Hierarchical Fractal Weyl Laws for Chaotic Resonance States in Open Mixed Systems

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In open chaotic systems the number of long-lived resonance states obeys a fractal Weyl law, which depends on the fractal dimension of the chaotic saddle. We study the generic case of a mixed phase space with regular and chaotic dynamics. We find a hierarchy of fractal Weyl laws, one for each region of the hierarchical decomposition of the chaotic phase-space component. This is based on our observation of hierarchical resonance states localizing on these regions. Numerically this is verified for the standard map and a hierarchical model system.

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It is just a century ago that Hermann Weyl published his celebrated theorem on the asymptotic distribution of eigenmodes of the Helmholtz equation in a bounded domain [1] which has found fundamental applications in the context of acoustics, optical cavities, and quantum billiards [2–4]. For a quantum billiard with a d-dimensional phase space the number $\mathcal{N}(k)$ of eigenmodes with a wave number below k is on average and in the limit of large k given by $\mathcal{N}(k) \sim k^{d/2}$ up to corrections of higher order [5–9]. Only recently, this fundamental question has been addressed for open scattering systems, where for the case of fully chaotic systems, a fractal Weyl law was found [10–23]. Because of the opening of the system one classically obtains a fractal chaotic saddle (sometimes also called repeller), which is the invariant set of points in phase space that do not escape, neither in the future nor in the past [24,25]. Its fractal dimension δ plays an important role quantum mechanically: The number $\mathcal{N}$ of long-lived resonance states is given by a fractal Weyl law,

$$\mathcal{N}(h) \sim h^{-\delta/2}, \tag{1}$$

which here is stated for open chaotic maps, where the k dependence is replaced by the dependence on the effective size of Planck's cell h.

Generic Hamiltonian systems exhibit a mixed phase space where regular and chaotic motion coexist [26], see Fig. 1(a). Regular resonance states of the open system obey a standard Weyl law, while for chaotic resonance states one would naively expect that their number follows the fractal Weyl law, Eq. (1). This ignores, however, that the dynamics in the chaotic region of generic two-dimensional maps is dominated by partial transport barriers, see Fig. 1(a). A partial barrier is a curve which decomposes phase space into two almost invariant regions. The small area, enclosed by the partial barrier and its preimage (dotted line in Fig. 1(a), magnification), consists of two parts of size Φ on opposite sides of the partial barrier, which are mapped to the other side in one iteration of the map. This flux Φ is the characteristic property of a partial barrier. There are infinitely many partial barriers which are hierarchically organized with decreasing fluxes towards the regular regions [27–31]. The partial barriers strongly impact the system's classical [27–33] and quantum mechanical [34–44] properties, and lead to, e.g., the localization of eigenstates in phase space [34–36,40,44] and fractal conductance fluctuations [37,38,42].

