Negativity as an Estimator of Entanglement Dimension

Christopher Eltschka
Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

Jens Siewert
Departamento de Química Física, Universidad del País Vasco UPV/EHU, E-48080 Bilbao, Spain
IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Spain
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Among all entanglement measures negativity arguably is the best known and most popular tool to quantify bipartite quantum correlations. It is easily computed for arbitrary states of a composite system and can therefore be applied to discuss entanglement in an ample variety of situations. However, as opposed to logarithmic negativity, its direct physical meaning has not been pointed out yet. We show that the negativity can be viewed as an estimator of how many degrees of freedom of two subsystems are entangled. As it is possible to give lower bounds for the negativity even in a device-independent setting, it is the appropriate quantity to certify quantumness of both parties in a bipartite system and to determine the minimum number of dimensions that contribute to the quantum correlations.

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Introduction.—Dimension, that is, the number of independent degrees of freedom, is a particularly important system parameter. It is relevant, for example, for the security of cryptography schemes and for the significance of Bell inequality violation [1,2]. In general, in information processing (both classical and quantum) the dimensionality may be regarded as a resource and is therefore crucial for system performance.

The device-independent characterization of physical systems [1–9] without *a priori* restrictions regarding the underlying structure of mathematical models is fundamental for our understanding of nature. The goal is to obtain the desired physical information based only on the statistics from certain measurement outcomes ("prepare and measurement scenario," Ref. [3]) without reference to internal properties or mechanisms of a device. In recent years numerous schemes for device-independent dimension testing and other system properties have been proposed. There are methods that detect the minimum number of classical or quantum degrees of freedom for a single system [3,7,8]. The dimensionality may be inferred also from Bell-inequality violation [1,2]. On the other hand, there are device-independent methods for multipartite entanglement detection [4–6,9]. In our Letter we propose direct counting of entangled dimensions based on a well-known entanglement measure for bipartite systems, the negativity, thereby elucidating the physical meaning of the latter. The method can be made device independent by invoking techniques from Refs. [6,9]. With our result we cannot draw any conclusion regarding the classical dimensions of the two local systems. However, since entanglement is possible only between quantum degrees of freedom we directly obtain the minimum number of quantum levels for both parties which then are certified to be quantum without further assumption.

To demonstrate this we first study a nontrivial family of mixed states that can be defined for any $d \times d$-dimensional bipartite system, the axisymmetric states. Their negativity provides a clear illustration for the central statement of our article. It is then easy to show that this statement holds for arbitrary states as well. Finally, we establish the link to the device-independent description that concludes our construction of a device-independent bound on the number of entangled dimensions for two-party systems.

Negativity.—The negativity was first used by Zyczkowski *et al.* [10] and subsequently introduced as an entanglement measure by Vidal and Werner [11]. Consider the state ρ of a bipartite system with finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The negativity is defined as

$$\mathcal{N}(\rho) = \frac{1}{2}(\|\rho^{T_A}\|_1 - 1), \tag{1}$$

where ρ^{T_A} denotes the partial transpose with respect to party A and $\|M\|_1 \equiv \mathrm{tr}\sqrt{M^\dagger M}$ is the trace norm of the matrix M. We slightly modify this definition by introducing the quantity

$$\mathcal{N}_{\mathrm{dim}}(\rho) = 2\mathcal{N}(\rho) + 1 \equiv \|\rho^{T_A}\|_1. \tag{2}$$

As our discussion proceeds it will turn out that the least integer greater than or equal to $\mathcal{N}_{\mathrm{dim}}$ is a lower bound to the number of entangled dimensions between the parties A and B.

We note that there is another entanglement measure closely related to the negativity, the so-called logarithmic negativity [11–13] $LN(\rho) = \log_2\|\rho^{T_A}\|_1$. The operational meaning of the logarithmic negativity is known: It is the entanglement cost of preparing a state under quantum operations preserving the positivity of the partial transpose [12,14]. It is remarkable that, in some sense, this physical meaning represents an asymptotic counterpart to the main statement of the present article concerning the entanglement dimension as single-copy property of entangled states.

