Rogue Waves Emerging from the Resonant Interaction of Three Waves

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We introduce a novel family of analytic solutions of the three-wave resonant interaction equations for the purpose of modeling unique events, i.e., "amplitude peaks" which are isolated in space and time. The description of these solutions is likely to be a crucial step in the understanding and forecasting of rogue waves in a variety of multicomponent wave dynamics, from oceanography to optics and from plasma physics to acoustics.

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Introduction.—The nature of rogue waves, mostly known as oceanic phenomena responsible for a large number of maritime disasters, has been discussed in the literature for decades [1–4]. A number of various approaches have been suggested to explain the high-impact power of these "monsters of the deep" [5], which appear visibly from nowhere and disappear without a trace. Theories may differ depending on the physical conditions where these waves appear [6,7].

As a matter of fact, a comprehensive understanding of the protean rogue wave phenomenon is still far from being achieved [5,8]. Indeed, these waves not only appear in oceans but also in the atmosphere [9], in optics [6], in plasmas [10], in superfluids [11], in Bose-Einstein condensates [12], and in capillary waves [13]. Peculiar aspects and common features of the multifaceted manifestations of rogue waves in their different physical realms are a subject of intense scientific debate [7]. New studies of rogue waves in any of these disciplines contribute to give a global view on a complex process that to a large extent remains unexplored [14].

Nonlinear dynamics is one of the theoretical frameworks that has been successful in predicting the basic features of rogue waves [15,16]. A formal prototypical description of a single rogue wave is provided by the so-called Peregrine soliton, a solution of the focusing nonlinear Schrödinger equation (NLSE) [17,18] which features a rational dependence on both space and time coordinates. Such a solution describes the growing evolution of a small, localized perturbation of a plane wave whose peak subsequently gets amplified by a maximal factor of 3 over the background and eventually decays and vanishes. After decades of debate [5,7], the Peregrine soliton has been observed experimentally only very recently in fiber optics [19], in water-wave tanks [20], and in plasmas [21]. Moreover, the Peregrine soliton turns out to be just the first of an infinite hierarchy of higher order rational solitons of the focusing NLSE with a progressively increasing peak amplitude. Again their amplitude over the background is expressed by the ratio of progressively higher degree polynomials and are therefore localized in both space and time [22,23]. Recent experiments in a water tank showed that these analytic solutions describe well the actual dynamics even in the case of steep waves [24]. These theoretical findings and experimental observations prove that the approach based on fundamental nonlinear models, such as NLSE, may be fruitful and appropriate to rogue wave description.

In a variety of physical contexts, several waves rather than a single one need to be considered in order to account for important resonant interaction processes. In these circumstances extreme waves should be described as solutions of coupled systems of equations rather than by the single-wave NLSE model. In this direction the investigation of solutions that are possible candidates as rogue waves has recently been extended to coupled NLSEs [25–28]. This weak resonant interaction of two waves has been shown to cause wave behaviors which could not be detected by the single Peregrine soliton. However, in order to account for the appearance and dynamics of extreme waves in strong resonant processes, the three-wave resonant interaction (TWRI) equations seem to be the fundamental and universal model. Indeed, this system describes the propagation and mixing of waves in weakly nonlinear and dispersive media. Applications of the TWRI system have been found in fluid dynamics (capillary-gravity waves, internal gravity waves, surface and internal waves) [29,30], in optics (parametric amplification, frequency conversion, stimulated Raman and Brillouin scattering) [31,32], in plasmas (plasma instability, laser-plasma interactions, radio frequency heating) [33,34], and in solid-state physics and acoustics [35].

In this Letter, we introduce a family of rational multicomponent solutions of the TWRI equations, in $1 + 1$ dimensions, which describe unique events, i.e., "amplitude peaks," which are distinctive of rogue waves as being much higher than the surrounding background and well isolated in both space and time. These solutions are expected to be crucial in forecasting and explaining extreme waves in a variety of multicomponent resonant processes (e.g., oceanography, optics, plasma physics).

