P H Y S I C A L   R E V I E W   L E T T E R S

Zwei-Dreibein Gravity: A Two-Frame-Field Model of 3D Massive Gravity

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We present a generally covariant and parity-invariant two-frame field ("zwei-dreibein") action for gravity in three space-time dimensions that propagates two massive spin-2 modes, unitarily, and we use Hamiltonian methods to confirm the absence of unphysical degrees of freedom. We show how zwei-dreibein gravity unifies previous "3D massive gravity" models and extends them, in the context of the AdS/CFT correspondence, to allow for a positive central charge consistent with bulk unitarity.

Einstein's theory of general relativity (GR) can be viewed as a field theory describing the interactions of a massless spin-2 particle, the graviton. This point of view suggests that GR might be generalized to allow for a small graviton mass. The natural starting point for such investigations is the massive spin-2 field theory constructed long ago by Fierz and Pauli, but it has proved difficult to find a consistent interacting version of this theory. One problem is the generic appearance of an unphysical scalar mode of negative energy: the Boulware-Deser ghost [1]. In the context of a three-dimensional (3D) spin-2 theory, many of the problems were resolved a few years ago by the construction of "new massive gravity" (NMG). Although this is a general covariant (diffeomorphism-invariant) theory, of fourth order in derivatives in its initial formulation, linearization yields a free field theory that is equivalent to the 3D version of the Fierz-Pauli (FP) massive spin-2 theory [2]. Moreover, the nonlinearities are precisely those for which the Boulware-Deser ghost is avoided [3].

In an initially parallel development, a ghost-free non-linear extension of the four-dimensional (4D) FP spin-2 field theory was constructed by de Rham, Gabadadze, and Tolley [4]. One unattractive feature of this "dRGT" model is the fact that it involves a fixed background metric, in addition to the dynamical metric, and so lacks the general covariance of Einstein's GR. General covariance can be restored, albeit at the cost of re-introducing a massless graviton, by considering an alternative "bimetric" gravity model [5] from which the dRGT model can be recovered as a truncation in which one metric is taken to be nondynamical. The structure of this bimetric model becomes quite simple when formulated in terms of two vierbeins rather than two metrics [6], i.e., when formulated as a "zwei-vierbein" model (see Ref. [7] for earlier related models). In this formulation, the absence of ghosts can be seen easily, although it has been shown that shock waves in these 4D massive gravity models propagate acausally [8–10].

Whereas the motivation for 4D massive gravity comes mainly from potential applications to cosmology, the motivation for 3D massive gravity stems from simplifying features of the quantum theory arising from the lower dimension. In particular, 3D massive gravity models provide a new arena for the AdS/CFT correspondence, in which a 3D quantum gravity theory in an asymptotically anti–de Sitter (AdS) space-time is conjectured to be equivalent to a 2D conformal field theory (CFT) on the AdS boundary. In the semiclassical approximation to the quantum gravity theory, the dual CFT has a large central charge that can be computed from the asymptotic symmetry algebra [11]. The AdS/CFT correspondence does not obviously apply to the 3D dRGT model because of its lack of general covariance. It does apply to NMG, but perturbative unitarity (in the bulk) holds if and only if the central charge of the boundary CFT is negative, which implies that the CFT is nonunitary.

This clash between bulk and boundary unitarity is a feature of all currently known generally covariant 3D models of massive gravity. In particular, the bimetric model of Ref. [12], although permitting a positive central charge, is not unitary in the bulk due to the Boulware-Deser ghost. In this Letter, we present a parity-preserving "zwei-dreibein" model of massive gravity that overcomes this problem. The general model of zwei-dreibein gravity (ZDG) has five continuous parameters and a choice of sign, but we find a parameter range for which the bulk theory is unitary and has a boundary CFT with positive central charge. For other choices of the parameters, we exhibit limits in which the dRGT model and NMG are recovered [5,13], the latter in its "Chern-Simons-like" form [14]. ZDG thus unifies these two rather different approaches to massive gravity in three dimensions, in addition to extending them in a way that resolves a major difficulty.