Axisymmetric states.—In studies of entanglement properties it is often useful to define families of states with a certain symmetry [15], such as the Werner states [16] and the isotropic states [17]. Here we introduce the axisymmetric states ρ^{axi} for two qudits which are all the states obeying the same symmetries as the maximally entangled state in d dimensions

$$|\Psi_d\rangle = \frac{1}{\sqrt{d}}(|11\rangle + |22\rangle + \cdots + |dd\rangle), \tag{3}$$

that is (i) exchange of the two qudits, (ii) simultaneous permutations of the basis states for both qudits, e.g., $|0\rangle_A \leftrightarrow |1\rangle_A$ and $|0\rangle_B \leftrightarrow |1\rangle_B$, (iii) coupled phase rotations

$$U(\varphi_1, \ldots, \varphi_{d-1}) = e^{i\sum_j \varphi_j \mathfrak{g}_j} \otimes e^{-i\sum_k \varphi_k \mathfrak{g}_k}$$

where $\mathfrak{g}_j$ ($j = 1, \ldots, d-1$) are the diagonal generators of the group SU(d).

Note that Ψ_d is the only pure state with these symmetries. Because of the invariance under phase rotations a state with these symmetries can be written as $|\psi\rangle = \sum_j a_j |jj\rangle$ with $j = 1, \ldots, d$. The symmetry with respect to simultaneous level permutations leads to $a_j = a_k$ ($j \neq k$), hence $a_j = 1/\sqrt{d}$ up to an irrelevant global phase. Apart from the maximally entangled state this family contains only (mostly full-rank) mixed states. For any $d \geqq 2$ these states are given by two real parameters x and y that describe the position of the state in a plane triangle (in close analogy to the Greenberger-Horne-Zeilinger symmetric states [18]), see Fig. 1. In order to determine the lengths of the triangle sides we choose the Euclidean metric of the triangle to coincide with the Hilbert-Schmidt metric of the density matrices. This enables us to deduce various physical facts from Fig. 1 merely by means of geometric intuition.

Axisymmetric states for $d \times d$ systems can be represented as $d^2 \times d^2$ matrices with diagonal elements

$$\rho^{\mathrm{axi}}_{jj,jj} = \frac{1}{d^2} + a, \qquad \rho^{\mathrm{axi}}_{jk,jk} = \frac{1}{d^2} - \frac{a}{d-1} \quad (j \neq k)$$

($j, k = 1, \ldots, d$) and off-diagonal entries

$$\rho^{\mathrm{axi}}_{jj,kk} = b \quad (j \neq k),$$

all other off-diagonal elements vanish due to the symmetry condition regarding local phase rotations. The ranges for the matrix elements are

$$-\frac{1}{d^2} \leqq a \leqq \frac{d-1}{d^2} \tag{4}$$

$$-\frac{1}{d-1}\left(\frac{1}{d^2} + a\right) \leqq b \leqq \left(\frac{1}{d^2} + a\right). \tag{5}$$

From Eqs. (4) and (5) we recognize the triangular shape of the set of axisymmetric states. With this choice of

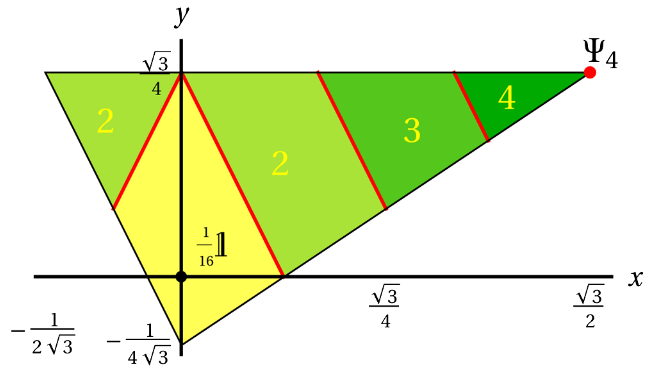

FIG. 1 (color online). The convex set of $d \times d$ axisymmetric states ρ^{axi}, here for $d = 4$. The family is characterized by two real parameters. While x is proportional to the off-diagonal element, y describes the asymmetry between the two types of diagonal elements. The upper right corner corresponds to the maximally entangled state $|\Psi_d\rangle$ (the only pure state), the completely mixed state $(1/d^2)\mathbb{1}_{d^2}$ resides at the origin. The states with local dimension d have d SLOCC classes corresponding to their Schmidt number k (indicated by the yellow numbers in the regions). The states with Schmidt number $\leqq k$ form the convex sets S_k and build a hierarchy $S_1 \subset S_2 \subset \cdots \subset S_d$. Note that Schmidt number $k = 1$ corresponds to separable states which are considered classical.

