Model-Independent Extraction of the Pole and Breit-Wigner Resonance Parameters

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We show that a slightly modified Breit-Wigner formula can successfully describe the total cross section even for the broad resonances, from the light $\rho(770)$ to the heavy Z boson. In addition to the mass, width, and branching fraction, we include another resonance parameter that turns out to be directly related to the pole residue phase. The new formula has two mathematically equivalent forms: one with the pole and the other with the Breit-Wigner parameters.

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Resonances are unstable particles usually observed as bell-shaped structures in scattering cross sections of their decay products. For a simple narrow resonance, its fundamental properties correspond to the visible cross-section features: mass M is at the peak position, and decay width Γ is the width of the bell shape. These parameters, along with the branching fraction x, are known as the Breit-Wigner (BW) parameters [1]. In reality, resonance peaks may be very broad, and the shape so deformed that it is not at all clear where exactly the mass is, or what would be the width of that resonance. In such cases, resonance parameters are treated as energy dependent functions. These functions are often defined differently for different resonances. For example, in the case of the $\rho(770)$ resonance in the $\pi\pi$ channel, modern analyses include the pion-pion P-wave potential barrier (momentum to three halves) in the energy dependent width [2], while in the case of the Z boson the width function is proportional to the energy squared [3].

With such model dependent parametrizations, the simple connection between the physical properties of a resonance and its model parameters is lost, and the choice of the "proper" resonance parameters becomes a matter of preference. There are many definitions for the Breit-Wigner mass, which is assumed by some to be the proper resonance physical property. Others will prefer the real part of the pole position in the complex energy plane. Some will even define the resonance mass to be something unrelated to these two most common definitions, as we will soon see, or assume that there is no difference between the poles and Breit-Wigner parameters whatsoever. All that makes the comparison between the cited resonance parameters quite confusing and potentially hinders the direct comparison between microscopic theoretical predictions (such as in [4]) and experimentally obtained resonance properties [5].

To clarify this situation we try to devise a simple model-independent formula for resonant scattering, with well-defined resonance physical properties, which will be capable of successfully fitting the realistic data for broad resonances.

In this Letter, we show how to dramatically improve the simple Breit-Wigner formula by incorporating just one additional (phase) parameter in it. This new formula has two equivalent forms that can be used to estimate either the pole or Breit-Wigner parameters in a model-independent way.

We begin our analysis by noting that the resonant cross section is commonly parametrized by a simple Breit-Wigner formula [1]

$$\sigma = \frac{4\pi}{q^2}\frac{2J+1}{(2s_1+1)(2s_2+1)}|A|^2, \tag{1}$$

where q is a c.m. momentum, J is the spin of the resonance, while s_1 and s_2 are spins of the two incoming particles. The resonant scattering amplitude A is given by

$$A = \frac{x\Gamma/2}{M - W - i\Gamma/2}, \tag{2}$$

where M is the resonant mass, Γ is the total decay width, x is the branching fraction to a particular channel (for inelastic scattering it is $\sqrt{x_{\mathrm{in}}x_{\mathrm{out}}}$), and W is the c.m. energy.

This simple parametrization cannot describe most of the realistic cross sections since the resonance shapes are seldom symmetric. To fix this, a background contribution is usually added. Unfortunately, there is no standard way to add background, but polynomials in W^2 (i.e., Mandelstam s) are commonly used in the literature (see, e.g., [6])

$$|A|^2 \to |A|^2 + \sum_{k=0}^{n} B_k W^{2k}. \tag{3}$$

To extract the resonance parameters, we do local fits (in energy) of this parametrization to a broad range of data points in the vicinity of the resonance peak. To estimate the proper order n of the polynomial background, we vary the endpoints of the data range and check the convergence of the physical fit parameters: M, Γ, and x. The goodness of the convergence is estimated by calculating $c_{n,l}$ parameters for each data range and for all polynomial orders n and l

$$c_{n,l} = \sum_{y=m,\Gamma,x} (y_l - y_n)^2/y_n^2. \tag{4}$$

Smaller $c_{n,l}$ means better convergence.

To avoid false positive convergence signals as much as possible, we demand good convergence not just for two, but also for three consecutive polynomial orders by using

$$c_n = c_{n,n+1} + c_{n,n+2}. \quad (5)$$

The final result is the one that has the smallest reduced χ^2_R among several fits (we use ten) with lowest convergence parameters c_n. When statistical errors turn out to be unrealistically small due to the data set issues, the spread in the extracted pole parameter values is used to estimate parameters errors.