The ZDG fields are a pair $\{e_I^a; I = 1, 2; a = 0, 1, 2\}$ of Lorentz-vector valued one-forms and a pair ω_I^a of

Lorentz-vector valued connection one-forms, from which we may construct pairs of torsion and curvature two-forms:

$$T_I^a = de_I^a + \varepsilon^{abc}\omega_{Ib}e_{Ic}, \qquad R_I^a = d\omega_I^a + \frac{1}{2}\varepsilon^{abc}\omega_{Ib}\omega_{Ic}. \tag{1}$$

We use a notation in which the exterior product of forms is implicit. It will be convenient to introduce a sign $\sigma = \pm 1$ and two independent positive mass parameters M_I, and to define [13]

$$M_{12} = (\sigma M_1 M_2)/(\sigma M_1 + M_2). \tag{2}$$

This is positive for $\sigma = 1$ and finite for finite M_I. It may have either sign when $\sigma = -1$, and in this case, we assume that $M_1 \neq M_2$. The Lagrangian three-form for ZDG can now be written as

$$\mathcal{L}_{\text{ZDG}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12}, \tag{3}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are Einstein-Cartan (EC) Lagrangian three-forms; i.e.,

$$\mathcal{L}_1 = -\sigma M_1 e_{1a}R_1^a - \frac{1}{6}m^2 M_1 \alpha_1 \varepsilon_{abc}e_1^a e_1^b e_1^c,$$
$$\mathcal{L}_2 = -M_2 e_{2a}R_2^a - \frac{1}{6}m^2 M_2 \alpha_2 \varepsilon_{abc}e_2^a e_2^b e_2^c, \tag{4}$$

where m is a further mass parameter and α_I are two dimensionless "cosmological" parameters. The two EC terms are coupled by the third term

$$\mathcal{L}_{12} = \frac{1}{2}m^2 M_{12}\varepsilon_{abc}(\beta_1 e_1^a e_1^b e_2^c + \beta_2 e_1^a e_2^b e_2^c), \tag{5}$$

where β_I are two dimensionless parameters.

The general ZDG model, as constructed above, depends on five independent continuous parameters (α_I, β_I) and the ratio M_1/M_2. The mass parameter m is convenient but inessential because we could consider $m^2(\alpha_I, \beta_I)$ as independent mass-squared parameters from which four dimensionless parameters can be found by taking ratios with, say, $M_1 M_2$. In the absence of the $\mathcal{L}_{12}$ term, the action is the sum of two EC actions and so has diffeomorphism and local Lorentz gauge invariance separately for the two sets of EC form fields; this is broken by $\mathcal{L}_{12}$ to the diagonal EC gauge invariance found by identifying the two sets of EC gauge parameters.

We now expand the two dreibeine about a common fixed dreibein $\bar{e}^a$ of a maximally symmetric background with cosmological constant Λ, and similarly for their respective spin connections:

$$e_1^a = \bar{e}^a + \kappa h_1^a, \qquad \omega_1^a = \bar{\omega}^a + \kappa v_1^a,$$
$$e_2^a = \gamma(\bar{e}^a + \kappa h_2^a), \qquad \omega_2^a = \bar{\omega}^a + \kappa v_2^a, \tag{6}$$

where γ is a constant and κ is a small expansion parameter. The ZDG action may now be expanded in powers of κ. Cancelation of the linear terms both fixes Λ

$$\Lambda/m^2 = -\gamma^2 \alpha_2 + (M_{12}/M_2)(\beta_1 + 2\gamma \beta_2) \tag{7}$$

and imposes the following quadratic constraint on γ:

$$[\alpha_2(\sigma M_1 + M_2) + \beta_2 M_2]\gamma^2 + 2(M_2\beta_1 - \sigma M_1\beta_2)\gamma$$
$$- \sigma[\alpha_1(\sigma M_1 + M_2) + \beta_1 M_1] = 0. \tag{8}$$

The quadratic terms in the expansion of the action may now be diagonalized, provided that

$$M_{\text{crit}} = \sigma M_1 + \gamma M_2 \neq 0, \tag{9}$$

by introducing the new one-form fields

$$h_+^a = (\sigma M_1 h_1^a + \gamma M_2 h_2^a)/M_{\text{crit}},$$
$$v_+^a = (\sigma M_1 v_1^a + \gamma M_2 v_2^a)/M_{\text{crit}}, \tag{10}$$

and

$$h_-^a = h_1^a - h_2^a, \qquad v_-^a = v_1^a - v_2^a. \tag{11}$$

In terms of these new fields, the quadratic Lagrangian three-form takes the form $\mathcal{L}^{(2)} = \mathcal{L}_+^{(2)} + \mathcal{L}_-^{(2)}$, with