TWRI equations and rogue waves.—The system of three coupled partial differential equations describing the resonant interaction of three waves in 1 + 1 dimensions reads as follows (in the notation of [36]):

$$\begin{aligned} E_{1t} + V_1 E_{1z} &= E_2^* E_3^*, \\ E_{2t} + V_2 E_{2z} &= -E_1^* E_3^*, \\ E_{3t} + V_3 E_{3z} &= E_1^* E_2^*, \end{aligned} \tag{1}$$

where each subscript variable stands for partial differentiation. $E_n = E_n(z, t)$, $n = 1, 2, 3$, are slowly varying, narrow band, complex envelopes of the waves at frequencies ω_n and wave numbers k_n, t is the evolution variable, and z is a second independent variable. Since we consider resonant interactions, the frequencies and momenta of the three waves must satisfy the equations $\omega_1 + \omega_2 + \omega_3 = 0$ and $k_1 + k_2 + k_3 = 0$. The coefficients V_n are the "group velocities" of the three waves and we assume the ordering $V_1 > V_2 > V_3$. With no loss of generality, we assume $V_3 = 0$ by writing these equations (1) in the reference frame moving with the same velocity of E_3. The signs of the coupling constants, with the minus sign only in the equation with the intermediate velocity V_2, correspond to the so-called "soliton-exchange" case in the terminology of [31].

It should be pointed out that the meaning of the slowly varying complex envelopes E_n and of the coordinates t, z depends on the particular applicative context (e.g., fluid dynamics [30], plasma physics [34], nonlinear optics [32], acoustics [35]).

TWRI equations, like NLSE, possess rational solutions with the property of representing, in each of the three waves E_n, amplitude peaks which are isolated in space and time. Similarly to the case of the NLSE, these solutions are local deformations of a nonvanishing background whose modulation instability is discussed in [37]. Such solutions can be expressed as

$$E_1 = 2q\delta_1\left[1 + \frac{3\sqrt{3}A^*\theta^*A_1}{|A|^2 + |A_1|^2 + |A_2|^2}\right]e^{iq(t-\nu_1 z)}, \tag{2a}$$

$$E_2 = 2q\delta_2\left[1 + \frac{3\sqrt{3}A\theta^*A_2^*}{|A|^2 + |A_1|^2 + |A_2|^2}\right]e^{iq(t+\nu_2 z)}, \tag{2b}$$

$$E_3 = 2iq\delta_3\left[1 + \frac{3\sqrt{3}\theta^*A_1^*A_2}{|A|^2 + |A_1|^2 + |A_2|^2}\right]e^{-iq[2t+(\nu_2-\nu_1)z]}, \tag{2c}$$

where

$$\theta = (-\sqrt{3} + i)/2, \qquad |\theta| = 1,$$
$$\delta_1 = \sqrt{(V_1 - V_2)/V_1}, \qquad \delta_2 = \sqrt{(V_1 - V_2)/V_2},$$
$$\delta_3 = (V_1 - V_2)/\sqrt{V_1 V_2},$$
$$A = \gamma_1 + \gamma_2\xi_1 + \gamma_3(\eta - i\theta^*),$$
$$A_1 = \gamma_1 + \gamma_2(\xi_1 + \theta^*) + \gamma_3(\eta + \theta^*\xi_1 + i\sqrt{3}),$$
$$A_2 = \gamma_1 + \gamma_2(\xi_1 + \theta) + \gamma_3(\eta + \theta\xi_1),$$
$$\xi_1 = -2q(t + i\rho_1 z), \qquad \eta = \frac{1}{2}\xi_1^2 - 2iq\rho_2 z,$$
$$\rho_1 = \theta/V_1 - \theta^*/V_2, \qquad \rho_2 = 1/V_1 - 1/V_2,$$
$$\nu_1 = 2/V_2 - 1/V_1, \qquad \nu_2 = 1/V_2 - 2/V_1.$$

The above expressions depend on the velocities V_1, V_2, the real "frequency" parameter q, and the complex parameters γ_1, γ_2, γ_3. Of course, the parameters q, ν_n of the background are restricted by conditions such as $|q| \ll \omega_n$, for $n = 1, 2$, $2|q| \ll \omega_3$, $|q\nu_1| \ll |k_1|$, $|q\nu_2| \ll |k_2|$, $|q(\nu_2 - \nu_1)| \ll |k_3|$, as we are considering narrow band envelopes.