To test this extraction approach, we analyze five broad resonances with well-known properties, and masses ranging from less then 1 GeV to almost 100 GeV. For $\Delta(1232)$ and $N(1440)$, we analyze the George Washington University [7] πN elastic partial-wave amplitudes squared. For $\rho(770)$ and Z boson we analyze the e^+e^- scattering ratio R (between hadronic and muonic channels) from the PDG compilation [5], and for $\Upsilon(11020)$ the new *BABAR* data [8].

Using the Breit-Wigner parametrization, Eq. (2), on broad resonances does not produce very good results. Therefore, in the advanced approaches, the resonance width Γ (and other parameters) is considered to be energy dependent, which drastically improves the fit. However, parametrization then becomes model dependent, obfuscating the connection between the model parameters and physical properties of the resonance. We want to find a simple model-independent form, as close to the original Breit-Wigner parametrization as possible, that will be capable of successfully fitting the realistic data for broad resonances. To do so, we assume that the numerator and the denominator in relation (2) are functions of energy, expand them, and keep only the linear terms. The amplitude A becomes

$$A = \frac{x_p \Gamma_p/2\, e^{i\theta_p}}{M_p - W - i\Gamma_p/2} + |A_B| e^{i\theta_B}, \quad (6)$$

which turns out to be the lowest order Laurent expansion of amplitude A about its pole position at $W = M_p - i\Gamma_p/2$. Therefore, M_p and Γ_p are the pole mass and width, while $x_p\Gamma_p/2$ and θ_p are the complex residue magnitude and phase, respectively. (Note that we use the standard convention for the residue phase θ_p, as used in PDG [5], which differs from the mathematical residue phase by $\pm\pi$.) Three additional fit parameters are the residue phase θ_p, the (coherently added) background magnitude $|A_B|$, and the background phase θ_B. We can extract only the relative phase $\delta_p = \theta_p - \theta_B$, since the absolute square of this amplitude will be compared to the data. In order to ease the numerical analysis, we rewrite the new parametrization in a compact form

$$|A|^2 = |A_B|^2 \frac{(\mu - W)^2 + \lambda^2}{(M_p - W)^2 + \Gamma_p^2/4}, \quad (7)$$

where μ and λ are simple fit parameters related to the pole parameters through

$$x_p \sin\delta_p = |A_B| \frac{\Gamma_p/2 - |\lambda|}{\Gamma_p/2}, \quad (8)$$

$$x_p \cos\delta_p = -|A_B| \frac{M_p - \mu}{\Gamma_p/2}. \quad (9)$$

Using Eq. (7), we should be able to extract the pole mass, width, branching fraction, the magnitude of the background amplitude, and the relative phase from the data. We again use the same polynomial background from relation (3) and convergence criteria from relations (4) and (5). However, at the very beginning of this analysis we stumbled upon a problem with our fits. When we fitted the $\Delta(1232)$ resonance, parameter λ was rather unstable, ranging from zero to several thousand MeV. In addition, fits often did not converge, even for carefully chosen initial values.

We looked into it more closely and realized that since $\Delta(1232)$ is almost an elastic resonance (decaying by more than 99% to the πN channel), λ should be zero due to the elastic two-body unitarity condition $A^\dagger A = \mathrm{Im}A$. When λ was set to zero, everything worked almost perfectly. It is important to note that x_p should be 1 for elastic resonances, again due to the unitarity, but setting λ to zero does not imply that x_p is 1.

Things became really interesting when we tried to extract Z boson parameters from e^+e^- scattering data. The unstable λ behavior seen in the case of $\Delta(1232)$ was observed again, even though the Z boson is definitely not an elastic resonance. The fit could not be stabilized, and eventually we tried $\lambda = 0$ again (x_p still can take care of inelasticity). This choice smoothed the fitting procedure, and the extracted resonance parameters were in excellent agreement with the PDG (pole) estimates [5]. Assuming that $\lambda = 0$ for other processes, we rewrite the amplitude defined in Eq. (6) as

$$A = x_p e^{i\eta}\left(\frac{\Gamma_p/2\, e^{2i\delta_p}}{M_p - W - i\Gamma_p/2} + e^{i\delta_p}\sin\delta_p\right), \quad (10)$$

with the unmeasurable overall phase η equal to $2\theta_B - \theta_p$. The square of this amplitude is then

$$|A|^2 = x_p^2 \frac{[(M_p - W)\sin\delta_p + \Gamma_p/2\cos\delta_p]^2}{(M_p - W)^2 + \Gamma_p^2/4}. \quad (11)$$