$$\mathcal{L}_+^{(2)} = -M_{\text{crit}}\left[h_{+a}\bar{D}v_+^a + \frac{1}{2}\varepsilon_{abc}\bar{e}^a v_+^b v_+^c\right.$$
$$\left. - \frac{1}{2}\Lambda\varepsilon_{abc}\bar{e}^a h_+^b h_+^c\right], \tag{12}$$

where $\bar{D}$ is the covariant exterior derivative with respect to the background and

$$\mathcal{L}_-^{(2)} = -\frac{\sigma\gamma M_1 M_2}{M_{\text{crit}}}\left[h_{-a}\bar{D}v_-^a + \frac{1}{2}\varepsilon_{abc}\bar{e}^a v_-^b v_-^c\right.$$
$$\left. + \frac{1}{2}(\mathcal{M}^2 - \Lambda)\varepsilon_{abc}\bar{e}^a h_-^b h_-^c\right], \tag{13}$$

where

$$\mathcal{M}^2 = m^2(\beta_1 + \gamma\beta_2)\frac{M_{\text{crit}}}{\sigma M_1 + M_2}. \tag{14}$$

The form fields $v_\pm^a$ appearing in the above quadratic Lagrangian three-forms are auxiliary and may be eliminated by their field equations. Eliminating v_+^a, we find that $\mathcal{L}_+^{(2)}$ becomes the quadratic approximation to the Einstein-Hilbert Lagrangian density in the AdS background; this does not propagate any modes. Eliminating v_-^a, we find that $\mathcal{L}_-^{(2)}$ is proportional to the Fierz-Pauli Lagrangian density, in the AdS background, for a spin-2 field with FP mass $\mathcal{M}$. Notice that Λ contributes to the mass term, which is zero when $\mathcal{M}^2 = \Lambda$; this is the "partially massless" case where the linearized theory acquires an additional gauge invariance. This case is not relevant when $\Lambda < 0$ because there will be a spin-2 tachyon unless $\mathcal{M}^2 > 0$. The parameters of the model are further restricted by the requirement of positive kinetic energy,

which amounts to bulk unitarity in the quantum theory; recalling that M_1, $M_2 > 0$, we must have

$$\frac{\sigma\gamma}{M_{\text{crit}}} > 0. \tag{15}$$

Although the quadratic Lagrangian $\mathcal{L}^{(2)}$ is not diagonalizable for $M_{\text{crit}} = 0$, we can still take the limit $M_{\text{crit}} \to 0$ in the field equations. The massive modes become formally massless in this limit. We shall not discuss this "critical" case here.

The fact that ZDG propagates just two physical modes (which happen to be spin-2 modes) implies that the dimension, per space point, of the physical phase space of the linearized theory is 4. This remains true in perturbation theory but does not exclude the appearance of additional degrees of freedom in other backgrounds. However, it is possible to determine the nonperturbative dimension of the physical phase space by Hamiltonian methods. As the action is Chern-Simons-like, being constructed as the integral of products of forms without an explicit metric, it is already first order, and a space-time split, e.g., $e^a_\mu = (e^a_0, e^a_i)$ ($i = 1, 2$), suffices to put it into a form that is "almost" Hamiltonian, with the Hamiltonian being a sum of Lagrange multipliers times constraint functions. However, the field equations will generically imply additional secondary constraints and these should be included, too (there are no tertiary constraints if one starts from a first-order Chern-Simons-like action [15]). In the case of ZDG, there are two secondary constraints:

$$
\begin{aligned}
0 &= \varepsilon^{ij} e_{1i} \cdot e_{2j}, \\
0 &= \varepsilon^{ij}[\beta_1(\omega_1 - \omega_2)_i \cdot e_{1j} + \beta_2(\omega_1 - \omega_2)_i \cdot e_{2j}],
\end{aligned} \tag{16}
$$

where the dot product notation implies contraction of the three-vectors with the Lorentz metric. In the NMG limit (to be discussed below), these reduce to the two secondary constraints found for that model in Ref. [14]. Each three-vector form field in the action adds $2 \times 3 = 6$ (per space point) to the phase-space dimension (from its space components) and contributes three primary constraints (its time components are the Lagrange multipliers). As we have four such fields, the initial phase space has dimension 24 (per space point), and there are 12 primary constraints, to which we must add the two secondary constraints, making a total of 14 constraints; of these, six are first class (corresponding to the six EC gauge invariances) and eight are second class. The constraints therefore reduce the phase-space dimension by $2 \times 6 + 8 = 20$, leaving a physical phase space of dimension $24 - 20 = 4$ (per space point), in agreement with the linearized analysis. The counting here is exactly the same as that given for NMG in Ref. [14], but the detailed verification of the fact that there are six first-class and eight second-class constraints (which we omit) is different.