Once the structural parameters (i.e., the characteristic velocities V_1 and V_2) are fixed, we are left with four independent parameters, q and γ_1, γ_2, γ_3.

However, not all of these parameters are essential, since some of them can be fixed without loosing generality by using appropriate symmetries of the TWRI equations (1). The parameter q merely rescales the wave amplitudes and the coordinates z and t. Thus, one can set $q = 1$.

Additionally, the three remaining parameters γ_1, γ_2, γ_3 are not all essential, as one (nonvanishing) of them can be given the unit value. Moreover, it can be shown that if $\gamma_2 = \gamma_3 = 0$, then the solution (2) represents plane wave backgrounds with no interest. Otherwise, if $\gamma_3 = 0$, then the parameter γ_1 can be made to vanish by using translation invariance, while, by the same argument, one can set $\gamma_2 = 0$ if $\gamma_3 \neq 0$, while γ_1 remains instead an essential parameter. Despite this simplification, we choose to keep all three parameters γ_1, γ_2, γ_3 and to play with them to better display a few aspects of the many properties of this family of solutions (2).

In Fig. 1, we first show the case $\gamma_2 \neq 0$, $\gamma_3 = 0$. The parameter γ_1 is so chosen as to put the peak at the origin of the (z, t) plane. As expected, the expression (2) describes amplitude peaks which are localized in both z and t. Interestingly, each component $|E_n|$ looks like a rogue wave whose maximum height is twice the background intensity while its minimum is zero; its eye-shaped distribution density shows one hump and two valleys. We also note that, as for the Peregrine soliton, the rational expression (2) is the ratio of two polynomials of second degree in the coordinates z, t.

In the case $\gamma_3 \neq 0$, the expression (2) may describe amplitudes with multiple peaks localized in z and t. Figure 2 shows two rogue waves with different structures

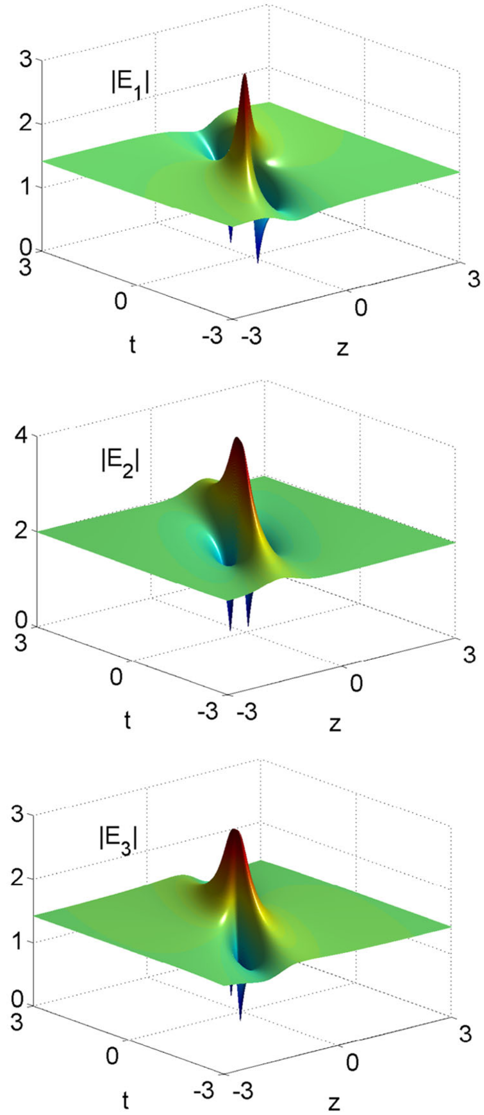

FIG. 1 (color online). Vector rogue waves envelope distributions $|E_1|$, $|E_2|$, and $|E_3|$ of (2). Here, $V_1 = 1$, $V_2 = 0.5$, $q = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0$.

