Spacetime and Flux Tube S-Matrices at Finite Coupling for $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We propose a nonperturbative formulation of planar scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, or, equivalently, polygonal Wilson loops. The construction is based on the operator product expansion approach and introduces a new decomposition of the Wilson loop in terms of fundamental building blocks named pentagon transitions. These transitions satisfy a simple relation to the worldsheet S matrix on top of the so-called Gubser-Klebanov-Polyakov vacuum which allows us to bootstrap them at any value of the coupling. In this Letter we present a subsector of the full solution which we call the gluonic part. We match our results with both weak and strong coupling data available in the literature.

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Introduction.—Computing the full S matrix of a four dimensional gauge theory at finite coupling might seem impossible. Conventional techniques, based on perturbation theory, soon become too cumbersome as the number of loops increases. Besides, the final results are typically much simpler than the intermediate steps would suggest. Both observations beg for an alternative nonperturbative approach. In the large N_c expansion, a dual *two-dimensional* string theory of 't Hooft surfaces emerges as such an alternative description. In some cases, these 't Hooft surfaces are integrable and their dynamics can be studied exactly. This is what happens in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and has led to the full solution of the problem of computing all two-point correlation functions of local operators [1]. Higher point correlation functions, Wilson loops (WL), and scattering amplitudes are considerably richer objects that depend on several external kinematics and probe string interactions. Since the string material is the same we expect integrability to help us compute these observables at any value of the coupling as well.

In this paper we consider planar scattering amplitudes or Null Polygon WLs in $\mathcal{N} = 4$ SYM theory (in this theory they are the same [2–4]). We identify a new object, called *pentagon transition*, as the building block of these WLs. The pentagon transitions arise naturally in the operator product expansion (OPE) construction [5] and completely determine the WL at any coupling. Remarkably, these transitions are directly related to the dynamics of the Gubser-Klebanov-Polyakov (GKP) flux tube [6,7] and can be computed exactly using integrability. In this Letter we present the ones describing the transition of the gluonic degrees of freedom.

Framing the wilson loop.—Our construction is based on a decomposition of a general polygon WL into simpler fundamental building blocks that we will denote as *square* and *pentagon* transitions. It starts with the observation that a polygon can be decomposed into a sequence of null squares as depicted in Fig. 1. Any two adjacent squares form a pentagon.

Of particular importance are the *middle squares* that arise as overlap of two consecutive pentagons. For an n-edged polygon there are $n - 4$ pentagons and $n - 5$ middle squares. Each one of these squares has three (Abelian) symmetries that are parametrized by a GKP time τ_i, space σ_i, and angle ϕ_i for rotations of the two-dimensional space transverse to the square. We coordinatize all conformally inequivalent polygons by acting with the symmetries of the ith middle square on all cusps located at the bottom of that square [8]. The set $\{\tau_i, \sigma_i, \phi_i\}_{i=1}^{n-5}$ so obtained is in one-to-one correspondence with the $3(n - 5)$ independent conformal cross ratios of an n-sided null polygon.

With the help of our sequence of squares and pentagons we can substract the well-known UV divergences of the WL by considering the ratio $\mathcal{W}$ defined in Fig. 2. Squares and pentagons WL have no cross ratios; their expectation

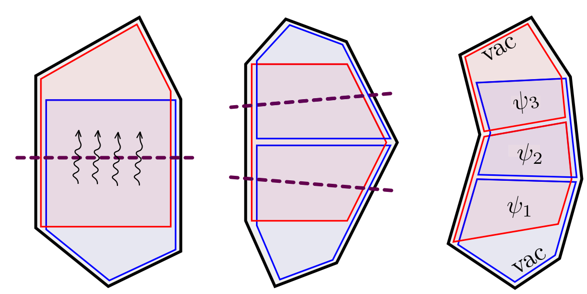

FIG. 1 (color online). Decomposition of n-sided null polygons into sequences of null squares and pentagons; here for (a) an hexagon ($n = 6$), (b) an heptagon ($n = 7$), and (c) an octagon ($n = 8$). Every middle square shares two of its opposite cusps with the big polygon; the positions of the other two cusps are fixed by the condition that they are null separated from their neighbors.