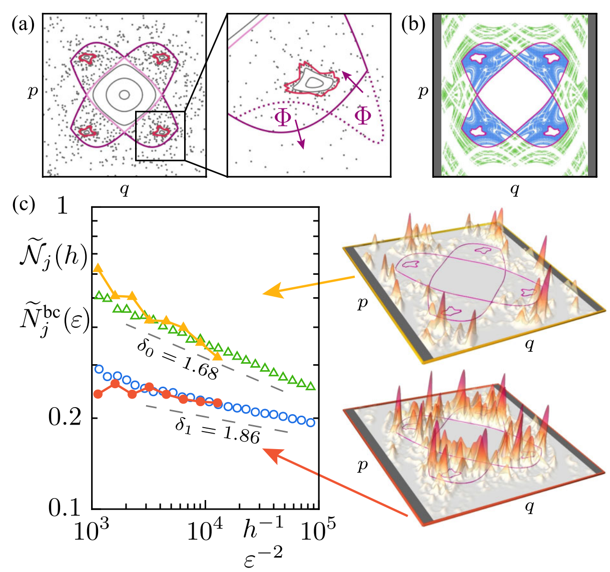

FIG. 1 (color online). (a) Phase space of the standard map at $\kappa = 2.9$ with regular (thin solid gray lines) and chaotic (gray points) orbits, three partial barriers (thick solid colored lines) and the preimage of the outermost partial barrier (dotted magenta line). (b) Chaotic saddle of the opened map (gray-shaded absorbing stripes) colored according to the regions (A_0: light green, A_1: dark blue). (c) Rescaled hierarchical fractal Weyl laws $\widetilde{\mathcal{N}}_j$ vs h^{-1} (filled symbols) counting hierarchical resonance states in the outer (A_0, triangles) and inner (A_1, circles) chaotic regions (with corresponding typical Husimi representations for $h = 1/1000$). Their power-law scaling is compared to the rescaled box-counting scaling $\widetilde{N}_j^{bc}$ vs ε^{-2} (open symbols) with fractal dimension δ_j in region A_j of the chaotic saddle.

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Classically, the chaotic saddle, see Fig. 1(b), in generic two-dimensional open maps gives rise to an individual fractal dimension for each region of the hierarchical decomposition of phase space [32]. It is important to stress that these are effective fractal dimensions, which are constant over several orders, while in the limit of arbitrarily small scales, they approach two [32,45]. Quantum mechanically, fractal Weyl laws for open systems with a mixed phase space have been investigated in Refs. [46–49], but the influence of the hierarchical phase-space structure remains to be studied. In particular, the individual effective fractal dimensions of the chaotic saddle have not been taken into account, so far.

In this Letter we propose a generalization of the Weyl law to open systems with a mixed phase space. We obtain hierarchical fractal Weyl laws,

$$\mathcal{N}_j(h) \sim h^{-\delta_j/2}, \tag{2}$$

one for each phase-space region A_j of the hierarchical decomposition of the chaotic component in a generic two-dimensional phase space. Here, δ_j denotes the effective fractal dimension of the chaotic saddle in each region. Quantum mechanically, this result is based on our observation of hierarchical resonance states, which predominantly localize on one of the regions A_j. Their number $\mathcal{N}_j$ follows the hierarchical fractal Weyl laws, Eq. (2). This holds over ranges of h where on the corresponding classical scale the effective fractal dimension δ_j is constant. In the semiclassical limit we expect a scaling h^{-1}. Equation (2) is confirmed for the generic standard map and a hierarchical model system.

Classical properties.—We first review the classical properties of the chaotic saddle in a generic mixed system and illustrate them for the prototypical example of the Chirikov standard map [50]. It is obtained from the kicked rotor Hamiltonian $H(q, p, t) = T(p) + V(q)\sum_{n\in\mathbb{Z}}\delta(t-n)$ with kinetic energy $T(p) = p^2/2$ and kick potential $V(q) = (\kappa/4\pi^2)\cos(2\pi q)$. At integer times t it leads to the symmetrized map $q_{t+1} = q_t + T'(p^*)$, $p_{t+1} = p^* - V'(q_{t+1})/2$ with $p^* = p_t - V'(q_t)/2$ on the torus $[0, 1) \times [-1/2, 1/2)$. We open the system by defining absorbing stripes of width 0.05 on the left and right, see Fig. 1(b). This leads to a chaotic saddle Γ, for which a finite-time approximation is shown in Fig. 1(b) for $\kappa = 2.9$. The chaotic saddle Γ of the open system is strongly structured by the presence of partial barriers. They originate from Cantori or stable and unstable manifolds of hyperbolic periodic orbits [31]. Partial barriers provide a hierarchical treelike decomposition [30] of the chaotic component of phase space into regions A_j: A typical orbit explores a region A_j before it enters a neighboring region A_k. The transition rate is approximately given by the ratio Φ/A_j where Φ is the flux across the partial barrier separating A_j and A_k. The route of escape from region A_j to the opening is determined by the treelike decomposition of phase space. It traverses the sequence of neighboring regions connecting A_j with the opening in A_0. The escape rate from a region A_j is dominated by the first transition rate, as subsequent transition rates are much larger. Figure 1(a) shows the outermost partial barrier separating the largest two regions A_0 and A_1 (which are quantum mechanically accessible), as well as its preimage illustrating its flux Φ. In addition, one can see the two partial barriers separating region A_1 from the chaotic region near the central island and near the period-four regular island chain. All three partial barriers are constructed from stable and unstable manifolds of a period 4 and a period 28 orbit.