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Now we choose the scale of $a \equiv \alpha y$ and $b \equiv \beta x$ such that the Euclidean metric for x and y agrees with the Hilbert-Schmidt metric in the space of density matrices. We define the Hilbert-Schmidt scalar product of two matrices M_1 and M_2 as $\langle M_1, M_2\rangle_{\mathrm{HS}} \equiv \mathrm{tr}(M_1^\dagger M_2)$. With this we find $\alpha = (\sqrt{d-1}/d)$ and $\beta = \sqrt{d(d-1)}^{-1}$ so that

$$-\frac{1}{d\sqrt{d-1}} \leqq y \leqq \frac{\sqrt{d-1}}{d} \tag{6}$$

$$-\frac{1}{\sqrt{d(d-1)}} \leqq x \leqq \sqrt{\frac{d-1}{d}}. \tag{7}$$

Entanglement of axisymmetric states.—Remarkably, many entanglement properties of axisymmetric states can be determined exactly. The entanglement class of a bipartite state with respect to stochastic local operations and classical communication (SLOCC) is given by its Schmidt number, the minimal required Schmidt rank for any pure-state decomposition of the state. By using the optimal Schmidt number witnesses [19]

$$\mathcal{W} = \frac{k-1}{d}\mathbb{1}_{d^2} - |\Psi_d\rangle\langle\Psi_d|$$

($2 \leqq k \leqq d$) we find for each state $\rho^{\mathrm{axi}}(x, y)$ the corresponding Schmidt number, cf. Fig. 1. Notably, the borders between the SLOCC classes for $x \geqq 0$ are straight lines parallel to the lower left side of the triangle. This is no

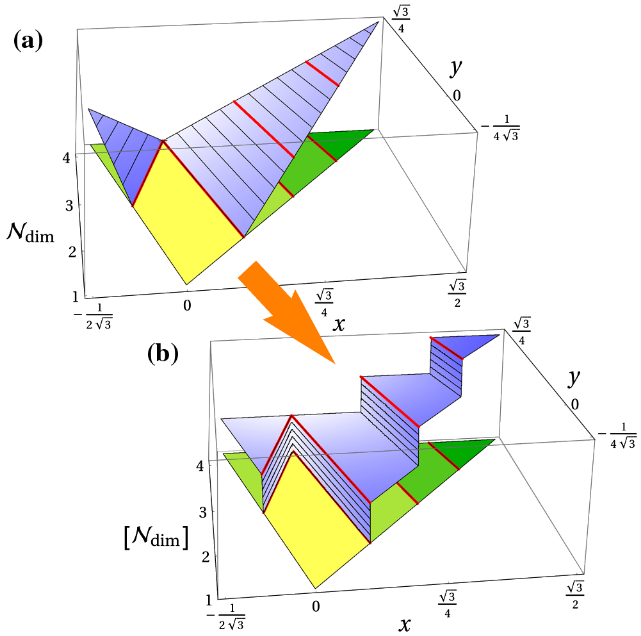

FIG. 2 (color online). Exact modified negativity $\mathcal{N}_{\rm dim}$ for $d \times d$ axisymmetric states $\rho^{\rm axi}$, again for $d = 4$. (a) The inclined triangular surfaces (blue) display $\mathcal{N}_{\rm dim}(x, y)$. It depends linearly on $|x|$ and y. Note that the borders between SLOCC classes (solid lines in the x-y plane, red) are projections of integer-value isolines of the modified negativity. (b) The ceiling function $\lceil \mathcal{N}_{\rm dim}(x, y)\rceil$ (staircaselike surface, blue) counts the Schmidt number of $\rho^{\rm axi}(x, y)$.

surprise since those lines correspond to states of constant overlap with the maximally entangled state Ψ_d. Moreover, we easily identify the compact convex sets S_k of states with a Schmidt number at most equal to k [19].

