# No-Go Theorems for $\psi$-Epistemic Models Based on a Continuity Assumption

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The quantum state $\psi$ is a mathematical object used to determine the probabilities of different outcomes when measuring a physical system. Its fundamental nature has been the subject of discussions since the inception of quantum theory. Is it ontic, that is, does it correspond to a real property of the physical system? Or is it epistemic, that is, does it merely represent our knowledge about the system? Assuming a natural continuity assumption and a weak separability assumption, we show here that epistemic interpretations of the quantum state are in contradiction with quantum theory. Our argument is different from the recent proof of Pusey, Barrett, and Rudolph and it already yields a nontrivial constraint on $\psi$-epistemic models using a single copy of the system in question.

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*Introduction.*—Quantum theory textbooks usually start from the hypothesis that to every physical system there corresponds a mathematical object—a ray in Hilbert space—called the quantum state. They then go on to deduce the multitude of quantitative predictions that make quantum theory so successful. But does the quantum state correspond to a real physical state or does it merely represent an observer's knowledge about the underlying reality? A major reason for doubting the reality of the quantum state is that it cannot be observed directly: it can only be reconstructed indirectly by lengthy state estimation procedures [1,2]. Furthermore, an epistemic interpretation of the quantum state could provide an intuitive explanation for many counterintuitive quantum phenomena and paradoxes, such as the measurement postulate and wave function collapse [3–5].

To formulate with precision the above question, we assume, following Ref. [6], that every quantum system possesses a real physical state (also called ontic state) denoted $\lambda$, which is independent of the observer. When a measurement is performed on the system, the probabilities to get different outcomes are determined by $\lambda$. If an ensemble of such systems is prepared, different members of the ensemble may be found in different states $\lambda$. A preparation procedure $Q$ therefore corresponds in general to a probability distribution $P(\lambda|Q)$ over the real states. The probability to obtain the outcome $r$ when preparation $Q$ is followed by measurement $M$ is $P(r|M,Q) = \sum_\lambda P(r|M,\lambda)P(\lambda|Q)$. Such a model will reproduce the quantum predictions if $P(r|M,Q) = \langle\psi_Q|\mathcal{M}_r|\psi_Q\rangle$, where $\psi_Q$ is the quantum state assigned by quantum theory to the preparation $Q$, and $\mathcal{M}_r$ is the quantum operator describing the measurement.

We can now distinguish two classes of models of the above type. A model is said to be $\psi$ ontic if the preparation of distinct pure quantum states always give rise to distinct real states. That is, for every $\lambda$ either $P(\lambda|Q) = 0$ or $P(\lambda|Q') = 0$ if the preparations $Q$ and $Q'$ correspond to different quantum states $|\psi_Q\rangle \neq |\psi_{Q'}\rangle$. In this case, every real state $\lambda$ is compatible with a unique pure quantum state. The quantum state is "encoded" in $\lambda$ and we can consider it to represent a real property of the system, akin, e.g., to the total energy of a system in classical physics [7]. In the second class of models, known as $\psi$-epistemic models, preparation of distinct pure quantum states may result in the same real state $\lambda$. Formally, there exist preparations $Q$ and $Q'$ corresponding to distinct quantum states $|\psi_Q\rangle \neq |\psi_{Q'}\rangle$ such that both $P(\lambda|Q) > 0$ and $P(\lambda|Q') > 0$ for some $\lambda$. In this case, the quantum state is not uniquely determined by the underlying real state and has a status analogous, e.g., to the Liouville distribution in statistical physics.

While nontrivial $\psi$-epistemic models exist in any fixed dimension $d$ [8,9], such models are necessarily highly contrived. Indeed, Pusey, Barrett, and Rudolph (PBR) have recently shown that the predictions of $\psi$-epistemic models are in contradiction with quantum theory under the assumption, termed preparation independence, that independently prepared pure quantum states correspond to product distributions over ontic states [10]. In the present work, we derive two alternative no-go theorems for $\psi$-epistemic models based on a natural assumption of continuity. Our approach shows that already at the level of a single system there exist strong constraints on $\psi$-epistemic models. Furthermore, our first no-go theorem readily translates in a simple experimental test, an implementation of which has been reported in Ref. [11] using high-dimensional attenuated coherent states of light traveling in an optical fibre.

