# Anti–de Sitter–Wave Solutions of Higher Derivative Theories

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(Received 16 May 2013; published 5 September 2013)

We show that the recently found anti–de Sitter (AdS)-plane and AdS-spherical wave solutions of quadratic curvature gravity also solve the most general higher derivative theory in $D$ dimensions. More generally, we show that the field equations of such theories reduce to an equation linear in the Ricci tensor for Kerr-Schild spacetimes having type-$N$ Weyl and type-$N$ traceless Ricci tensors.

DOI: 10.1103/PhysRevLett.111.101101 PACS numbers: 04.50.−h, 04.20.Jb, 04.30.−w

There is a vast literature on the exact solutions of four-dimensional Einstein's gravity. But, as more powers of curvature are added, or computed in a microscopic theory such as string theory, to get a better UV behaved theory, the field equations become highly nontrivial and so solutions are not easy to find. In fact, only a few classes of solutions are known: For example, see [1–5] for solutions in low energy string theory. $AdS_5 \times S^5$, which played a major role in AdS/CFT, is also expected to be an exact solution of string theory [6]. In this Letter, we present new asymptotically anti–de Sitter (AdS) solutions, which are AdS-plane and AdS-spherical waves, to $D$-dimensional, for $D \geq 3$, generic gravity theories based on the Riemann tensor and its arbitrary number of covariant derivatives which are in some sense natural geometric extensions of Einstein's gravity. Certain low energy string theory actions constitute a subclass of this theory once all the nongravitational fields are turned off [7]. Asymptotically, AdS solutions in higher derivative theories are relevant in the context of generic gravity-gauge theory dualities and holography. Here, we shall provide such solutions.

Using a theorem given in [5], we first prove that any spacetime with type-$N$ Weyl and type-$N$ traceless Ricci tensors, where the metric is in the Kerr-Schild form, the field equations of the most general higher derivative theory reduce to a linear equation for the traceless Ricci tensor. These spacetimes have constant scalar invariants. Furthermore, using the type-$N$ property of the traceless Ricci tensor in these field equations, in the AdS background, we obtain a linear partial differential equation for a metric function $V$. This result implies that the AdS-wave metrics are universal in the sense defined in [5].

As a special case, the field equations of the theory which depends on the contractions of the Riemann tensor but not on its derivatives, $f(R^{\mu\nu}_{\rho\sigma})$ theory, are also highly cumbersome, but using our general result for type-$N$ spacetimes under certain assumptions, they reduce to those of the quadratic gravity. Then, taking the metric to be in the Kundt subclass of type-$N$ spacetimes, we show that AdS-plane wave [8,9] and AdS-spherical wave [10] solutions of the quadratic gravity theory are also the solutions of the $f(R^{\mu\nu}_{\rho\sigma})$ theory. Logarithmic terms arising in the solutions of the quadratic gravity exist also in some $f(R^{\mu\nu}_{\rho\sigma})$ theories corresponding to the generalizations of critical gravity [11,12]. As an application of our result, we show that any type-$N$ Einstein space ($R_{\mu\nu} = (R/D)g_{\mu\nu}$) solves the field equations of the Lanczos-Lovelock theory. In addition, for a special choice of the parameters, any spacetime metric having type-$N$ Weyl and type-$N$ traceless Ricci tensors with a constant Ricci scalar solves a specific Lanczos-Lovelock theory.

To find exact asymptotically AdS solutions of a generic higher derivative theory, we shall make use of general results of [13–17] which utilize the boost weight formalism.

*Type-N Weyl and traceless Ricci tensors.*—The Weyl tensor of type-$N$ spacetimes has been studied in detail in [13–16]. Let $\ell$, $\mathbf{n}$, and $\mathbf{m}^{(i)}$, $(i = 2, \ldots D-1)$ be a null tetrad frame with

$$\ell_\alpha \ell^\alpha = n^\alpha n_\alpha = \ell^\alpha m^{(i)}_\alpha = n^\alpha m^{(i)}_\alpha = 0, \qquad \ell^\alpha n_\alpha = 1, \qquad m^{(i)\alpha} m^{(j)}_\alpha = \delta_{ij}, \tag{1}$$

where $\alpha = 0, 1, 2, \cdots D-1$ and $i, j = 2, 3, \cdots D-1$. Then, the metric takes the form

$$g_{\mu\nu} = \ell_\mu n_\nu + \ell_\nu n_\mu + \delta_{ij} m^{(i)}_\mu m^{(j)}_\nu. \tag{2}$$