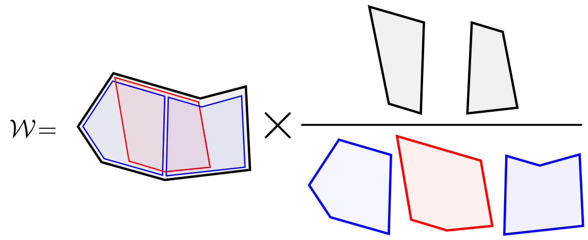

FIG. 2 (color online). We construct the conformally invariant and UV finite ratio $\mathcal{W}$ by dividing the v.e.v. of the WL by all the pentagons in the decomposition and multiplying it by all the middle squares. This generalizes the ratios considered in [5,8].

values are fixed by conformal symmetry [9] and given by the BDS ansatz [10]. We lose therefore no information by considering the conformally invariant ratios $\mathcal{W}$.

We move now to the dynamics depicted in Fig. 1(c). We start with the GKP vacuum in the bottom and evolve it all the way to the top where it is overlapped with the vacuum again. In between, we decompose the flux tube state in the ith middle square over a basis of GKP eigenstates ψ_i. Each eigenstate ψ_i propagates trivially in the corresponding square for a time τ_i. It then undergoes a pentagon transition P to the consecutive square where it is decomposed again and so on:

$$\text{vacuum} \to \psi_1 \to \ldots \to \psi_{n-5} \to \text{vacuum}. \tag{1}$$

In particular, for the hexagon we have vacuum $\to \psi_1 \to$ vacuum while for the heptagon we have vacuum $\to \psi_1 \to \psi_2 \to$ vacuum. The latter thus stands as the first polygon that contains nontrivial transitions between arbitrary states.

Following this picture we can write any n-sided WL as

$$\mathcal{W} = \sum_{\psi_i} e^{\sum_j(-E_j\tau_j + ip_j\sigma_j + im_j\phi_j)} \times P(0|\psi_1)P(\psi_1|\psi_2)\ldots P(\psi_{n-6}|\psi_{n-5})P(\psi_{n-5}|0). \tag{2}$$

The eigenstates ψ_i have definite energy E_i, momentum p_i and $U(1)$ charge m_i. They are N-particle states made out of N excitations on top of the GKP flux tube. The total energy of the state E_i is nothing but the sum of the energies of these excitations, and similarly for other charges. A useful way to parametrize energy and momentum of an excitation is through a Bethe rapidity u such that a state is defined by a set of rapidities $\mathbf{u} = \{u_1, \ldots, u_N\}$. An extra index $\mathbf{a} = \{a_1, \ldots, a_N\}$ is needed to specify the flavors of the excitations, which can be gluons, fermions, etc. [11]. Using these labels we can be more explicit and rewrite (2) as

$$\mathcal{W}_{\text{hex}} = \sum_{\mathbf{a}} \int d\mathbf{u} P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0) e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi},$$

$$\mathcal{W}_{\text{hep}} = \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u} d\mathbf{v} P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v}) P_{\mathbf{b}}(\bar{\mathbf{v}}|0) \times e^{-E(\mathbf{u})\tau_1 + ip(\mathbf{u})\sigma_1 + im_1\phi_1 - E(\mathbf{v})\tau_2 + ip(\mathbf{v})\sigma_2 + im_2\phi_2} \tag{3}$$

for the hexagon and heptagon WL, respectively. The generalization to higher polygon WL is straightforward. Note that here we introduced the notations $\bar{\mathbf{u}} = \{-u_N, \ldots, -u_1\}$ and, more importantly, the measure

$$d\mathbf{u} = \mathcal{N}_{\mathbf{a}} \prod_{j=1}^{N} \mu_{a_j}(u_j) \frac{du_j}{2\pi}, \tag{4}$$

where $\mathcal{N}_{\mathbf{a}}$ is a symmetry factor (which is equal to $1/N!$ for identical particles, for example).

