Physics of the Gluon-Helicity Contribution to Proton Spin

Xiangdong Ji,[1,2,3] Jian-Hui Zhang,[1] and Yong Zhao[3]
[1]*INPAC, Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China*
[2]*Center for High-Energy Physics, Peking University, Beijing 100080, People's Republic of China*
[3]*Maryland Center for Fundamental Physics, University of Maryland, College Park, Maryland 20742, USA*
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The total gluon helicity in a polarized proton, measurable in high-energy scattering, is shown to be the large momentum limit of a gauge-invariant but nonlocal, frame-dependent gluon spin $\vec{E}\times\vec{A}_\perp$ in QCD. This opens a door for a nonperturbative calculation of this quantity in lattice QCD and also justifies using free-field expressions in the light-cone gauge as physical observables.

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The total gluon helicity $\Delta G = \int dx \Delta g(x)$ in a longitudinally polarized proton is an important physical quantity that has motivated much experimental effort to measure in high-energy scattering [1–4]. It helps to understand, among other things, how the helicity of a fast-traveling proton is composed of partons' helicity and orbital angular momentum [5–9]. The factorization theorems in quantum chromodynamics (QCD) indicate that ΔG is a matrix element of a complicated light-cone correlation operator of the gluon fields [10] and has a simple physical interpretation only in the light-cone gauge $A^+ = 0$ natural for parton physics [5]. This state of understanding makes computing ΔG in lattice QCD infeasible and raises the fundamental question about the gauge invariance of the gauge particle spin in a bound state [11].

In this Letter, we report a breakthrough in understanding the physics of ΔG and correspondingly leading to a practical way to its calculation in lattice QCD. We find that ΔG can be obtained by boosting a matrix element of the gluon spin operator $\vec{E}\times\vec{A}_\perp$ to the infinite momentum frame (IMF), where $\vec{A}_\perp$ is the transverse part of the gauge potential. This operator was first proposed in Ref. [12] as the gauge-invariant gluon spin, but has been criticized as physically uninteresting because of its frame dependence [13]. The physics behind the IMF limit we propose here goes back to the well-known Weizsäcker-Williams equivalent photon picture for high-energy scattering [11]. But of course, there are subtleties in taking the IMF limit. In particular, the matrix element has a singular dependence on the bound state momentum in perturbation theory as it approaches infinity. Moreover, the anomalous dimension of $\vec{E}\times\vec{A}_\perp$ does not coincide with that of the nonlocal light-cone correlation in the factorization theorems [14]. Therefore, we will provide a well-defined procedure, or a matching condition, for the limiting procedure. In particular, we will show in a one-loop example how to obtain ΔG from the IMF limit of a frame-dependent, time-independent matrix element. This example helps to demonstrate that a nonperturbative ΔG can be recovered from a frame-dependent lattice matrix element of a Euclidean space operator $\vec{E}\times\vec{A}_\perp$.

The difficulty in understanding and calculating ΔG is easy to appreciate. Through QCD factorization, it has been shown that ΔG is a matrix element of a nonlocal operator involving light-cone correlation [10],

$$\Delta G = \int dx \frac{i}{2xP^+} \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \langle PS| F_a^{+\alpha}(\xi^-) \times \mathcal{L}^{ab}(\xi^-, 0)\tilde{F}^+_{\alpha,b}(0)|PS\rangle, \quad (1)$$

where $|PS\rangle$ is a proton plane-wave state with momentum P^μ and polarization S^μ, $\tilde{F}^{\alpha\beta} = (1/2)\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$, and $\mathcal{L}(\xi^-,0) = P\exp[-ig\int_0^{\xi^-}\mathcal{A}^+(\eta^-,0_\perp)d\eta^-]$ with $\mathcal{A}^+ \equiv T^c A_c^+$ is a light-cone gauge link defined in the adjoint representation. The light-front coordinates are defined as $\xi^\pm = (\xi^0 \pm \xi^3)/\sqrt{2}$. It is usually difficult to see the above operator as the gluon spin or helicity. However, in the light-cone gauge $A^+ = 0$, the whole operator collapses into $\vec{E}\times\vec{A}$, the textbook definition of the gauge particle spin [11], which is known to be gauge dependent, and ΔG can be regarded as the number of gluon partons with helicity in the direction of the proton helicity minus that with opposite helicity. Because of the explicit presence of the real time in ξ^-, one cannot evaluate the above expression in lattice QCD. An early attempt to get the gluon helicity on lattice was to calculate the matrix element of $F_{\mu\nu}\tilde{F}^{\mu\nu}$ [15], but there is no demonstrated connection between this and ΔG.