In the next step, we calculate the negativity for axisymmetric states. To this end we consider the eigenvalue problem for the partial transpose of $\rho^{\rm axi}$. It results in $d(d-1)/2$ identical eigenvalue problems for 2×2 matrices

$$\begin{pmatrix} \frac{1}{d^2} - \frac{a}{d-1} & b \\ b & \frac{1}{d^2} - \frac{a}{d-1} \end{pmatrix}$$

which have the eigenvalues

$$\lambda_\pm = \frac{1}{d^2} - \frac{a}{d-1} \pm |b|.$$

Adding the absolute negative eigenvalues and rewriting a and b in terms of x and y leads to

$$\mathcal{N} = \max\left\{0, \frac{1}{2}\left(\sqrt{d(d-1)}|x| + \sqrt{d-1}y - \frac{d-1}{d}\right)\right\}. \tag{8}$$

From this we find the exact $\mathcal{N}_{\rm dim}$ for the entangled axisymmetric states

$$\mathcal{N}_{\rm dim}(\rho^{\rm axi}(x, y)) = \sqrt{d(d-1)}|x| + \sqrt{d-1}y + \frac{1}{d} \tag{9}$$

which is noteworthy in several respects. First, the negativity is a linear function of $|x|$ and y (see Fig. 2). A state has nonvanishing negativity if and only if it is not separable. Consequently, there are no entangled axisymmetric states with positive partial transpose. Further, and most importantly, the borders between SLOCC classes correspond to isolines for integer values of the negativity. With the ceiling function $\lceil x\rceil$, the smallest integer greater than or equal to x, we see that for axisymmetric states $\rho^{\rm axi}(x, y)$

$$\text{SLOCC class } k = \lceil \mathcal{N}_{\rm dim}(x, y)\rceil. \tag{10}$$

However, the SLOCC class, that is, the minimum required Schmidt rank of the pure states in the decomposition of $\rho^{\rm axi}$, counts the number of degrees of freedom in which subsystems A and B are entangled. In consequence, our result implies that for axisymmetric states the modified negativity $\mathcal{N}_{\rm dim}$ is a precise counter of entangled dimensions.

Dimension estimator for arbitrary states.—Naturally the question arises to what extent this statement holds for all bipartite states. Because of the existence of entangled states with positive partial transpose [20] it is clear that the negativity cannot be a precise counter of entangled dimensions for arbitrary states. It is worth noticing that even for pure states the dimension estimate from the negativity can turn out much smaller than the Schmidt rank (for example, for very asymmetric coefficients, as in $|\psi\rangle = \sqrt{1-\varepsilon}|11\rangle + \sqrt{\varepsilon}|\varphi\rangle$ with $\varepsilon \ll 1$ and $|\varphi\rangle$ in the orthogonal complement of $|11\rangle$). This is because the negativity is continuous in a small "dimension admixture" while the Schmidt rank is not.

An interesting consequence is that the dimension indicated by the negativity is an effective number in the sense that weakly entangled degrees of freedom give only a weak contribution.

In the following we prove that, while not being an exact counter, the modified negativity $\mathcal{N}_{\rm dim}$ is always a lower bound to the Schmidt number.

To this end, we explicitly show again how to calculate the negativity for pure entangled states of Schmidt rank k. Any such state is locally equivalent (that is, equivalent under SLOCC) to $|\Psi_k\rangle$, the maximally entangled state of Schmidt rank k. Considering the partial transpose of $|\Psi_k\rangle\langle\Psi_k|$

$$|\Psi_k\rangle\langle\Psi_k| = \frac{1}{k}\sum_{\alpha\beta}|\alpha\alpha\rangle\langle\beta\beta| \xrightarrow{T_A} \frac{1}{k}\sum_{\alpha\beta}|\beta\alpha\rangle\langle\alpha\beta|$$