Constraints on $\psi$-epistemic models at the level of single quantum systems have also been obtained in Ref. [12] using an assumption termed ontic indifference. This assumption is in fact closely related to the one presented here. In Sec. 2 of the Supplemental Material [13], we show how to use our approach to recover, in a simple and clear way, those of Ref. [12]. Arguments using single quantum systems have also been used in Refs. [14,15] to show that $\psi$-epistemic models cannot be "maximally epistemic" (in a sense defined in Refs. [14,15]).

 

*No-go theorems for ψ-epistemic models.*—The key motivation behind our result is that $\psi$-epistemic models should satisfy a form of continuity. Indeed, we assign an ontic status to $\psi$ if a variation of $\psi$ necessarily implies a variation of the underlying reality $\lambda$, and we assign it an epistemic status if a variation of $\psi$ does not necessarily imply a variation of $\lambda$. It is then natural to assume a form of continuity for $\psi$-epistemic models: a slight change of $\psi$ induces a slight change in the corresponding ensemble of $\lambda$'s in such a way that at least some $\lambda$'s from the initial ensemble will also belong to the perturbed ensemble. We use a slightly stronger form of continuity which asserts that there are real states $\lambda$ in the initial ensemble that will remain part of the perturbed ensemble, no matter how we perturb the initial state, provided this perturbation is small enough. Models that violate this condition are presumably very contrived. Formally this continuity condition is defined as follows (see Fig. 1 for a depiction of the difference between $\psi$-ontic and $\delta$-continuous $\psi$-epistemic models).

*Definition (δ continuity).*—Let $\delta > 0$ and let $B_\psi^\delta$ be the ball of radius $\delta$ centered on $|\psi\rangle$; i.e., $B_\psi^\delta$ is the set of states $|\phi\rangle$ such that $|\langle\phi|\psi\rangle| \geq 1 - \delta$. We say that a model is $\delta$ continuous if for any preparation $Q$, there exists an ontic state $\lambda$ (which can depend on $Q$) such that for all preparations $Q'$ corresponding to quantum states $|\phi_{Q'}\rangle$ in the ball $B_{\psi_Q}^\delta$ centered on the state $|\psi_Q\rangle$, we have $P(\lambda|Q') > 0$.

Note that for notational simplicity we formulate our results in the case where the set $\Lambda = \{\lambda\}$ of real states is finite or denumerable. The generalization of Theorems 1 and 2 below to measurable spaces is given in Sec. 1 of the Supplemental Material [13]. This generalization is important since reproducing the predictions of even a single qubit requires an infinite, and probably even uncountably infinite, number of real states (see Refs. [16–18] for evidence to this effect).

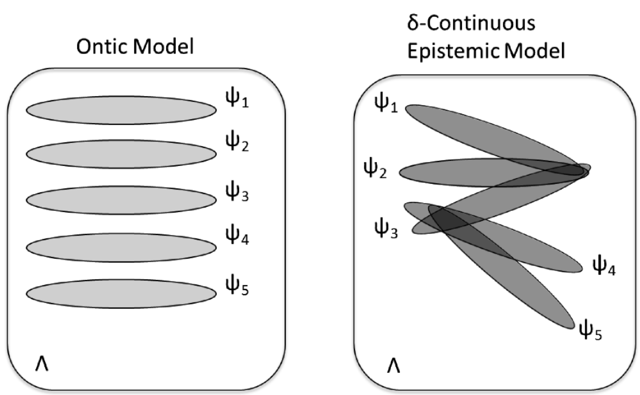


FIG. 1. Illustration of $\psi$-ontic and $\delta$-continuous $\psi$-epistemic models. Depicted is the space $\Lambda$ of ontic states, as well as the support of the probability distribution $P(\lambda|Q_k)$ for preparation $Q_k$ associated to distinct pure states $\psi_k$, $k = 1, \ldots, 5$. In $\psi$-ontic models (left) distinct quantum states give rise to probability distribution $P(\lambda|Q_k)$ with no overlap. In $\delta$-continuous $\psi$-epistemic models (right), states that are close to each other (such as $\{\psi_1, \psi_2, \psi_3\}$ and $\{\psi_3, \psi_4, \psi_5\}$) all share common ontic states. However states that are further from each other (such as $\psi_1$ and $\{\psi_4, \psi_5\}$) do not necessarily have common ontic states $\lambda$.

Note also that the above definition introduces a connection between the overlap of quantum states and the overlap of distributions in the ontic space of $\lambda$'s. This is extremely natural if we do not introduce a privileged direction in Hilbert space (i.e., a preferred basis), since then the properties of $\psi$-epistemic models can only depend on the geometry of the Hilbert space.

Our first result is a constraint on $\delta$-continuous models for single systems.