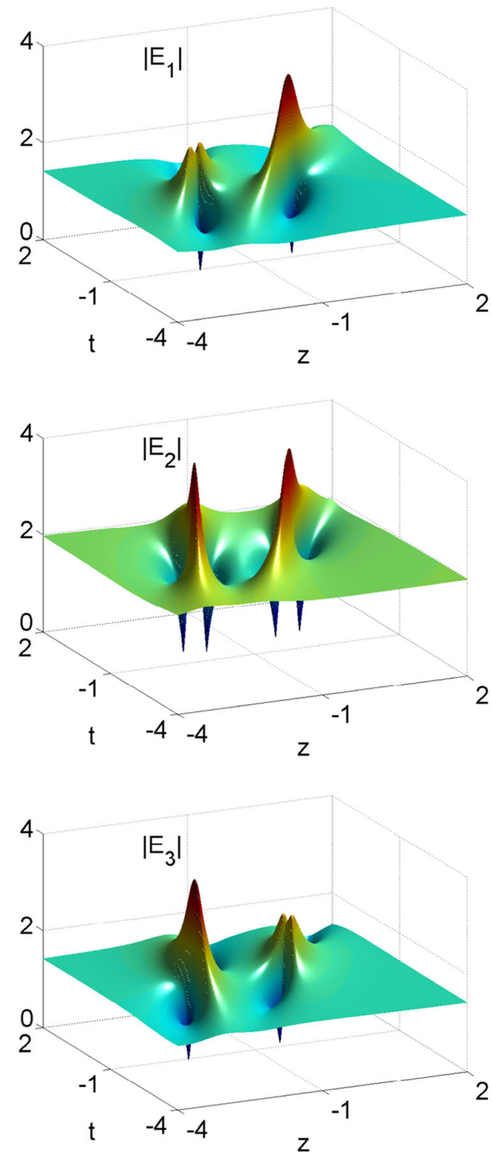

FIG. 2 (color online). Vector rogue waves envelope distributions $|E_1|$, $|E_2|$, and $|E_3|$ of (2). Here, $V_1 = 1$, $V_2 = 0.5$, $q = 1$, $\gamma_1 = 2$, $\gamma_2 = 7$, $\gamma_3 = 1.5 + i$.

in each one of the three components $|E_n|$. Figure 2 shows, in the E_1 component, a bright rogue wave with an eye-shaped distribution (a hump and two valleys), together with a wave with a four-petaled distribution (two humps and two valleys around a center, and the center value is almost equal to that of the background). The four-petaled wave in the E_1 component corresponds to eye-shaped rogue waves in E_2 and E_3 components. Splitting of the maxima in Fig. 2 is a phenomenon that deserves attention. It is interesting, as it looks like higher order solitons but appears from the same solution. By decreasing the value $|\gamma_3/\gamma_2|$, in each component these rogue waves separate. By increasing $|\gamma_3/\gamma_2|$, these rogue waves instead merge, giving birth to higher-amplitude vector rogue wave solutions (see Fig. 3). The maximum value of the humps is more than 3 times the plane wave's background for some components, and the minimum value of the valleys is zero.

Notice that effective energy exchanges take place between waves E_1, E_2, and E_3 in TWRI during their interaction. Figure 4 reports a typical evolution of the effective energy $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$ versus t. Effective energies $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$ are obtained according to the definition

$$\bar{I}_n = \frac{1}{2}\int (|E_n|^2 - |E_{n0}|^2)dz,$$

where $E_{n0} = \lim_{z\to\infty} E_n$, $n = 1, 2, 3$, is the plane wave background. The energy transfer between the waves can enhance the peak amplitude in some of the wave