The Weyl tensor of type-$N$ spacetimes, expressed in the above frame, where $\ell_\mu$ is the Weyl aligned null direction, takes the form

$$C_{\alpha\beta\gamma\delta} = 4\Omega'_{ij} \ell_{\{\alpha} m^{(i)}_\beta \ell_\gamma m^{(j)}_{\delta\}}, \tag{3}$$

which transforms under scale transformations with boost weight -2 (In the above expressions, we used the notations of [13]).

Type-$N$ property alone is not sufficient to reduce the field equations of higher derivative gravity theories to a solvable form, we make a further assumption that the spacetime is radiating and the Ricci scalar is constant.

 

For radiating type-$N$ spacetimes, the Ricci tensor is in the form

$$R_{\mu\nu} = \rho \ell_\mu \ell_\nu + \frac{R}{D} g_{\mu\nu}, \tag{4}$$

where $\rho$ is a scalar function and $R$ is the Ricci scalar. Taking $R$ to be a constant and using the Bianchi identity, one obtains

$$\nabla^\mu (\rho \ell_\mu \ell_\nu) = 0. \tag{5}$$

Notice that the traceless part,

$$S_{\mu\nu} = R_{\mu\nu} - \frac{R}{D} g_{\mu\nu} = \rho \ell_\mu \ell_\nu, \tag{6}$$

of the Ricci tensor is of type $N$. Following [5], and paraphrasing the statement given in that work we have the following result:

*Theorem.*—Consider a Kundt spacetime for which the Weyl and the traceless Ricci tensors be of type $N$, that is having the forms (3) and (6), respectively, and all scalar invariants be constant. Then, any second rank symmetric tensor constructed from the Riemann tensor and its covariant derivatives can be written as a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher orders of $S_{\mu\nu}$ (such as, for example, $\Box^n S_{\mu\nu}$).

The proof of the above statement, which we can only sketch here, depends heavily on the boost-weight formalism [5,18,19]. In particular, a Kundt spacetime with type-$N$ traceless Ricci and type-$N$ Weyl tensors is degenerate; that is, all positive boost weight components of the Riemann tensor and its covariant derivatives, $\nabla^{(n)}$ (Riemann), are zero [19]. This further implies, that all tensors $\nabla^{(n)}$ (Riemann), $n \geq 1$, are also of maximal boost order $-2$. Since the components of a two tensor can only have boost weights $-2, \ldots, +2$, and using the same arguments as in [5] the traceless part of any second rank symmetric tensor can only have boost-weight $-2$, and, hence, also be of type $N$. Furthermore, using the appendix of [5], the symmetric tensor can only be linear in $g_{\mu\nu}$ or linear in the contractions of curvature tensors of the form $\nabla^{(n)}$ (Riemann). One can put any contraction into the sought form by changing the order of covariant derivatives with the help of the Ricci identity;

$$[\nabla_\mu, \nabla_\nu] T_{\alpha\beta\ldots\gamma} = R^\lambda_{\mu\nu\alpha} T_{\lambda\beta\ldots\gamma} + \cdots + R^\lambda_{\mu\nu\gamma} T_{\alpha\beta\ldots\lambda},$$

and by using the contractions of the Bianchi identity, that are $\nabla^\beta R_{\alpha\beta\mu\nu} = \nabla_\nu R_{\alpha\mu} - \nabla_\mu R_{\alpha\nu}$ and $\nabla_\alpha R^\alpha_\mu = 0$. Then, one can now see that the above theorem follows.

