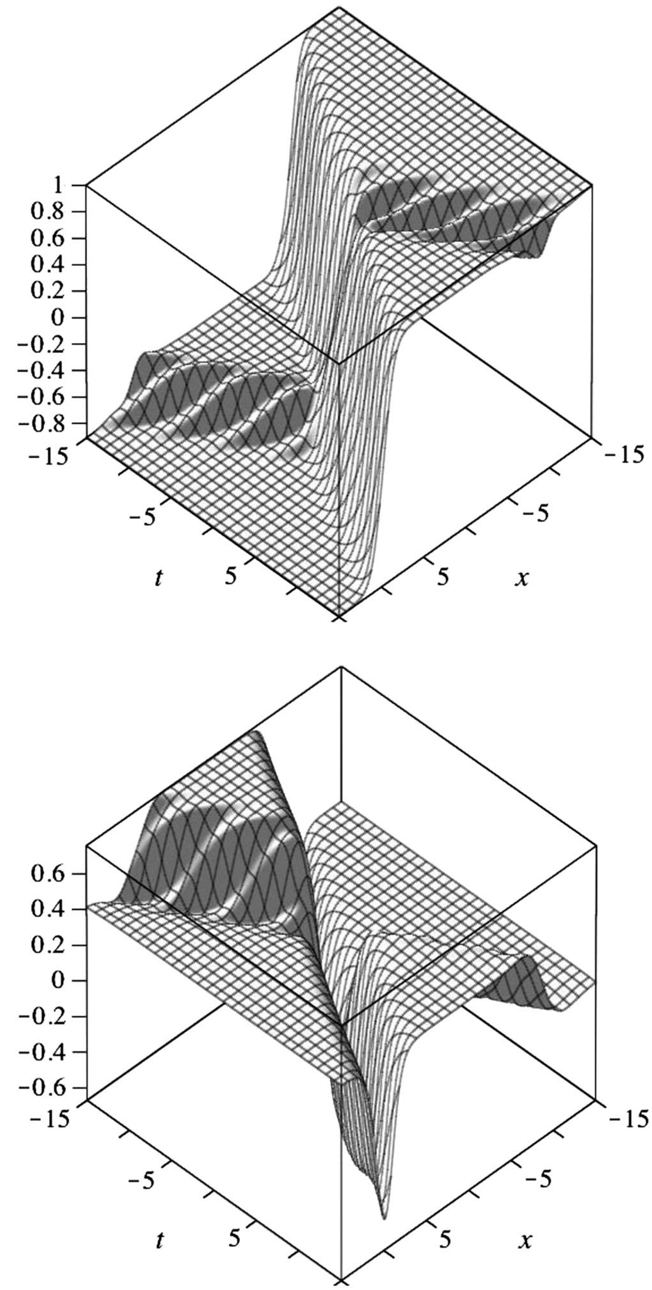

FIG. 2. The real (upper) and imaginary (lower) parts of the condensate $\Delta(x, t) = S - iP$ for the scattering of two twisted NJL_2 breathers. The objects breathe as they scatter, and $|\Delta|^2 \to 1$ asymptotically.

general off-diagonal mixing matrix ω, we obtain a novel four-breather solution in which all four twisted-kink constituent kinks "breathe." The pattern should now be clear. Choosing different boost parameters η_i gives a solution describing scattering of twisted-kink constituents. If some of the η_i are equal, the solution describes bound combinations of twisted-kink constituents, which are baryons if ω is diagonal and breathers if ω is off diagonal. The fermion filling-fraction consistency condition (20) can always be solved for any given choice of spectral parameters ζ_i and mixing matrix ω.

In the special case where all $\phi_i = \pi/2$, the V_i are real, and we have $B_{ij} = \eta_i\eta_j V_i V_j/(\eta_i + \eta_j)$, and $A_{ij} = -B_{ij}$, which agrees with the known multikink scattering solutions of GN_2 [24] and whose nonrelativistic limit agrees with the Schrödinger results of Kay-Moses and Nogami-Warke [28,29]. For NJL_2, taking all $\eta_i = 1$, we obtain the static solution $\Delta = \det(\mathbf{1} + \hat{A})/\det(\mathbf{1} + \hat{B})$, with

$$\hat{B}_{ij} = \frac{e^{(x-x_i)\sin\phi_i + (x-x_j)\sin\phi_j}}{2\sin(\frac{\phi_i+\phi_j}{2})}, \qquad \hat{A}_{ij} = \frac{\hat{B}_{ij}}{e^{i(\phi_i+\phi_j)}}, \tag{26}$$

where we have removed phases from the determinant, which is possible for diagonal ω. This is a compact solution for the algebraic system found recently in Ref. [20] for self-consistent *static* multitwisted kinks in NJL_2. Taking the η_i different, our solution [Eqs. (14), (15), and (20)] gives the full time-dependent generalization.

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