Using the box-counting method [51] one can associate a fractal dimension δ_j with the intersection $\Gamma \cap A_j$ of the chaotic saddle Γ with each of the regions A_j. The number $N_j^{\mathrm{bc}}(\varepsilon)$ of occupied boxes of side length ε scales like $N_j^{\mathrm{bc}}(\varepsilon) \sim \varepsilon^{-\delta_j}$, see Fig. 1(c), with $\delta_0 = 1.68$ and $\delta_1 = 1.86$. To emphasize the difference between such dimensions close to two, the ordinate is rescaled by ε^2, yielding the rescaled counting function $\tilde{N}_j^{\mathrm{bc}}(\varepsilon) = \varepsilon^2 N_j^{\mathrm{bc}}(\varepsilon)$. The increase of δ_j towards two when going deeper into the hierarchy can be qualitatively understood by adapting the Kantz–Grassberger relation [52] from fully chaotic systems.

Hierarchical resonance states.—We now present the essential quantum effect that resonance states localize predominantly on one of the regions A_j. The closed quantum system is described by the time-evolution operator $U = \exp\{-(i/2\hbar)V(q)\}\exp\{-(i/\hbar)T(p)\}\exp\{-(i/2\hbar)V(q)\}$. The corresponding open quantum system is given by $U_{\mathrm{open}} = PUP$, where P is a projector on all positions not in the absorbing regions. The resonance states ψ are given by $U_{\mathrm{open}}\psi = \exp[-i(\varphi - i\gamma/2)]\psi$. Regular resonance states are predominantly located in the regular region. Chaotic resonance states are predominantly located in either of the hierarchical regions A_j, see Fig. 1(c). Hence, we will call them hierarchical resonance states (of region A_j). Such a localization of chaotic eigenstates on different sides of a partial barrier is well known for closed quantum systems [27,36,44]. Chaotic eigenstates localized in the hierarchical region of a mixed phase space were termed hierarchical states [40]. They require that the classical flux Φ across a partial barrier is small compared to the size h of a Planck cell, i.e., $\Phi \ll h$. In the opposite case, eigenstates would be equidistributed ignoring the partial barrier [27,36,44]. Quite surprisingly, in open quantum systems we find that this condition from closed systems is irrelevant for hierarchical resonance states. In the standard map at $\kappa = 2.9$ we have $\Phi \approx 1/80$, and for $h = 1/1000$, such that the condition $\Phi \ll h$ is violated, typical resonance states still predominantly localize in one of the regions A_j, as shown in Fig. 1(c). This is still the case for $h = 1/12\,800$, see Fig. 2. This crucial phenomenon for our study highlights the strong impact of the opening.

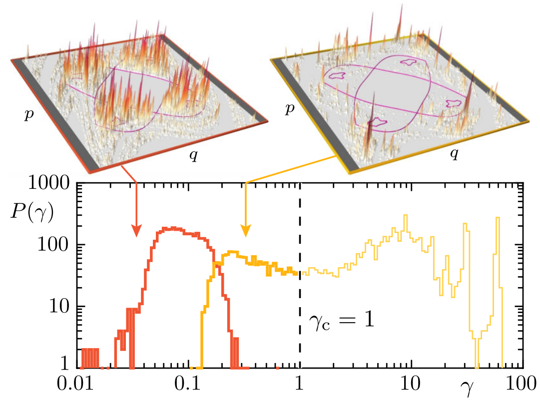

FIG. 2 (color online). Distributions $P(\gamma)$ of decay rates γ for hierarchical resonance states of the standard map at $\kappa = 2.9$ located in regions A_0 (right, yellow) and A_1 (left, red) for $1/h = 12800$ and corresponding Husimi representations of typical states. Short-lived states ($\gamma > \gamma_c$) are not counted in the fractal Weyl law.

One can qualitatively understand this localization of hierarchical resonance states in the following way: The localization on an almost invariant region A_j seems plausible in view of the semiclassical eigenfunction hypothesis for invariant regions [53–55]. However, eigenstates localized on neighboring regions hybridize, if their coupling due to the flux Φ is larger than their energy spacing. In closed systems this happens for $\Phi > h$ [27,36,44]. In open systems, though, the distance of resonance energies in the complex plane is larger due to their imaginary part. In fact, it is much larger due to the different decay rates of resonance states of neighboring regions A_j corresponding to their different classical escape rates. Therefore the localization of resonance states on regions A_j is possible in open systems, even if the criterion for the closed system, $\Phi \ll h$, is not fulfilled. Such a line of reasoning is reminiscent of the considerations on resonance trapping in fully chaotic systems [56–58]. This impact of the opening will be studied quantitatively in the future.