An immediate consequence of the above theorem is the reduction of the field equation corresponding to the most general action;

$$I = \int d^D x \sqrt{-g} f(g^{\alpha\beta}, R^\mu_{\nu\gamma\sigma}, \nabla_\rho R^\mu_{\nu\gamma\sigma}, \ldots, (\nabla_{\rho_1} \nabla_{\rho_2} \ldots \nabla_{\rho_M}) R^\mu_{\nu\gamma\sigma}, \ldots), \tag{7}$$

to the form

$$E_{\mu\nu} = e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0, \tag{8}$$

where $e$ and $a_n$'s are constants depending on the explicit form of the action. Here, $N$ is related to the number of covariant derivatives of the Riemann tensor or its contractions in the action: For the low-energy limit of the full string theory, $N$ goes to infinity. Hence, we reduced the field equations of the most general higher derivative theories to a form linear in the Ricci tensor. Furthermore, the trace part of (8) is

$$e = 0, \tag{9}$$

which gives the relation between the cosmological constant and the parameters of the theory. To have full solutions of (8), there must exist at least a real solution to (9). We assume that there is such a solution. The traceless part is

$$\sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0. \tag{10}$$

This is still hard to solve: despite its appearance it is a nonlinear equation. But, for the Kundt type of Kerr-Schild metrics $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V \ell_\mu \ell_\nu$ and $S_{\mu\nu} = \rho \ell_\mu \ell_\nu$ (see [18,20] for more details), one gets $\Box^n S_{\mu\nu} = \ell_\mu \ell_\nu \mathcal{O}^n \rho$ for all $n = 0, 1, 2, \ldots, N$, then contractions of $\Box^n S_{\mu\nu}$ vanish identically. This nice property of the Kerr-Schild metrics leads to a further reduction of the field equation. There are two explicit metrics, the AdS-plane [8,9] and the AdS-spherical waves [10], which are of the Kerr-Schild form and belong to the Kundt class of type-$N$ spacetimes. For these metrics, $\bar{g}_{\mu\nu}$ is the AdS background metric and the function $V$ satisfies $\ell^\mu \partial_\mu V = 0$. Then, $\rho$ has the form

$$\rho \equiv \mathcal{O} V = \left[ \bar{\Box} + 2\xi_\mu \partial^\mu + \frac{1}{2} \xi_\mu \xi^\mu + \frac{2R(D-2)}{D(D-1)} \right] V, \tag{11}$$

where $\bar{\Box}$ is the Laplace-Beltrami operator of the AdS background, and $\xi_\mu$ arises in $\nabla_\mu \ell_\nu = \ell_{(\mu} \xi_{\nu)}$. Because of (11), Eq. (10) reduces to

$$\sum_{n=0}^{N} a_n \mathcal{O}^{n+1} V = 0. \tag{12}$$

For $N > 1$, this equation can be factorized as

$$\prod_{n=0}^{N} (\mathcal{O} + b_n) \mathcal{O} V = 0, \tag{13}$$

where some $b_n$'s are real and some are complex constants in general (complex $b_n$'s come in complex conjugate pairs). Then, the most general solution is

$$V = \Re\left( \sum_{i=0}^{N} V_i \right), \tag{14}$$

where $\Re$ represents the real part and $V_i$'s solve $(\mathcal{O} + b_i) V_i = 0$, $i = 0, 1, 2, \ldots, N$. Solutions of such linear partial differential equations are given in [8,10] in the AdS background. As a conclusion, we can say that the AdS-wave metrics found recently in [8,10] solve the field equations of the most general higher derivative theories. To give explicit exact solutions, we have to know the constants $b_i$, $i = 0, 1, 2, \ldots$ in terms of the parameters of the theory. For this purpose, we shall consider some special cases below. Note that in the above discussion, we have considered only the negative cosmological constant background metric, AdS. With a positive cosmological constant background metric, dS, it is not possible to obtain analogous explicit metrics given in [8–10]. For a flat background metric, the solutions reduce to the plane wave solutions of generic gravity theories [1,2].