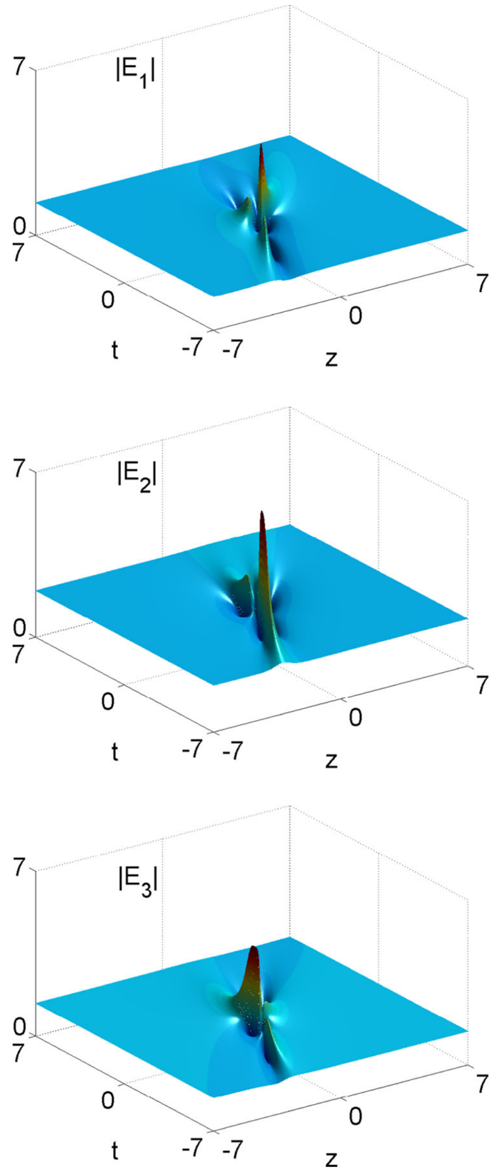

FIG. 3 (color online). Vector rogue waves envelope distributions $|E_1|$, $|E_2|$, and $|E_3|$ of (2). Here, $V_1 = 1$, $V_2 = 0.5$, $q = 1$, $\gamma_1 = 1$, $\gamma_2 = 0.1$, $\gamma_3 = i$.

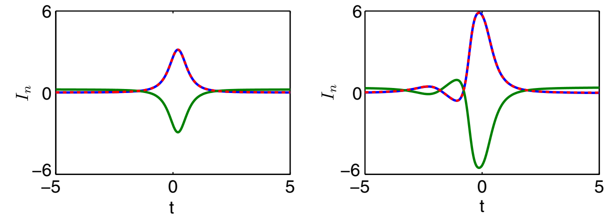

FIG. 4 (color online). Effective energy evolution $\bar{I}_1$ (blue line), $\bar{I}_2$ (green line), and $\bar{I}_3$ (red line) versus t, for the case reported in Fig. 1 (left) and Fig. 3 (right).

components. This wave behavior is completely different from what happens in coupled NLSEs, where energy exchanges are forbidden [38].

Let us briefly discuss the experimental condition in nonlinear optics for the observation of TWRI rogue waves. In fact, nonlinear optics has recently been seen as a fertile, reproducible, and safe ground to experimentally develop the knowledge of rogue waves [6,7,14,19]. One may consider a TWRI optical spatial noncollinear scheme with type II second-harmonic generation in a 3 cm long birefringent KTP crystal (e.g., see the experimental setup of Ref. [32]). Spatial diffractionless 6 mm waist beams, mimicking quasi-plane waves, at 1064 nm (o wave and e wave) and at 532 nm (e wave) would lead to TWRI modulational instability evidence and rogue wave dynamics with peak field intensities of tens of $\mathrm{MW/cm^2}$.

As a final remark, for the computation of expression (2), we notice that this is based on the Darboux technique applied to the Lax pair associated with the TWRI equations. This method is well known and does not need to be detailed here to any extent. The relevant literature is rather vast and we refer to Ref. [39] for the formalism we have adopted and to Refs. [25,40] for the basic arguments to follow for the construction of rational solutions.

Conclusions.—We have reported the explicit analytic expression of solutions of the equations describing the resonant interaction of three waves. These solutions have the important property of describing rogue wave events. Several articles have recently been devoted to rogue waves as rational solutions of multicomponent systems of coupled wave equations: vector NLSEs [25–28], Davey-Stewartson equation [41], and coupled Hirota systems [42]. The present step in this direction, dealing with rogue wave solutions of the TWRI equations which represent a fundamental and universal model for the description of strong resonant interactions, seems to be a crucial stride to controlling and forecasting extreme-wave dynamics in multicomponent wave systems, with a broad variety of applications.

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