Any 3D gravity model admitting an AdS vacuum will also admit the asymptotically AdS black hole metric found by Bañados, Teitelboim, and Zanelli (BTZ) as solutions of 3D GR [16]. Generically, the mass (and entropy) of BTZ black holes is positive whenever the central charge of the dual CFT is positive. Therefore, in the NMG model, these BTZ black holes have negative mass whenever the bulk spin-2 modes have positive energy. As we shall see, ZDG overcomes this problem.

The central charge of the boundary CFT for GR follows directly from the results of Brown and Henneaux on asymptotic symmetries in AdS$_3$ [11]; their result is $c = 24\pi\ell M_P$, where ℓ is the AdS$_3$ radius (so $\Lambda = -1/\ell^2$) and $M_P = 1/(16\pi G)$, where G is the 3D Newton constant, which has dimensions of inverse mass in units for which the speed of light is unity. A similar computation for ZDG, which we have verified using Hamiltonian methods, shows that

$$c = 12\pi\ell M_{\text{crit}} \quad \text{(2DG)}. \tag{17}$$

We note that for $M_2 = 0$ and $\sigma = 1$, this reduces to the Brown-Henneaux result, using that in our normalization of Eq. (4) the Planck mass is $M_P = M_1/2$. It was to be expected that the central charge would be proportional to M_{crit} because it should vanish for the critical gravity case.

We are now in a position to determine whether there is a parameter range for ZDG for which perturbative unitarity in the bulk is compatible with positive central charge of the boundary CFT. When $\sigma = -1$, the bulk unitarity condition $-\gamma/M_{\text{crit}} > 0$ is incompatible with the condition $M_{\text{crit}} > 0$ unless $\gamma < 0$, but then $M_{\text{crit}} < 0$ from its definition, so we require both $\sigma = 1$ and an AdS vacuum with $\gamma > 0$ for the compatibility of $c > 0$ with bulk unitarity. The absence of tachyons $\mathcal{M}^2 > 0$ then requires that

$$\beta_1 + \gamma\beta_2 > 0. \tag{18}$$

Of course, this result applies only when there is an AdS vacuum, so we also need to check that Eq. (8) allows real positive solutions for γ such that $\Lambda < 0$.

A simple explicit case satisfying all the above conditions can found by setting

$$M_1 = M_2, \qquad \beta_I = 1, \qquad \alpha_I = \frac{3}{2} + \zeta, \tag{19}$$

where ζ is a positive constant. For this choice, the quadratic equation (8) reduces to $\gamma^2 = 1$, and choosing $\gamma = 1$, we get an AdS vacuum with $(\ell m)^{-2} = \zeta$. Furthermore, $\gamma \approx 1$ for any "nearby" ZDG model, with slightly different parameters, which are themselves constrained only by inequalities that have been satisfied but not saturated. It follows that the above explicit model is one of an open set of models in the ZDG parameter space with similar "good" properties; these properties are *not* the result of any fine-tuning of parameters that could be destabilized by perturbative quantum corrections. There could also be higher-derivative quantum corrections, of course, but these

can be dealt with, in perturbation theory, in the same way as in GR.

We conclude with a discussion of how the general ZDG model unifies previous models of massive 3D gravity. To see the relation to the 3D dRGT model, we set $\sigma = 1$ and write

$$e_2^a = \bar{e}_2^a + \lambda h_2^a, \qquad \omega_2^a = \bar{\omega}_2^a + \lambda v_2^a, \qquad (20)$$

where $\bar{e}_2^a$ is a "reference dreibein" and $\bar{\omega}_2^a$ the corresponding (zero torsion) reference spin connection, and λ is a constant. If we now take $\lambda \to 0$, keeping fixed $\lambda^2 M_2 = M_1 \equiv 2M_P$, then the ZDG Lagrangian three-form reduces to

$$\mathcal{L} = 2M_P \mathcal{L}_0(h_2, v_2) + 2M_P \mathcal{L}_{\mathrm{dRGT}}(e_1, \omega_1), \quad (21)$$

where the first term is the quadratic approximation to the 3D EC action (in the reference background) for the fluctuations (h_2^a, v_2^a); because of its linearized EC gauge invariances, this term propagates no modes. In the second term, we have, after renaming (e_1^a, ω_1^a) as (e^a, ω^a),

$$\mathcal{L}_{\mathrm{dRGT}} = -e_a R^a - \alpha_1 \frac{m^2}{6} \varepsilon^{abc} e_a e_b e_c$$
$$+ \frac{m^2}{2} \varepsilon^{abc}(\beta_1 e_a e_b \bar{e}_c + \beta_2 e_a \bar{e}_b \bar{e}_c), \quad (22)$$

which is the dreibein form of the 3D dRGT model.