*Type-N spacetimes in quadratic gravity.*—For type-$N$ Weyl and type-$N$ traceless Ricci tensors, and using (5), the field equations of quadratic gravity [21]

$$\begin{aligned}&\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}\right)\\&\quad+2\alpha R\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)+(2\alpha+\beta)(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R\\&\quad+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+2\beta\left(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho}\\&\quad+2\gamma\Big[RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu^{\sigma\rho\tau}-2R_{\mu\sigma}R_\nu^\sigma\\&\quad-\frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\sigma\rho}-4R^2_{\sigma\rho}+R^2)\Big]=0, \qquad (15)\end{aligned}$$

reduce to the following simplified equations [15,16],

$$(\beta\Box+c)(\rho\ell_\mu\ell_\nu)=0, \qquad (16)$$

and a trace equation that gives a relation between the constant $R$ and the parameters of the theory. Here, $c$ is given as [21]

$$c\equiv\frac{1}{\kappa}+2R\alpha+\frac{2(D-2)}{D(D-1)}R\beta+\frac{2(D-3)(D-4)}{D(D-1)}R\gamma. \qquad (17)$$

Exact solutions, the AdS-wave metrics, of this have been reported recently [8–10].

*$f(R^{\mu\nu}_{\rho\sigma})$ theory.*—Consider the theory with an action that depends on the Riemann tensor and its contractions but not on the derivatives of the Riemann tensor:

$$I=\int d^Dx\sqrt{-g}f(R^{\mu\nu}_{\rho\sigma}). \qquad (18)$$

For this case, $N=1$ and hence (10) reduces to

$$(a\Box+b)(\rho\ell_\mu\ell_\nu)=0, \qquad (19)$$

where $a$ and $b$ are constants. In a generic higher curvature theory, the constant $a$ is not zero. However, if the theory parameters are tuned properly, $a$ can be zero and an example of this case, the Lanczos-Lovelock theory, is considered below. On the other hand, if $a$ is not zero, for AdS-wave metrics in the Kerr-Schild form, the field equations reduce to a fourth order linear partial differential equation with constant coefficients of the form

$$\left(\bar\Box-\frac{2R}{D(D-1)}-M^2\right)\left(\bar\Box-\frac{2R}{D(D-1)}\right)(V\ell_\mu\ell_\nu)=0, \qquad (20)$$

where $\bar\Box$ is the Laplace-Beltrami operator of the AdS background and

$$M^2=-\frac{b}{a}-\frac{2R}{D(D-1)}, \qquad (21)$$

which corresponds to the mass of the spin-2 excitation in the linearized version of the $f(R^{\mu\nu}_{\rho\sigma})$ theory about AdS [22]. For the AdS-plane wave,

$$d\bar s^2=-\frac{R}{D(D-1)z^2}\left(2dudv+\sum_{m=1}^{D-3}(dx^m)^2+dz^2\right), \qquad (22)$$

where $u$ and $v$ are null coordinates, and $z$ is a spatial coordinate with $z>0$. Then,

$$\ell_\mu=(1,0,0,\ldots,0), \qquad \xi_\mu=\frac{2}{z}\delta^z_\mu, \qquad (23)$$

and one can find a specific solution [8]

$$\begin{aligned}V(u,z)&=c_1(u)z^{D-3}+c_2(u)\frac{1}{z^2}\\&\quad+z^{(D-5)/2}(c_3(u)z^\nu+c_4(u)z^{-\nu}), \qquad (24)\end{aligned}$$

where $c_i$ are arbitrary functions and

$$\nu=\frac{1}{2}\sqrt{(D-1)^2-\frac{4D(D-1)M^2}{R}}.$$

Thus, the solution takes the form

$$\begin{aligned}ds^2&=-\frac{R}{D(D-1)z^2}\left(2dudv+\sum_{m=1}^{D-3}(dx^m)^2+dz^2\right)\\&\quad+V(u,z)du^2. \qquad (25)\end{aligned}$$

In addition, the most general solution of (20) was also given in [8]. Note that for the AdS-plane wave, $\ell_\mu$ is proportional to the null Killing vector $\zeta_\mu$, i.e., $\zeta_\mu=(1/z)\ell_\mu$.