To find the physics of ΔG without committing to the light-cone gauge, let us examine the operator structure a bit further. For simplicity, we first consider the U(1) gauge theory (quantum electrodynamics, or QED) so that the gauge link is absent. Carrying out the integration over the longitudinal momentum, the gauge-invariant photon "spin" operator becomes

$$\hat{S}_\gamma^{\rm inv}(0) = i\int\frac{dx}{x}\int\frac{d^2k_\perp}{(2\pi)^3}\int d\xi^- d^2\xi_\perp e^{-i(xP^+\xi^- - \vec{k}_\perp\cdot\vec{\xi}_\perp)}[ixP^+A^i(\xi^-,\vec{\xi}_\perp) - ik_\perp^i A^+(\xi^-,\vec{\xi}_\perp)]\tilde{F}_i^+(0)$$
$$= -\int\frac{dk^+d^2k_\perp}{(2\pi)^3}\left[\tilde{A}^i(k^+,\vec{k}_\perp) - \frac{k_\perp^i}{k^+}\tilde{A}^+(k^+,\vec{k}_\perp)\right]\tilde{F}_i^+(0) = \left[\vec{E}(0)\times\left(\vec{A}(0) - \frac{1}{\nabla^+}\vec{\nabla}A^+(\xi^-)\right)\right]^3, \quad (2)$$

where $\nabla^+ = \partial/\partial\xi^-$, $E^i = F^{i+}$, $k^+ = xP^+$, and the third component of a vector is interpreted in the usual sense of the cross product. The ξ^- coordinate in A^+ is taken to 0 after operating with the inverse derivative which is independent of the boundary condition because $\vec{\nabla}A^+(\infty) = 0$.

The above operator is just the IMF limit of $\vec{E}\times\vec{A}_\perp$, where $E^i = F^{i0}$ is the electric field in ordinary frame, and $A_\perp^i = (\delta^{ij} - \nabla^i\nabla^j/\nabla^2)A^j$ is the transverse part of the gauge potential and is invariant under gauge transformation. The rule of taking the IMF limit of an operator is as follows: For any vector V^μ, define $V^\pm = (V^0 \pm V^3)/\sqrt{2}$. If the boost is along the 3-direction, then the components of the vector go like $V^+ \to V^+\Lambda$, $V^- \to V^-/\Lambda$, and $V_\perp \to V_\perp$. Thus, $\nabla^2 \to (\nabla^+)^2\Lambda^2$ and $\vec{\nabla}\cdot\vec{A} \to \nabla^+A^+\Lambda^2$ for the leading components. Using these rules, one finds that $\vec{E}\times\vec{A}_\perp \to \vec{E}\times(\vec{A} - (1/\nabla^+)\vec{\nabla}A^+)$, which is exactly the above operator in Eq. (2).

It is known that in electromagnetic theory the vector potential can be uniquely separated into the longitudinal and transverse parts, $\vec{A} = \vec{A}_\parallel + \vec{A}_\perp$, and the transverse part is gauge invariant [16]. Given $\vec{A}$, $\vec{A}_\perp$ can be uniquely constructed as a functional of $\vec{A}$ with an appropriate boundary condition. Thus, $\vec{E}\times\vec{A}_\perp$ is a gauge-invariant operator and can be regarded as the gauge-invariant part of the gauge particle spin [12]. $\vec{E}\times\vec{A}_\perp$ is a nonlocal operator in that it depends on the gauge potential over all space.