To see the relation of ZDG to NMG, we set $\sigma = -1$ and write

$$e_2^a = e_1^a + \frac{\lambda}{m^2} f^a, \qquad \omega_2^a = \omega_1^a - \lambda h^a. \quad (23)$$

We now consider the "flow"

$$M_1(\lambda) = 2\left(1 + \frac{1}{\lambda}\right)M_P, \qquad M_2(\lambda) = \frac{2}{\lambda}M_P,$$

$$\alpha_1(\lambda) = -\frac{\lambda_0}{m^2}\lambda + \frac{1}{\lambda}, \qquad \alpha_2(\lambda) = 2\left(1 + \frac{1}{\lambda}\right), \qquad (24)$$

$$\beta_1(\lambda) = 0, \qquad \beta_2(\lambda) = 1,$$

and send $\lambda \to 0$ for fixed Planck mass M_P. This is the first-order formulation of the limit considered in Ref. [13], which exists only if the kinetic terms for e_1^a and e_2^a have opposite sign, i.e., only if $\sigma = -1$. Then, the $\lambda \to 0$ limit of $\mathcal{L}_{\mathrm{ZDG}}$ exists, too, with the final result that

$$\mathcal{L} = 2M_P\left[e_a R^a + \frac{\lambda_0}{6} \varepsilon^{abc} e_a e_b e_c + h_a T^a \right.$$
$$\left. - \frac{1}{m^2}\left(f_a R^a + \frac{1}{2} \varepsilon^{abc} e_a f_b f_c\right)\right]. \quad (25)$$

This is the Chern-Simons-like action for NMG [14]. The second dreibein has now become an auxiliary field; by eliminating it, we recover the higher-derivative action for NMG.

As a consistency check, we now verify that the central charge (17) of ZDG reduces in the above limit to the known central charge of NMG. From Eqs. (7) and (8), we learn that

$$\Lambda(\lambda) = \Lambda_0 + \mathcal{O}(\lambda), \qquad \gamma(\lambda) = 1 - \frac{\Lambda_0}{2m^2}\lambda + \mathcal{O}(\lambda^2), \qquad (26)$$

where Λ_0 is the cosmological constant in NMG as determined by the NMG field equations in terms of the cosmological parameter λ_0. Now, we insert Eq. (24) into Eq. (17), use the NMG relation $\lambda_0 = \Lambda_0^2/(4m^2) - \Lambda_0$ [2], and then write $\Lambda_0 = -1/\ell^2$ to deduce that

$$c(\lambda) = -24\pi\ell M_P\left(1 - \frac{1}{2\ell^2 m^2}\right) + \mathcal{O}(\lambda). \qquad (27)$$

The limit $\lambda \to 0$ indeed gives the NMG central charge.

We recall that NMG has a parity-violating extension to a "general massive gravity" (GMG) model in which the two spin-2 modes have unequal masses [2]. It would be natural to suppose that ZDG is also a special case of a more general parity-violating theory that propagates two massive spin-2 modes with unequal masses. By taking one mass to infinity, we would then have a generalization of "topologically massive gravity" [17] and hence of "chiral gravity" [18]. However, although it is not difficult to find generalizations of ZDG that have a limit to GMG, those that we have considered have additional degrees of freedom that only go away in the GMG limit, so this remains an open problem.

In this Letter, we have presented a zwei-dreibein model of 3D massive gravity that both incorporates earlier parity-invariant models, such as NMG, and extends them in such a way as to resolve the clash between bulk and boundary unitarity in the context of the AdS/CFT correspondence. Moreover, this is achieved without the need to fine-tune parameters, so it is a result that is robust against the quantum renormalization of parameters. As both bulk and boundary unitarity are essential for quantum consistency, the model constructed here may be the first candidate for a semiclassical approximation to a consistent quantum theory of 3D massive gravity.

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