it is evident that $\mathcal{N}_{\rm dim}(\Psi_k) = 2(1/k)(k(k-1)/2) + 1 = k$. Now, since according to Ref. [11] the negativity is a convex function of the state ρ we find for an arbitrary state of Schmidt number k

$$\mathcal{N}_{\rm dim}(\rho) \leqq \sum_j p_j \mathcal{N}_{\rm dim}(\psi_j) \leqq \sum_j p_j k = k \tag{11}$$

for $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, since in that case $\mathcal{N}_{\rm dim}(\psi_j) \leqq k$. We mention that these estimates are valid for arbitrary bipartite systems with $d \times d'$ dimensions, both for $d = d'$ and for $d \neq d'$. This is because the Schmidt rank of a pure $d \times d'$ state cannot exceed the smaller of the two local dimensions. This concludes the proof that the modified negativity $\mathcal{N}_{\rm dim}$ is an estimator for the number of entangled dimensions of arbitrary two-party states. ■

While the dimension-counting property of $\mathcal{N}_{\rm dim}$ is comprehensible for pure states it is not so easy to develop an intuition for mixed states. After all, this property relies on the convexity of the negativity that is not obvious either. The axisymmetric states at least provide an illustration showing that precise dimension counting is possible also for mixed states.

Device-independent dimension estimate.—It has yet to be discussed that a lower bound on the entangled dimensions via the negativity, or $\mathcal{N}_{\rm dim}$, can be obtained in a device-independent setting. This technique has recently been worked out by Moroder *et al.* [9] and we sketch only the main idea here. A device-independent scenario implies that a number of generalized measurements are carried out on the subsystems A and B. While the detailed actions A_i, B_j of the measurement devices on the true state ρ_{AB} are unknown to the observers, the outcomes for each party labeled by i and j, are mutually exclusive. One also defines $A_0 = \mathbb{1}_A$ and $B_0 = \mathbb{1}_B$. The observers "see" ρ_{AB} only via their preparation-measurement setup, and (partially) determine the Hermitian matrix

$$\chi_{ij,kl}[\rho_{AB}] = \mathrm{tr}[\rho_{AB}(A_k^\dagger A_i \otimes B_l^\dagger B_j)] \qquad (12)$$

with orthonormal bases $\{|i\rangle_{\tilde{A}}\}$, $\{|j\rangle_{\tilde{B}}\}$ in the outcome spaces $\tilde{A}$ and $\tilde{B}$. This matrix depends linearly on ρ_{AB} and is positive whenever the true state ρ_{AB} is positive. Correspondingly, whenever $\rho_{AB}^{T_A}$ is positive, χ^{T_A} is positive, too.

The possibility of estimating the negativity relies on its variational expression [11]: $\mathcal{N}(\rho_{AB}) = \min\{\mathrm{tr}\sigma{:}\sigma^{T_A} \geqq 0, (\rho_{AB} - \sigma)^{T_A} \geqq 0\}$. The properties of χ mentioned above mean that the conditions for minimization hold also for $\chi[\rho_{AB}]$ and $\chi[\sigma]$. Moreover, the optimized quantity tr σ equals $\chi_{0000}[\sigma]$. Therefore, minimizing $\chi_{0000}[\sigma]$ over all matrices χ consistent with the measurement outcomes (and the condition tr $\rho_{AB} = 1$) will give a lower bound for the negativity $\mathcal{N}(\rho_{AB})$.

Evidently, our findings are useful for characterizing a test system with unknown quantum dimension. By entangling it with an auxiliary system of known dimension and measuring the negativity, a lower bound to the number of quantum levels in the test system can be found.

In principle, this method can be applied also to multipartite systems where it may yield information regarding the quantum dimension of the various bipartitions. However, in general it will not help to characterize specific SLOCC classes. For example, it is not possible to distinguish between a biseparable mixed three-qubit state (with entanglement in all bipartitions) and a W state just by means of the negativity.

We conclude by mentioning that the results regarding the negativity hold also for the convex-roof extended negativity [21] because it is the largest convex function that coincides with the negativity on pure states [22]. However, while improving the estimate in Eq. (11), the negativity would forfeit its most important asset, namely, that it can be calculated easily.

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