For the present study it is sufficient to observe that the great majority of chaotic resonance states is predominantly located in one of the regions A_j, allowing their classification. Numerically, we use their relative local Husimi weight in A_j (in the case of A_0 excluding the area of the opening) and discard states with more than 50% Husimi weight in the regular region and the deep hierarchical region (A_j, $j \geq 2$). This classification is supported by the distribution of the decay rates γ of the corresponding resonance states, see Fig. 2. States which are located deeper in the hierarchy have smaller decay rates. In fact, the two distributions for regions A_0 and A_1 have a small overlap, only. Note that an alternative classification of resonance states purely based on their decay rates γ, would fail deeper in the hierarchy, as the treelike structure allows for different regions A_j having strongly overlapping decay rate distributions.

Hierarchical fractal Weyl laws.—For each region A_j of the hierarchical phase space we now relate the number $\mathcal{N}_j$ of hierarchical resonance states of that region to the fractal dimension δ_j of the chaotic saddle in that region. To this end we use the fractal Weyl law of fully chaotic systems [12,13], Eq. (1), individually for each region A_j. This gives our main result that in open systems with a mixed phase space one obtains a hierarchy of fractal Weyl laws, one for each phase-space region A_j, Eq. (2). We stress that this result is based on the surprising existence of hierarchical resonance states. Note that as a consequence of Eq. (2) the total number of long-lived hierarchical resonance states is a superposition of power laws with different exponents and not a single power law.

To give an intuitive understanding of the hierarchical fractal Weyl laws, let us recall the interpretation of the fractal Weyl law [12], and apply it in the presence of a hierarchical phase space. The number of quantum states localizing on a particular phase-space region is given by the number of Planck cells necessary to cover the chaotic saddle in that region. Using the scaling, $N_j^{\rm bc}(\varepsilon) \sim \varepsilon^{-\delta_j}$, of the number of boxes $N_j^{\rm bc}$ to cover the chaotic saddle in region A_j and the identification of the box area ε^2 with the Planck cell area h directly leads to Eq. (2). This holds for values of h not too small, such that on the corresponding classical scale the effective fractal dimension δ_j still remains constant. Asymptotically ($\varepsilon \to 0$), all δ_j approach two [32,45]. Therefore, in the semiclassical limit ($h \to 0$), we expect an individual resonance state to extend over all regions A_j and that their number scales as h^{-1}.

Standard map.—The numerical investigation of the standard map supports the existence of hierarchical fractal Weyl laws, as we now show. By the classification of resonance states we are able to determine the number $\mathcal{N}_j(h)$ of long-lived hierarchical resonance states associated with a particular region A_j depending on h. We restrict ourselves to the consideration of small h such that $\Phi/h \gtrsim 10$ where quantum mechanics can very well mimic classical transport in phase space [44]. Short-lived states are discarded by defining an arbitrary cutoff rate $\gamma_c = 1$, as usual for the fractal Weyl law [13]. In globally chaotic systems the particular choice of γ_c (within a reasonable range) does not influence the power-law exponent of the fractal Weyl law but its prefactor only [14]. Here this merely affects resonance states of the outermost region A_0. We obtain distinct behavior for each rescaled counting function $\widetilde{\mathcal{N}}_j(h) = f_j h \mathcal{N}_j(h)$, see Fig. 1(c), corresponding to the previous classical rescaling. We fitted prefactors f_j to the quantum results to better demonstrate their scaling with power laws in agreement with the classical counterparts (both prefactors f_j are of order one: $f_0 = 2.6$, $f_1 = 0.85$). Apart from the smallest values of $1/h$, one observes the power-law scaling of Eq. (2) and good agreement with the box-counting results for the fractal dimensions