The measure and the pentagon transitions are not independent. Instead they are related as

$$\text{Res}_{v=u} P_{aa}(u|v) = \frac{i}{\mu_a(u)}. \tag{5}$$

This relation is understood as follows. We can think of a pentagon transition as a pentagon WL with local operators inserted along its edges. In the simplest case considered here we have an insertion at a position σ_1 on the bottom edge and at σ_2 on the top edge. Taking the residue at $u = v$ is equivalent to studying the $\sigma_1, \sigma_2 \to -\infty$ limit of the pentagon transition with $\sigma_1 - \sigma_2$ fixed. It is conformally equivalent to the flattening of the pentagon into the square WL and translates into (5).

Finally, we note that, contrary to pentagon WL with no insertions, the pentagon transitions are not fixed by conformal symmetry. Remarkably enough, as we will see below, they can be fixed exactly using integrability.

The pentagon transition.—We are interested in the transitions involving the gluon excitations F, $\bar{F}$ which are the twist-one components of the Faraday tensor $F_{\mu\nu}$. We further choose the normalization in which their creation amplitude is $P_a(0|u) = 1$ with $a = F, \bar{F}$. We shall denote by $P(u|v) \equiv P_{FF}(u|v) = P_{\bar{F}\bar{F}}(u|v)$ and $\bar{P}(u|v) \equiv P_{F\bar{F}}(u|v) = P_{\bar{F}F}(u|v)$ their two possible transitions on the pentagon. Our conjecture is that these transitions obey three important axioms.

The first axiom reads $P(u|v) = P(-v|-u)$ and directly follows from the reflection symmetry of the pentagon. This transformation exchanges top and bottom edges and reverses the orientation of the pentagon, or equivalently swaps the two rapidities and flips their signs.

We dub the second axiom as the *fundamental relation*:

$$P(u|v) = S(u, v)P(v|u), \tag{6}$$

where $S(u, v)$ is the S matrix for the scattering of two Fs on the GKP background. A similar equation holds for $\bar{P}$ using the S matrix $\bar{S}$ between F and $\bar{F}$. All these S matrices can be computed exactly using integrability [12] and, in particular, $(u - v - i)S(u, v) = (u - v + i)\bar{S}(u, v)$.

The fundamental relation (6) is reminiscent of the Watson's equation [13] for form factors in integrable QFT. The analogy is, however, a bit dangerous since in our case the two excitations live on different edges of the pentagon. If both were on the same edge then it would be natural to expect an S matrix upon exchanging momenta; this would be basically built into the two particle Bethe wave function. Hence, to gain some better intuition about the origin of (6) we first need to understand how to move excitations between the different edges of the pentagon. The third and last axiom is precisely about that. It is depicted in Fig. 3 and reads

$$P(u^{-\gamma}|v) = \bar{P}(v|u), \tag{7}$$

where γ: $u^{-\gamma} \to u$ is a mirror transformation such that $E(u^{-\gamma}) = -ip(u)$, $p(u^{-\gamma}) = -iE(u)$. The latter transformation [14] entails crossing the Zhukowsky cuts $x(u^{-\gamma} \pm i/2) = g^2/x(u \pm i/2)$, where $x = 1/2(u + \sqrt{u^2 - 4g^2})$ and $g = \sqrt{\lambda}/4\pi$ with $\lambda = g_{\mathrm{YM}}^2 N_c$ is the 't Hooft coupling.

As a corollary of our axioms one can easily check that $\bar{P}(u^{2\gamma}|v)/P(u^{-3\gamma}|v) = S(v,u)$. This equation has a neat interpretation: we can bring a particle from the bottom to the top of the polygon either through the left by using $u \to u^{2\gamma}$ or through the right by $u \to u^{-3\gamma}$. Both give us two F's on the top but depending on which option we choose we end up with u to the left or to the right of the top excitation v. To compare both options we have to permute the two excitations thereby acquiring an S matrix factor. This self-consistency check provides further motivation for the fundamental relation (6).