We know that for $\Delta(1232)$, the overall phase η is zero due to unitarity, which means that $\theta_B = \delta_p$ and $\theta_p = 2\delta_p$. We compare our results for $2\delta_p$ to published results of θ_p for other analyzed resonances to check whether the same relation is valid for them as well. The Roper resonance

TABLE I. Resonance pole parameters extracted by using the pole formula, Eq. (11). Our $2\delta_p$ is compared to the residue phase θ_p from the literature. The PDG pole estimates are from Ref. [5]. The ρ meson pole and the Z boson residue phase are estimated by an analytic continuation of the Gounaris-Sakurai [2] and Breit-Wigner [5] parametrizations, respectively.

Resonance	M_p/MeV	Γ_p/MeV	x_p/%	$2\delta_p/^\circ$
$\rho(770)$	762 ± 1	138 ± 2	0.71	1 ± 1
Pole	763	144	⋯	⋯
$\Delta(1232)$	1211 ± 1	102 ± 1	103 ± 1	-47 ± 1
PDG pole	1210 ± 1	100 ± 2	104 ± 2	-47 ± 1
$N(1440)$	1362 ± 5	191 ± 10	61 ± 4	-81 ± 10
PDG pole	1365 ± 15	190 ± 30	65 ± 10	$-85\pm^{15}_{10}$
$\Upsilon(11020)$	$11\,000 \pm 2$	43 ± 6	0.10	-52 ± 8
BABAR [8]	$10\,996 \pm 2$	37 ± 3	⋯	⋯
$Z(91188)$	$91\,167 \pm 6$	2493 ± 5	15.4	-2.2 ± 0.2
PDG pole	$91\,162 \pm 2$	2494 ± 2	⋯	-2.35

$N(1440)$ is the πN resonance with the πN branching fraction x estimated to 65%, and the $2\delta_p$ value of $-81°$ is surprisingly close to the newest residue phase estimate $-85°$ from [5]. For the Z boson, these two values are even closer: $2\delta_p$ is $-2.2°$, while θ_p is $-2.35°$. The extracted masses and widths are much closer to the pole parameters listed in the literature than to the Breit-Wigner ones, as can be seen in Table I. The best fits for all analyzed resonances are shown in Figs. 1–5.

The new pole parametrization, Eq. (10), may be used instead of Eq. (2), since it works much better and adds only one quite important parameter δ_p. This parameter is the main ingredient of the shape of the resonance contribution to the cross section. When δ_p is equal to zero, the new pole and the old simple Breit-Wigner parametrization are exactly the same.

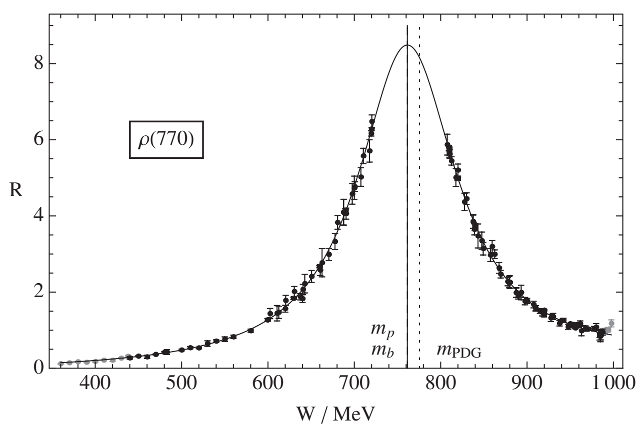

FIG. 1. Fitting pole parametrization, Eq. (11), to the data [5]. The pole and BW masses have almost the same value, which is quite different from PDG estimate (dotted line). [We removed the data from the peak to eliminate the influence of the $\omega(782)$ resonance.]