*Theorem 1*:–There are no $\delta$-continuous models with $\delta \geq 1 - \sqrt{(d-1)/d}$ reproducing the measurement statistics of quantum states in a Hilbert space of dimension $d$.

*Proof.*—Consider $d$ preparations $Q_k$ $(k = 1, \ldots, d)$ corresponding to distinct quantum states $|\psi_k\rangle$ all contained in a ball of radius $\delta$. By definition of a $\delta$-continuous model, there is at least one $\lambda$ for which $\min_k P(\lambda|Q_k) > 0$ and thus

$$\epsilon \equiv \sum_\lambda \min_k P(\lambda|Q_k) > 0. \tag{1}$$

This last quantity can be viewed as a measure of the extent to which distributions over real states overlap in the neighborhood of a given quantum state. It was also introduced in Ref. [10] where it was shown to be related to the variational distance between the distributions $P(\lambda|Q_k)$.

Suppose now that a measurement $M$ yielding one of the possible outcomes $r = 1, \ldots, d$ is made on each of the prepared systems. A $\delta$-continuous model then makes the prediction

$$\begin{aligned}\sum_k P(k|M, Q_k) &= \sum_k \sum_\lambda P(k|M, \lambda) P(\lambda|Q_k) \\ &\geq \sum_k \sum_\lambda P(k|M, \lambda) \min_k P(\lambda|Q_k) \\ &= \sum_\lambda \min_k P(\lambda|Q_k) = \epsilon > 0.\end{aligned} \tag{2}$$

According to quantum theory, however, there exist states in a Hilbert space of dimension $d$ contained in a ball of radius $\delta = 1 - \sqrt{(d-1)/d}$ such that the left-hand side of Eq. (2) is equal to 0. To show this, let $\{|j\rangle : j = 1, \ldots, d\}$ be a basis of the Hilbert space. Consider the $d$ distinct states $|\psi_k\rangle = (1/\sqrt{d-1}) \sum_{j \neq k} |j\rangle$. These states are all at mutual distance $|\langle\psi_k|\psi\rangle| = \sqrt{(d-1)/d}$ from the state $|\psi\rangle = (1/\sqrt{d}) \sum_j |j\rangle$. Let the measurement $M$ be the measurement in the basis $\{|j\rangle\}$. Then $P(k|M, Q_k) = 0$ for all $k = 1, \ldots, d$ and thus $\sum_k P(k|M, Q_k) = 0$. (These states and measurements were considered in the $d = 3$ case in Ref. [19]). ■

Note that the above result also applies if we only require $\delta$ continuity to hold around some fixed quantum states rather than for all states in Hilbert space. Interestingly, the $\psi$-epistemic model of Ref. [8] for Hilbert spaces of dimension $d$ is $\delta$ continuous around a specific state with a value of $\delta$ saturating the above bound.

Though one expects a $\psi$-epistemic model to be $\delta$ continuous for some value of $\delta$, the bound derived in Theorem 1 may *a priori* seem arbitrary. This motivates the following definition.

*Definition (Continuity).*—A $\psi$-epistemic model is continuous if there exists a nonzero $\delta > 0$ such that it is $\delta$ continuous.

Note that continuous $\psi$-epistemic models are easy to construct. For instance the model introduced in Ref. [9] is continuous. And taking convex combinations of the $\psi$-epistemic models of Ref. [8] centered around different states easily yields a continuous model.

Our second result shows that there are no $\psi$-epistemic models that are both continuous and which satisfy the following separability assumption. A similar condition was independently introduced in Ref. [20], where it is called "compactness." Though weaker than the preparation independence assumption explicitly used by PBR, it is already sufficient to derive their main result.

*Definition (Separability).*—Let $Q$ be the preparation of a physical system yielding with nonzero probability $P(\lambda|Q) > 0$ the real state $\lambda$. A model is separable if $n$ independent copies $Q^n = (Q, \ldots, Q)$ of the preparation devices yield with nonzero probability $P(\vec{\lambda} = \lambda^n|Q^n) > 0$ a system in the joint real state $\lambda^n = (\lambda, \ldots, \lambda)$, for any positive integer $n$.

*Theorem 2*:–Separable continuous $\psi$-epistemic models cannot reproduce the measurement statistics of quantum states in a Hilbert space of dimension $d \geq 3$.