Solution for gauge field.—Equations (6) and (7) are the main axioms of the pentagon transitions. A solution to these equations reads

$$P(u|v)^2 = \left[\frac{f(u,v)}{g^2(u-v)(u-v-i)}\right]^\eta \frac{S(u,v)}{S(u^\gamma,v)}, \tag{8}$$

with $\eta = 1$. The symmetric function $f(u,v) = f(v,u)$ here is given by

$$f = \left(x^+ - \frac{g^2}{y^-}\right)\left(x^- - \frac{g^2}{y^+}\right)\left(y^+ - \frac{g^2}{x^+}\right)\left(y^- - \frac{g^2}{x^-}\right), \tag{9}$$

when written in terms of the Zhukowsky variables $x^\pm = x(u \pm i/2)$ and $y^\pm = x(v \pm i/2)$. For $\bar{P}(u|v)$ we have the same as in (8) but with $\eta = -1$. One easily verifies that the expression (8) solves the relations (6) and (7) using unitarity, $S(u,v)S(v,u) = 1$, the mirror invariance [5,7] of the flux tube dynamics, $S(u^\gamma, v^\gamma) = S(u,v)$, and the crossing

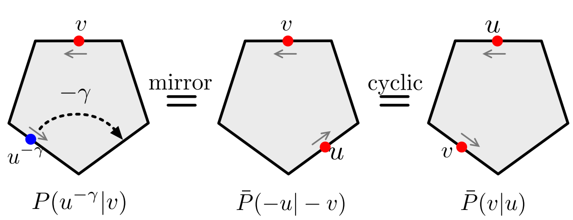

FIG. 3 (color online). Under a mirror transformation $u \to u^{-\gamma}$ an excitation is sent to the neighboring edge on its right. This is consistent with exchanging GKP space and time (in the bottom square). Combining this transformation with a cyclic rotation leads to (7). Note that the mirror transformation swaps P and $\bar{P}$. This fact admits a simple geometrical explanation [12].

relation obeyed by the gluon S matrix, $(u-v-i)S(u^\gamma,v) = (u-v+i)S(v^\gamma,u)$. Equation (8) renders the connection between the space-time and the flux-tube S matrices explicit. We now consider the weak and strong coupling limits of these finite coupling conjectures.

Perturbative regime.—To leading order at weak coupling we find that

$$P(u|v) = -\frac{(u^2+\frac{1}{4})\Gamma(iu-iv)(v^2+\frac{1}{4})}{g^2\Gamma(\frac{3}{2}+iu)\Gamma(\frac{3}{2}-iv)} + O(g^0), \tag{10}$$

from which we read that $\mu(u) \simeq -\pi g^2 \mathrm{sech}(\pi u)/(u^2+1/4)$, using (5). For the $\bar{P}$ transition we simply multiply (10) by the factor $g^2(u-v)(u-v-i)/(u^2+1/4)(v^2+1/4)$. The fact that P and $\bar{P}$ start at different loop orders is to be expected since at leading order gluons preserve their helicity, see e.g., [15].

We proceed with some perturbative checks of these results. The first one concerns the hexagon WL. Its leading OPE behavior is governed to any order in perturbation theory by the exchange of a twist-one gluon, $\mathcal{W}_{\mathrm{hex}} = 1 + e^{-\tau} f(\tau,\sigma,\phi) + \mathcal{O}(e^{-2\tau})$. Using our expression (3) we can compute the latter quantity to all loops,

$$f(\tau,\sigma,\phi) = 2\cos(\phi)\int_{-\infty}^{\infty}\frac{du}{2\pi}\mu(u)e^{-\gamma(u)\tau + ip(u)\sigma}, \tag{11}$$

where $\gamma(u) = E(u) - 1$ is the anomalous energy of the F excitation [11]. The relation (11) suffices to determine the two unfixed constants α_1 and α_2 in the hexagon three loops result [16] to be exactly as derived by the $\bar{Q}$ equation in [17]. Similarly, we can compute the leading OPE behaviour for general n-sided polygons. For example, for the heptagon $\mathcal{W}_{\mathrm{hep}}$ we have a double expansion in $e^{-\tau_1}$ and $e^{-\tau_2}$. The term proportional to $e^{-\tau_1-\tau_2}$ is particularly interesting because it is governed by the single gluon transitions. It is given by

$$\int\frac{dudv}{(2\pi)^2}\mu(u)\mu(v)e^{-\gamma(u)\tau_1 + ip(u)\sigma_1 - \gamma(v)\tau_2 + ip(v)\sigma_2} \times 2[\cos(\phi_1+\phi_2)P(-u|v) + \cos(\phi_1-\phi_2)\bar{P}(-u|v)]. \tag{12}$$

We matched this all loop relation against the symbol of the two-loop MHV heptagon [18] and found perfect agreement. We stress that, when put together, these checks of both P and $\bar{P}$ provide nontrivial evidence for our ansatz since these two transitions are related by a mirror transformation which is nonperturbative in the coupling.