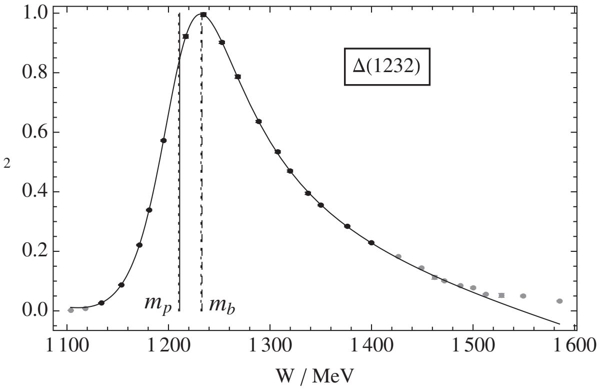

FIG. 2. Fitting pole parametrization, Eq. (11), to the SAID data [7]. Pole (solid line) and BW mass (dashed line) are clearly distinct, while the PDG estimates (dotted lines) are indistinguishable from them. All the data in this figure are analyzed, but only the black data points are used in the best fit.

We began this study in the first place to find an improved Breit-Wigner parametrization, but ended up with a pole parametrization instead. We followed the original notion of Breit and Wigner, that the resonance mass is at the peak position [1], and noted the convenient form of our pole parametrization, Eq. (10), that looks very similar to a single-channel elastic amplitude (apart from $x_p \neq 1$ and $\eta \neq 0$). We now define the new Breit-Wigner parameters as a single-channel K-matrix pole M_b, residue Γ_b, branching fraction x_b, and background phase δ_b,

$$K = \frac{\Gamma_b/2}{M_b - W} + \tan\delta_b, \tag{12}$$

$$A = x_b \frac{K}{1 - iK}, \tag{13}$$

$$|A|^2 = x_b^2 \frac{(\Gamma_b/2 + \tan\delta_b)^2}{(M_b - W)^2 + (\Gamma_b/2 + \tan\delta_b)^2}. \tag{14}$$

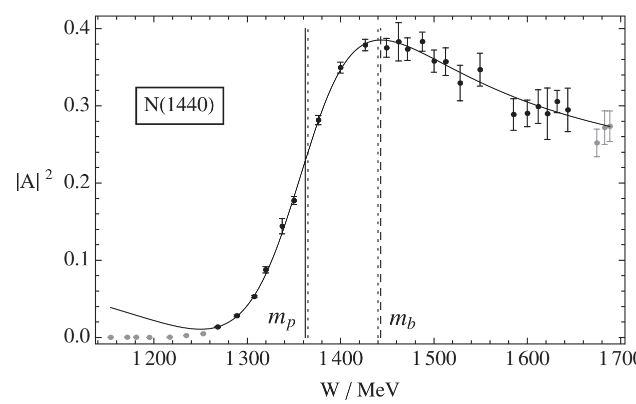

FIG. 3. Fitting pole parametrization, Eq. (11), to the SAID data [7]. This resonance has the largest difference between pole and BW mass. PDG estimates (dotted lines) are consistent with pole (solid line) and BW (dashed line) parameters.

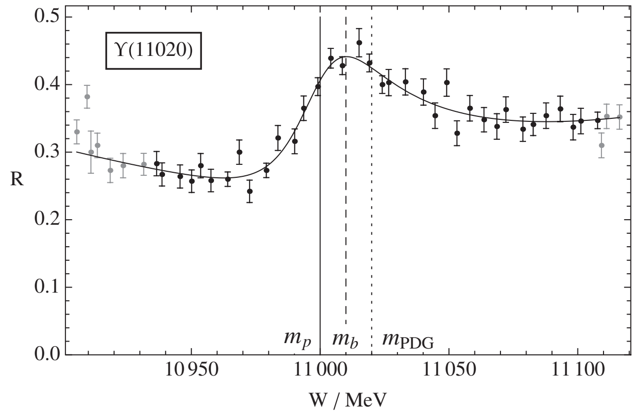

FIG. 4. Fitting pole parametrization, Eq. (11), to the *BABAR* data [8]. Pole and BW masses are clearly distinct, but PDG estimate coincides with neither of them.