It is important to realize that separating $\vec{A}$ and $\vec{E}$ into longitudinal and transverse parts is in general not a physically meaningful thing to do. In the first place, the physics of $\vec{E}$ is to apply force to electric charge, and there is no charge that responds separately to $\vec{E}_\parallel$ and $\vec{E}_\perp$. Second, in a different frame, one sees different transverse and longitudinal separations, and therefore the notion has no Lorentz covariance [13]. As we shall see, the frame dependence of both parts is dynamical and cannot be calculated without solving the theory. However, there are two exceptions where the separation is meaningful. The first case concerns the radiation field [17]. For free radiation, by separating out the unphysical degrees of freedom, one simplifies the quantization procedure significantly. In the laser beam, this separation allows one to talk about the gauge-invariant photon spin and orbital angular momentum [18]. The second case is the IMF, which is our interest here. In the IMF, $E_\parallel \ll E_\perp$, and the electromagnetic field can be regarded as free radiation. This was recognized long ago by Weizsäcker and Williams, in the name of equivalent photon approximation [11]. Only then, $\vec{E}\times\vec{A}_\perp$ can be understood as a physical quantity, where $\vec{E}$ can also be replaced by $\vec{E}_\perp$.

Therefore, the photon helicity measurable in high-energy scattering can be calculated as the IMF limit of a matrix element of the static operator $\vec{E}\times\vec{A}_\perp$. To calculate the matrix element of the time-independent, albeit nonlocal, operator is a standard practice in lattice QCD. It will be dynamically dependent on the momentum of the external particle [13]. In fact, the dependence is singular in the leading order perturbation theory. In the IMF limit, the matrix element diverges. To obtain the physically interesting finite light-cone matrix element, one has to find a matching condition, which we will come to after establishing a similar connection in QCD.

The case for QCD is a bit more complicated. Separating $\vec{A}$ into longitudinal and transverse parts requires generalizing the observations in QED to similar conditions in QCD, which was considered long ago [19] (see also Ref. [12]). Clearly, we would like to have $\vec{A}_\perp$ transform covariantly under gauge transformation,

$$\vec{A}_\perp \to U(x)\vec{A}_\perp U^\dagger(x), \quad (3)$$

where $\vec{A}_\perp \equiv T_a\vec{A}_\perp^a$, so it is easy to construct gauge-invariant quantities with $\vec{A}_\perp$. Second, we require $\vec{A}_\parallel$ to produce null magnetic field, as it does in QED. This condition is [19]

$$\partial^i A_\parallel^{j,a} - \partial^j A_\parallel^{i,a} - gf^{abc}A_\parallel^{i,b}A_\parallel^{j,c} = 0, \quad (4)$$

which is a nonlinear equation to solve for $A_\parallel^i$ as a functional of A^i. Moreover, the transverse part of the gauge potential satisfies a generalized Coulomb condition [19]

$$\partial^i A_\perp^i = ig[A^i, A_\perp^i]. \quad (5)$$

We can then go through a similar derivation as in the QED case and find [20]

$$\hat{S}_g^{\rm inv}(0) = \left[\vec{E}^a(0)\times\left(\vec{A}^a(0) - \frac{1}{\nabla^+}(\vec{\nabla}A^{+,b})\mathcal{L}^{ba}(\xi^-,0)\right)\right]^3, \quad (6)$$

where the inverse derivative acts on everything after it and takes the ξ^- coordinate in the gauge link to 0. To get the linear term in A^μ, the quadratic term in the gauge field cancels with the derivative acting on the gauge link.