*Proof.*—The idea of the proof is to fix an arbitrarily small $\delta > 0$, and choose specific states $|\phi_k\rangle$, all within the ball of radius $\delta$. Because of $\delta$ continuity, these states share a common ontic state. Using separability, the states $|\phi_k^{\otimes n}\rangle$ also share a common ontic state. By taking $n$ large enough, the distance between the tensor products $|\phi_k^{\otimes n}\rangle$ becomes large enough that we can apply Theorem 1. In more detail we proceed as follows.

Consider $d \geq 3$ preparations $Q_k$ corresponding to the $d$ distinct states $|\phi_k\rangle = \alpha|k\rangle + (\beta/\sqrt{d})\sum_{i=1}^{d}|i\rangle$ with $k = 1, \ldots, d$, $\alpha = -\sqrt{1-[(d-2)/(d-1)]^{1/n}}$, and $\beta = -\alpha/\sqrt{d} + \sqrt{\alpha^2/d + [(d-2)/(d-1)]^{1/n}}$. It is easily checked that these states are normalized $\langle\phi_k|\phi_k\rangle = 1$, have mutual scalar product $|\langle\phi_k|\phi_l\rangle| = [(d-2)/(d-1)]^{1/n}$ for $k \neq l$, and are all at distance $|\langle\phi_k|\phi\rangle| = 1 - \delta_{nd}$ from the state $|\phi\rangle = (1/\sqrt{d})\sum_{i=1}^{d}|i\rangle$, where $\delta_{nd} = 1 - \sqrt{1-(d-1)\alpha^2/d}$. In a $\delta_{nd}$-continuous model, these states share at least a common real state $\lambda$. The separability assumption then implies that the states $|\phi_k^{\otimes n}\rangle$ also share a common real state, and thus that $\epsilon_n \equiv \sum_\lambda \min_k P(\lambda|Q_k^n) > 0$. By the same argument as in Theorem 1, it then follows that if a measurement $M$ yielding one of the possible outcomes $r = 1, \ldots, d$ is performed on each of these systems, the quantity $\sum_k P(k|M, Q_k^n) \geq \epsilon_n > 0$.

Note now that the $d$ $n$-fold copies $|\phi_k^{\otimes n}\rangle$ are normalized and have mutual scalar product $|\langle\phi_k^{\otimes n}|\phi_l^{\otimes n}\rangle| = (d-2)/(d-1)$ for $k \neq l$. There therefore exists a unitary transformation $U$ in the subspace $S_d \subset \mathbb{C}_d^{\otimes n}$ spanned by the $d$ states $|\phi_k^{\otimes n}\rangle$ that carries out the transformation $U|\phi_k^{\otimes n}\rangle = |\psi_k\rangle$, where $|\psi_k\rangle = 1/(\sqrt{d-1})(\sum_{j=1}^{d}|j\rangle - |k\rangle)$ for some basis $\{|j\rangle\}$. The states $|\psi_k\rangle$ are identical to the states used in the proof of Theorem 1. It follows that there exists a $d$-outcome measurement $M$ in $\mathbb{C}_d^{\otimes n}$, which applied on the states $|\phi_k^{\otimes n}\rangle$ gives the same statistics as the measurement in the basis $\{|j\rangle\}$ applied on the states $|\psi_k\rangle$ of Theorem 1. We can therefore find a measurement such that $P(k|M, Q_k^n) = 0$ and thus $\sum_k P(k|M, Q_k^n) = 0$ in contradiction with the prediction of a $\delta_{nd}$-continuous separable $\psi$-epistemic model.

We have thus shown that one can exclude $\delta$-continuous separable models with $\delta \geq \delta_{nd}$ for any positive integers $d$ and $n$. For large $n$ this bound behaves as $\delta \gtrsim \gamma/n$, with $\gamma = (d-1)[\log(d-1) - \log(d-2)]/(2d)$, thereby implying by taking $n$ arbitrarily large that no $\psi$-epistemic model for Hilbert spaces of dimension $d \geq 3$ can satisfy both the assumptions of continuity and separability. ∎