In this form, x_b and δ_b will be mathematically equal to x_p and δ_p, respectively. When we fit parametrization, Eq. (14), to the data, extracted fit parameters M_b, Γ_b, and x_b are consistent with Breit-Wigner parameters from PDG [5], as is clearly visible from Table II. As expected, x_b and δ_b have almost exactly equal values as their pole counterparts x_p and δ_p in Table I. Furthermore, the extracted pole and Breit-Wigner parameters are interrelated through Manley relations [9]

$$M_b = M_p - \Gamma_p/2 \tan\delta_p, \quad (15)$$

$$\Gamma_b = \Gamma_p/\cos^2\delta_p. \quad (16)$$

These Breit-Wigner parameters are uniquely defined and model independent, with a directly observable mass as the peak of the squared amplitude $|A|^2$. However, they strongly depend on phase δ_p, which may change from reaction to reaction. That means that for the same pole position, there will be different Breit-Wigner masses and widths in different channels. Therefore, it is more practical to have one (pole) mass and width for each resonance in particle data tables [5] than to stockpile different (Breit-Wigner) masses and widths for each process in which the resonance contributes.

In our study, there are two resonances that show systematic discrepancy between the PDG mass estimates [5] and our results presented here: the $\rho(770)$ and the $\Upsilon(11020)$. For ρ meson an alternative parametrization by Gounaris and Sakurai [2] is used, where mass and width are defined somewhat unconventionally to take into account its mixing with $\omega(782)$ and $\rho(1450)$. The $\Upsilon(11020)$ mass and width were fit parameters to a Gaussian with a relativistic tail [10]. For both resonances cited values in the PDG tables are neither masses consistent with the original Breit-Wigner idea, being at the peak of the resonance, nor the pole positions. Since the resonance parameters are collected in PDG tables to be used as an input for various models and for comparison between theory and experiment, placing all these resonance parameters in a single table may generally create considerable confusion. This confusion is evident in the ρ meson case where in the table with predominantly Gounaris-Sakurai masses (about 775 MeV) one can find pole masses (roughly 760 MeV).

In conclusion, we have shown here that the original Breit-Wigner formula may be drastically improved by including a single additional (phase) parameter δ_p. Our results suggest that parameter δ_p seems to be equal to the half of the resonance residue phase θ_p, regardless of the resonance inelasticity. This new formula has two equivalent forms that can be used to estimate either the pole or the Breit-Wigner parameters in a model-independent way. Having both forms enabled us to learn that in the PDG tables [5] there are values that do not correspond either to the pole or to Breit-Wigner parameters. Such an outcome undermines the proper matching between microscopic theories (e.g., lattice QCD [4]) and experiment.

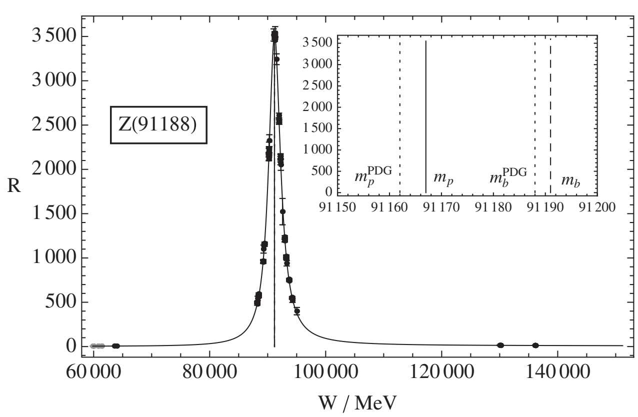

FIG. 5. Fitting pole parametrization, Eq. (11), to the data [5]. The pole and BW parameters are consistent with their PDG estimates (dotted lines).

TABLE II. Resonance parameters extracted by using new Breit-Wigner formula, Eq. (14). PDG estimates are from Ref. [5].

Resonance	M_b/MeV	Γ_b/MeV	x_b/%	$2\delta_b/^\circ$
$\rho(770)$	761 ± 1	139 ± 2	0.71	0 ± 1
PDG	775.5 ± 0.3	146.2 ± 0.7	0.69	···
$\Delta(1232)$	1233 ± 1	120 ± 1	102 ± 1	-46 ± 1
PDG-BW	1232 ± 2	117 ± 3	100	···
$N(1440)$	1443 ± 2	325 ± 11	61 ± 4	-80 ± 2
PDG-BW	1440^{+30}_{-20}	300^{+150}_{-100}	65 ± 10	···
$\Upsilon(11020)$	$11\,010 \pm 2$	53 ± 8	0.10	-52 ± 8
PDG	$11\,019 \pm 8$	79 ± 16	···	···
$Z(91188)$	$91\,191 \pm 5$	2494 ± 5	15.4	-2.2 ± 0.2
PDG-BW	$91\,188 \pm 2$	2495 ± 2	15.3	···

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