It is a bit involved to show that the expression in the parentheses above is indeed $\vec{A}_\perp$ in the IMF. Since $\vec{A}_\perp = \vec{A} - \vec{A}_\parallel$, we just need to solve for $\vec{A}_\parallel$. After solving

Eqs. (4) and (5) order by order in g, we find that $A_\perp^+$ vanishes, and thus $A_\parallel^+ = A^+$ in the IMF. Substituting this into Eq. (4), we obtain a first-order inhomogeneous linear equation for $A_\parallel^i$,

$$\partial^+ A_\parallel^{i,a} - g f^{abc} A^{+,b} A_\parallel^{i,c} = \partial^i A^{+,a}. \tag{7}$$

Its solution is easy to construct as a geometric series expansion

$$A_\parallel^{i,a} = \frac{1}{\nabla^+}\left[1 + \left(-ig\mathcal{A}^+\frac{1}{\nabla^+}\right) + \cdots + \left(-ig\mathcal{A}^+\frac{1}{\nabla^+}\right)^n + \cdots\right]^{ab}(\partial^i A^{+,b}). \tag{8}$$

By commuting $\partial^i A^+$ systematically to the front of the expression, one finds

$$A_\parallel^{i,a}(\xi^-) = \frac{1}{\nabla^+}[(\partial^i A^{+,b})\mathcal{L}^{ba}(\xi'^-, \xi^-)], \tag{9}$$

where the coordinate ξ'^- in the gauge link is taken to ξ^- after operating with the inverse derivative.

Alternatively, one can multiply a gauge link $\mathcal{L}$ on both sides of Eq. (7) and find after some manipulations

$$\partial^+(A_\parallel^{i,a}\mathcal{L}^{ad}) = (\partial^i A^{+,a})\mathcal{L}^{ad}. \tag{10}$$

It is then straightforward to see that the solution for $A_\parallel^{i,a}$ is formally given by Eq. (9). Clearly, $\vec{A}_\parallel^a(0)$ is just the part subtracted from $\vec{A}^a(0)$ in the gluon-helicity operator in Eq. (6). Therefore, we established the same conclusion in QCD that the gluon-helicity operator is the IMF limit of the gauge-invariant, nonlocal gluon spin operator $(\vec{E}\times\vec{A}_\perp)^3$.

Now we show that the matrix element of $\hat{S}_\gamma^{\rm inv} = (\vec{E}\times\vec{A}_\perp)^3$ depends on the choice of frames dynamically. By "dynamically" we mean that the frame dependence cannot be obtained from Lorentz transformation and is a function of dynamic details. Let us consider the example of photon spin in a free electron state $|p, s\rangle$. A simple perturbative calculation of Fig. 1 yields

$$\langle p, s|\hat{S}_\gamma^{\rm inv}|p, s\rangle = \frac{\alpha_{\rm em}}{4\pi}\left[\frac{5}{3}D + \frac{31}{9} + 2\int_0^1 dx\sqrt{1-x}\ln\left(1 + x\frac{\vec{p}^2}{m^2}\right)\right]\bar{u}(p, s)\Sigma^3 u(p, s), \tag{11}$$

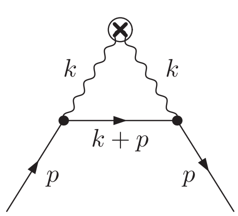

FIG. 1. Matrix element of $\vec{E}\times\vec{A}_\perp$ in an asymptotic electron state.

where $p^\mu = (p^0, 0, 0, p^3)$, $D = 1/\epsilon - \gamma_E + \ln 4\pi + \ln(\mu^2/m^2)$, and m is the mass of the electron used to regularize the collinear divergence. The result has a nontrivial dependence on the electron momentum $\vec{p}^2$, which makes its physical interpretation less straightforward. In particular, the result is divergent in the IMF limit. The ultraviolet (UV) part of the matrix element is consistent with that found in Ref. [14]. Our result also shows that the statement in Ref. [21] about the frame independence is incorrect.