On the other hand, for the AdS-spherical wave, one obtains

$$\begin{aligned}ds^2&=-\frac{R}{D(D-1)z^2}\left[-dt^2+\sum_{i=1}^{D-1}(dx^i)^2\right]\\&\quad+2V(\ell_\mu dx^\mu)^2, \qquad (26)\end{aligned}$$

where

$$\ell_\mu=\left(1,\frac{x^i}{r}\right), \qquad \xi_\mu=-\frac{1}{r}\ell_\mu+\frac{2}{r}\delta^t_\mu+\frac{2}{z}\delta^z_\mu, \qquad (27)$$

with $r^2=\sum_{i=1}^{D-1}(x^i)^2$ and $x^{D-1}=z$. The most general $V$ as a solution of (20) was given in [10] whose explicit form is not particularly illuminating to depict here. In this case, there exists no null Killing vector fields. For $M^2 \neq 0$, $V$ decays sufficiently fast for both AdS-wave solutions and they are asymptotically AdS. When $M^2 = 0$, there are logarithmic solutions which spoil the asymptotically AdS structure in both cases. Let us give two concrete $f(R^{\mu\nu}_{\rho\sigma})$ theories as examples.

*Cubic gravity generated by string theory.*—In [7], it was shown that the bosonic string has the following effective Lagrangian density at $O[(\alpha')^2]$

$$f_{\text{eff}} = R + \frac{\alpha'}{4}\Big(R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2\Big) + \frac{(\alpha')^2}{24}\Big(-2R^{\mu\alpha\nu\beta}R^{\lambda\gamma}_{\mu\nu}R_{\alpha\gamma\beta\lambda} + R^{\mu\nu}_{\alpha\beta}R^{\gamma\lambda}_{\mu\nu}R^{\alpha\beta}_{\gamma\lambda}\Big), \quad (28)$$

where $\alpha'$ is the usual inverse string tension. Using the results of [23], the field equations of (28) for the AdS-plane and AdS-spherical wave metrics reduce to (20) with $a$ and $M^2$ given as

$$a = \frac{7\alpha'^2 R}{4D(D-1)}, \quad (29)$$

$$M^2 = -\frac{4D(D-1)}{7\alpha'^2 R} - \frac{2(D-3)(D-4)}{7\alpha'} + \frac{9D-29}{7D(D-1)}R. \quad (30)$$

Therefore, the solutions quoted above are the solutions of (28) with this $M^2$.

*Lanczos-Lovelock theory.*—Another special case of the $f$ (Riemann) theory is the Lanczos-Lovelock theory given with the Lagrangian density

$$f_{\text{L-L}} = \sum_{n=0}^{[(D-1)/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \prod_{p=1}^{n} R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \quad (31)$$

where $a_n$'s are dimensionful constants, $[(D-1)/2]$ corresponds to the integer part of $(D-1)/2$ and $\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}}$ is the generalized Kronecker delta. In this case, since the constant $a$ in (19) vanishes identically, the field equations reduce to

$$b\rho = 0, \quad (32)$$

where $b$ was calculated in [24] as

$$b = 2(D-3)! \sum_{n=0}^{[(D-1)/2]} a_n \frac{n}{(D-2n-1)!}\left[\frac{2R}{D(D-1)}\right]^{n-1}. \quad (33)$$

Equation (32) gives two subclasses. The first class corresponds to $\rho = 0$ which is of type-$N$ Einstein space. The second class corresponds to $b = 0$ which gives a relation between the parameters of the theory. In this subclass, any type-$N$ radiating metric with a constant Ricci scalar is an exact solution of the theory. AdS-wave metrics are exact solutions of the Lancsoz-Lovelock theory. We note that the Lanczos-Lovelock theory is free of logarithmic solutions.

*Conclusion.*—Type-$N$ radiating spacetimes simplify the field equations of higher derivative theories. This simplification rests on the result that any second rank symmetric tensor constructed from contractions of type-$N$ Weyl and type-$N$ traceless Ricci tensors, and their covariant derivatives, reduces to a simpler form. To find explicit solutions, we showed that one has to consider a subclass that is the Kundt spacetime. Hence, Kerr-Schild metrics reported recently as the solutions of the quadratic gravity, the AdS-plane [8] and AdS-spherical wave metrics [10], which belong to the Kundt subclass of type-$N$ spacetimes, solve the field equations of any higher curvature gravity theory exactly. We gave a cubic theory coming from the string theory and the Lanczos-Lovelock theory as explicit examples. A more detailed version of this work will be communicated elsewhere.

This work is partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK).

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