It is interesting to compare how these no-go theorems could be used in practice to test $\psi$-epistemic models (see Refs. [11,21] for actual tests). Experimental tests that rule out $\psi$-epistemic models for smaller values of the continuity parameter $\delta$ are clearly stronger. We thus consider how resources scale as $\delta \to 0$. If we use the construction of Theorem 1, then we need to use systems of dimension $d = O(1/(2\delta))$, and the resources needed to test $\psi$-epistemic models increase as $O(1/\delta)$. If we use Theorem 2, and take, e.g., the dimension $d = 3$, then one needs to prepare three states $|\phi_1^{\otimes n}\rangle$, $|\phi_2^{\otimes n}\rangle$, $|\phi_3^{\otimes n}\rangle$, where the number of copies of each state is $n = O(\ln 2/(3\delta))$. Again the resources needed increase as $O(1/\delta)$. Finally, we could also test $\delta$-continuous $\psi$-epistemic models using the construction given in PBR [8]. In this case we need to prepare $2^n$ distinct states, each of which is a product state of $n$ qubits, with $n = O(\sqrt{2}\ln 2/\sqrt{\delta})$. The resources required for the application of the PBR construction therefore grow exponentially in $1/\sqrt{\delta}$. Experimental tests based on Theorems 1 and 2 thus seem much easier than those based on the PBR construction.

Note that our theoretical arguments and that of PBR rely on the fact that certain quantum probabilities are exactly equal to zero. However, these arguments are robust against small deviations from these predictions, as expected in an experimental implementation where noise is inevitably present. Indeed, Eq. (2) implies that the observed value $\epsilon_{\exp} = \sum_k P(k|M, Q_k)$ provides an upper bound on the overlap $\epsilon = \sum_\lambda \min_k P(\lambda|Q_k)$ of the ontic distributions $P(\lambda|Q_k)$. A small value of $\epsilon_{\exp}$ therefore translates into a strong constraint on continuous $\psi$-epistemic models, since it implies that these distributions have only a small common overlap. Similarly $\epsilon_{\exp}^n = \sum_k P(k|M, Q_k^n)$ in Theorem 2 upper bounds the overlap $\epsilon_n = \sum_{\vec{\lambda}} \min_k P(\vec{\lambda}|Q_k^n)$ of the

$n$-copy joint distributions $P(\vec{\lambda}|Q_k^n)$. This last quantity can simply be related to the single-copy overlap $\epsilon$ if we further make the preparation independence assumption of PBR that joint distributions $P(\vec{\lambda}|Q_k^n) = P(\lambda_1, \ldots \lambda_n|Q_k^n) = P(\lambda_1|Q_k) \ldots P(\lambda_n|Q_k)$ are the product of individual distributions, which then implies $\epsilon_n = \epsilon^n$. Note that a comparison of the sensitivity of the different tests to experimental noise is possible, but goes beyond the scope of the present work. It would require a detailed modeling of the state preparation and measurement procedures.

*Discussion.*—In his seminal paper on the probabilistic interpretation of quantum theory, Born gave the wave function a functional interpretation: a mathematical object from which the probabilities of different measurement outcomes can be determined [22]. But the fundamental nature of this object, a real physical wave or a summary of our knowledge about physical systems, is a question that has divided physicists ever since. A precise formulation of these two alternatives, opening the way to clearcut answers, was provided by Harrigan and Spekkens [6]. If the wave function corresponds to a real, ontic, property of physical systems, the preparation of a system in different pure quantum states should always result in different physical states. If, on the other hand, the wave function has an epistemic status, such preparations should sometimes result in the same underlying physical state. PBR have recently introduced a no-go theorem that, given certain assumptions, rules out this latter possibility [10], thus awarding ontic status to the wave function. This result can also be seen as a constraint on the structure of possible extensions or generalizations of quantum theory. If they reproduce the quantum predictions and satisfy these assumptions, then such theories can only supplement the wave function $\psi$ with additional variables $\lambda'$; i.e., a system should be described by a physical state of the form $\lambda = (\psi, \lambda')$.

The theorem of PBR relies on two main assumptions. The first, which is also unquestioned in the present work, is that a system has a real and objective state $\lambda$ that is independent of the observer. The second is an assumption of "preparation independence," which states that independently prepared systems are described by independent product distributions over real states. It can be replaced by the weaker separability assumption used here.

In the present work, we reached the same conclusion as PBR using a simple argument that relies on a natural assumption of continuity. This notion of continuity captures the intuition that in a model where the quantum state is epistemic, a small variation of $\psi$ does not necessarily imply a variation of the underlying real state $\lambda$. We derived a fundamental limit on the degree of continuity of $\psi$-epistemic models, as parametrized by a quantity $\delta$, already at the level of single quantum systems (Theorem 1). Combining our continuity assumption with a separability assumption, we then showed that no $\psi$-epistemic model can reproduce all the predictions of quantum theory (Theorem 2).

Besides their simplicity and the fact that they already constrain $\psi$-epistemic models for single quantum systems, an interest of our results is that they are easy to implement experimentally. Such an experimental test based on Theorem 1 has been reported in Ref. [11].

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