However, the measurable photon helicity can be obtained by the same matrix element by going to the IMF limit first before the loop-momentum integration. In this limit, the external momentum dependence is dropped out, and the matrix element becomes

$$\langle p, s|\hat{S}_\gamma^{\rm inv}|p, s\rangle = \frac{\alpha_{\rm em}}{4\pi}(3D + 7)\bar{u}(p, s)\Sigma^3 u(p, s). \tag{12}$$

This is exactly the same as that computed using the factorization expression in Eq. (1) or using $\vec{E}\times\vec{A}$ (notice the full $\vec{A}$), in the light-cone gauge $A^+ = 0$ [22]. The UV property of the matrix element is the same as that derived by Altarelli and Parisi (AP) [23]. In fact, in the original derivation of the AP evolution equation, a finite frame result was used to obtain parton physics in the IMF limit. Our result also indicates the claim that gluons carry only about $1/5$ of the nucleon momentum in Ref. [12] is incorrect, because the matrix elements of the quark and gluon momentum operators were calculated in the finite momentum frame. We have verified that if they are boosted to the IMF, we can get the standard mixing matrix in Ref. [24]. Meanwhile, it should be pointed out that the result in Ref. [25] is standard because they used the light-cone gauge condition.

The above calculation shows that the IMF and UV limits are not exchangeable. However, since the collinear-divergent part is the same, the difference is a perturbatively calculable quantity. This turns out to be the key for a nonperturbative computation of the gluon helicity. Since lattice QCD cannot handle the real time dependence, a direct calculation of Eq. (1) is infeasible. However, one can get the same matrix element by studying the matrix element of $\vec{S}_g^{\rm inv} = \vec{E}\times\vec{A}_\perp$ as a function of external momentum $\vec{P}$. The largest momentum attainable on a lattice is of order $1/a$, with a being the lattice spacing. On the other hand, the matrix element also has UV dependence on $1/a$, and can be calculated in perturbation theory. Thus, we can calculate the matrix element at the largest momentum $\sim 1/a$, matching the results of the two different limits in a perturbative way. For instance, in the above one-loop example with a finite ϵ, one can match the two results by setting

$$\ln\vec{p}^2 = (D + \ln m^2) + \frac{16}{3} - 2\ln 2. \tag{13}$$

The result will be accurate up to controllable power corrections of type M^2/P^2, where M is the nucleon mass. On the lattice, $D + \ln m^2$ is replaced by $-\ln a^2$, and the matching condition becomes $\ln(pa)^2 = \text{const}$. We will explore the issue of the lattice calculation and matching conditions in more detail in a separate publication [26].

At last, it is useful to consider the frame dependence of the angular momentum sum rule, which has recently been strongly advocated in the literature [12,21,27]. The primary goal is to find a simple free-field form of the angular momentum decomposition so that the individual parts have simple physical interpretation. The closest gauge-invariant form involves the expression [12]

$$\vec{J} = \int d^3x \psi^\dagger \frac{1}{2}\vec{\Sigma}\psi + \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i}(\vec{\nabla} - ig\vec{A}_\parallel)\psi + \int d^3x \vec{E}_a \times \vec{A}^a_\perp + \int d^3x E^i_a \vec{x} \times \vec{\nabla} A^{i,a}_\perp. \quad (14)$$

This result is frame dependent and not physically interesting in general. However, in the IMF one has $A^{i,a}_\parallel(\xi^-) = (1/\nabla^+)[\nabla^i A^{+,b}\mathcal{L}^{ba}(\xi'^-, \xi^-)]$; the above decomposition is the same as the Jaffe-Manohar result in the light-cone gauge $A^+ = 0$ [5]. Therefore, this serves to justify that the light-cone gauge is the natural choice in the IMF, where free-field expressions such as $\vec{E} \times \vec{A}$ attain physical significance. In particular, the corresponding matrix elements are physically measurable.

To conclude, we have shown that the total gluon helicity measured in high-energy scattering is the IMF limit of a matrix element of a gauge-invariant operator. This limit does not commute with the UV limit in quantum field theory, and therefore the two operators have different anomalous dimensions at first sight. However, they can be related through a matching condition. This allows an otherwise infeasible lattice QCD calculation of light-cone correlations. We have also explained why free-field theory expressions in the light-cone gauge are physically meaningful, as they correspond to the IMF limit of gauge-invariant but nonlocal expressions in interacting theories.

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