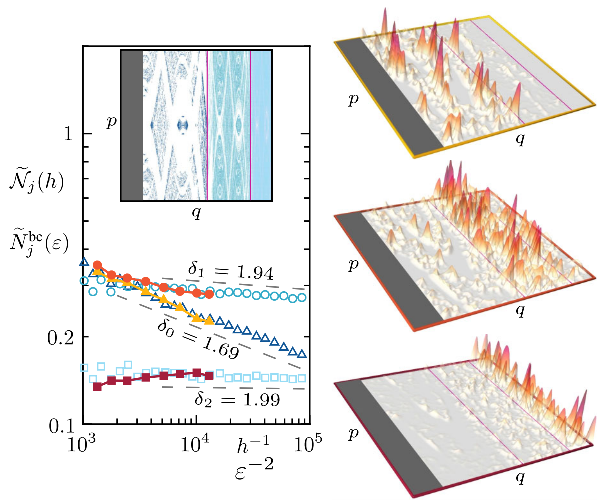

FIG. 3 (color online). Rescaled hierarchical fractal Weyl laws $\widetilde{\mathcal{N}}_j$ vs h^{-1} (filled symbols) counting hierarchical resonance states in the outer (A_0, triangles), central (A_1, circles), and inner (A_2, squares) chaotic regions for the hierarchical model system, and corresponding typical Husimi representations (right) for $h = 1/1115$. Comparison to rescaled box-counting scaling $\tilde{N}_j^{\mathrm{bc}}$ vs ε^{-2} (open symbols) of the chaotic saddle (inset) with fractal dimension δ_j in region A_j.

δ_j of $\Gamma \cap A_j$. Note that the deviations between corresponding classical and quantum power-law exponents are much smaller than the differences between the exponents associated with different regions A_j of the hierarchy. Figure 1(c) confirms for two regions A_j of the standard map that they give rise to hierarchical fractal Weyl laws. Note that the shape and position of the absorbing region modifies the fractal dimension of the chaotic saddle and the power-law exponent of the fractal Weyl law, but their relation remains valid (not shown).

Hierarchical model system.—To verify the hierarchical fractal Weyl laws for more than two regions, we suggest the following system that models the hierarchical structure of partial barriers in a generic mixed phase space, similar in spirit to a one-dimensional model [32] and a Markov chain [29]. The numerics for the corresponding quantum model allows for studying three regions.

We first define a composed symplectic map $C \circ M$ on the phase space $[0, 1) \times [0, 1)$. It models b partial barriers at the positions $q_1 < \cdots < q_b$ as straight lines in the p direction, giving a decomposition into $b + 1$ regions $A_j = [q_j, q_{j+1}) \times [0, 1)$ with $q_0 = 0$ and $q_{b+1} = 1$. The map M describes the uncoupled dynamics being sufficiently mixing in each region A_j. We choose the standard map at kicking strength $\kappa = 10$ acting individually on each of the regions A_j after appropriate rescaling. The map C couples these regions mimicking the turnstile mechanism of a partial barrier with flux Φ_j by exchanging the areas $[q_j - \Phi_j, q_j) \times [0, 1)$ with their neighboring areas $[q_j, q_j + \Phi_j) \times [0, 1)$. Finally, we open the system by defining the absorbing region $[0, \Phi_0) \times [0, 1)$. Here, we use $b = 2$, $q_1 = 4/7$, $q_2 = 6/7$, $\Phi_0 = 1/7$, $\Phi_1 = 1/28$, and $\Phi_2 = 1/112$.

Figure 3 shows the results for the hierarchical model system: We obtain the fractal dimensions $\delta_0 = 1.69$, $\delta_1 = 1.94$, and $\delta_2 = 1.99$. Quantum mechanically, we again find hierarchical resonance states predominantly localizing on one of the regions A_j, even though $h \ll \Phi_1, \Phi_2$. Their number follows the proposed hierarchical fractal Weyl laws according to Eq. (2). For the rescaled numbers $\widetilde{\mathcal{N}}_j$ in Fig. 3 we use prefactors $f_0 = 1.75$, $f_1 = 1.55$, and $f_2 = 0.8$, which are of order one.

An experimental verification of the hierarchical fractal Weyl laws should be feasible using microwave cavities as in a recent study on chaotic resonance states [59]. A future challenge is the study of fractal Weyl laws in higher dimensional systems with a generic phase space.

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