Strong coupling.—At strong coupling $\sqrt{\lambda} \gg 1$ the gluonic excitations become relativistic particles of mass $\sqrt{2}$ [7,19]. Our expression (8) predicts that in this limit their pentagon transitions can be written in terms of a kernel K as

$$P(u|v) = 1 + \frac{2\pi}{\sqrt{\lambda}}K(\theta,\theta') + O(1/\lambda), \tag{13}$$

whose expression is

$$K(\theta,\theta')=\frac{i\cosh(2\theta)\cosh(2\theta')}{2\sinh(2\theta-2\theta')}\left[\sqrt{2}\cosh\left(\theta-\theta'-i\frac{\pi}{4}\right)+1\right], \tag{14}$$

and where θ is the hyperbolic rapidity, related to u by $u = 2g\tanh(2\theta) + O(g^0)$. The fact that the P transition starts with 1 at strong coupling is essential to match with the string theory prediction [12] and is directly related to the exponential form of the amplitude at strong coupling [2]. For the $\bar{P}$ transition we have $\bar{K}(\theta,\theta') = K(\theta' - i\pi/2, \theta)$, while for the measure we find that $\mu(u) = -1 + \cdots$, i.e.,

$$\mu(u)\frac{du}{2\pi} = -\frac{\sqrt{\lambda}}{2\pi} \times \frac{d\theta}{\pi\cosh^2(2\theta)} + \cdots. \tag{15}$$

Let us comment now on how these expressions are reproduced from the strong coupling Y system solution. As found in [5,20,21], the strong coupling result reads

$$\log\langle W\rangle = -\frac{\sqrt{\lambda}}{2\pi}[A_{\rm div} + A_{\text{BDS-like}} + A_0 + YY_{\rm cr}] \tag{16}$$

where $A_{\rm div}$, $A_{\text{BDS-like}}$, and A_0 are simple, explicitly known, functions of both the positions of the cusps of the polygon and the UV cutoff. The most nontrivial part of the result is the critical Yang-Yang functional $YY_{\rm cr}$. Remarkably, when considering our ratio defined in Fig. 2, it is this part and this part only that remains, i.e., $\mathcal{W} = \exp(-\sqrt{\lambda}YY_{\rm cr}/2\pi)$, while all the other contributions cancel out exactly. Now, to read off the transitions and measures we expand $\mathcal{W}$ at large τ_i using the Y-system prediction for $YY_{\rm cr}$. It turns out that there is no order of limits issue: the various kernels $K_{ab}(\theta,\theta')$ and measures $\mu_a(\theta)$ obtained from our expressions as in (15) and (13) are in direct correspondence with the measures and (linear combination of the) kernels appearing in the strong coupling thermodynamic Bethe ansatz, see appendix F of [5]. For example, the gluonic measure (15) governs the leading large τ contribution to the hexagon WL and it matches perfectly with the stringy result $R_{\sqrt{2}}$ given in equation (4.2) in [5].

Discussion.—The decomposition (2) breaks down the computation of scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory into fundamental building blocks which we dubbed pentagon transitions. These transitions obey a set of bootstrap equations (6) and (7) that can be solved thanks to the integrability of the GKP flux tube.

In this Letter we presented a conjecture for the gluonic transitions at any value of the coupling; see (8). We have similar conjectures for all the single particle transitions as well as for bound states [12]. For MHV amplitudes, the gluonic transitions considered in this paper dominate both at weak and strong coupling. For N^kMHV amplitudes, scalar and fermion transitions will certainly play a more important role [4,12].

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