Universality for Moving Stripes: A Hydrodynamic Theory of Polar Active Smectics

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We present a theory of moving stripes ("polar active smectics"), both with and without number conservation. The latter is described by a compact anisotropic Kardar-Parisi-Zhang equation, which implies smectic order is quasilong ranged in $d = 2$ and long ranged in $d = 3$. In $d = 2$ the smectic disorders via a Kosterlitz-Thouless transition, which can be driven by *either* increasing the noise *or* varying certain nonlinearities. For the number-conserving case, giant number fluctuations are greatly suppressed by the smectic order, which is long ranged in $d = 3$. Nonlinear effects become important in $d = 2$.

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Phases of nonequilibrium systems that spontaneously break both translational and rotational symmetry by forming moving stripe patterns have been extensively studied for many years, both theoretically and experimentally [1]. More recently, such "polar active smectic phases" have been found [2] to be quite generic in simulations of models of active particles [3] at high density with repulsive interactions. Since such models are known to provide a good description of a number of experimental systems, including *in vitro* experiments in which microtubules are bound to a substrate by molecular motors [4] and insect swarms [5], this strongly suggests that polar active smectic phases may occur in such systems. Moving layers are also ubiquitous [6] as the order-disorder transition is approached in, e.g., the "Vicsek model" [7] of flocking.

In this Letter, we formulate the hydrodynamic theory of such *polar* active smectic phases, and find that they differ considerably from their active but apolar (i.e., nonmoving) analogs [8]. We restrict ourselves here to "active smectics *A*," meaning phases with the average particle velocity along the mean layer normal. "Active smectics *C*" [9], with other relative orientations of particle velocity and the layers, will be considered elsewhere [10].

Given the enormous number of experimental realizations of such moving stripe systems, including convection in binary fluid mixtures [11], parametric waves in shaken fluids [12], and Belousov-Zhabotinsky chemical reaction-diffusion systems [13], to name just three, there are innumerable opportunities to compare our theory with experiments. However, most of our predictions relate to fluctuations induced by either thermal or active noise. Thermal noise may be unobservably small in many macroscopic realizations, while active noise may simply be absent. Active noise *is* present in both the simulations of [2] and in the experimental systems that those simulations mimic.

We consider two cases: first, with no conserved quantities, and second, with only the particle number conserved. Systems with momentum conservation, which have radically different hydrodynamics, will not be considered here.

For the non-number-conserving case (hereafter the "Malthusian" case [14]), we find that the hydrodynamics of the polar active smectic phase can be described by a "compact" anisotropic Kardar-Parisi-Zhang (KPZ) equation [15,16] (here "compact" means a model with topological defects). This theory predicts that a stable two-dimensional (2D) smectic phase, which cannot exist in equilibrium systems [17], is possible in active systems. Strikingly, this new phase *only* occurs if certain nonlinearities have opposite signs.

In particular, we find quasi-long-ranged smectic order in $d = 2$, and long-ranged order in $d = 3$. The latter should be contrasted with equilibrium smectics which have only quasi-long-ranged smectic order in $d = 3$ [18].

Fluctuations of the smectic layers can be described by the displacement $u(\vec{r}, t)$ of the layers along z, the coordinate along the mean layer normal. We find that in $d = 2$, fluctuations in the relative position of widely separated layers diverge logarithmically with that separation:

$$\langle[u(\vec{r}, t) - u(\vec{r}', t)]^2\rangle = C \ln\left[\frac{(r_\perp - r'_\perp)^2 + \gamma(z - z')^2}{a^2}\right], \tag{1}$$

where γ and C are nonuniversal (i.e., they vary from system to system), $O(1)$ constants, $\perp$ denotes components perpendicular to z, and a is the mean spacing of the layers. Equation (1) implies quasisharp Bragg peaks in light or x-ray scattering, which are proportional to the Fourier transformation of density correlations; that is,

$$\begin{aligned}\langle|\rho(\vec{q}, t)|^2\rangle &= \frac{1}{L^d}\int_{\vec{r},\vec{r}'}\langle\rho(\vec{r}, t)\rho(\vec{r}', t)\rangle e^{i\vec{q}\cdot(\vec{r}-\vec{r}')} \\ &= \sum_n c_n[(q_z - nq_0)^2 + \gamma q_\perp^2]^{(-2+n^2\eta)/2},\end{aligned} \tag{2}$$

where n is an integer denoting the order of the Bragg peak, the fundamental wave number $q_0 \equiv 2\pi/a$, L is the spatial linear extent of the system, the c_n are L independent constants, and $\eta \equiv Cq_0^2$. In addition to scattering experiments, this prediction can be tested experimentally and in simulations by, e.g., numerically Fourier transforming microscope images or the positions of simulated particles [19,20].

For a finite system, this divergence is cut off at $|\vec{\delta q}| \sim 1/L$, where $\vec{\delta q} \equiv \vec{q} - nq_0\hat{z}$. This implies the nth peak in $\langle|\rho(\vec{q}, t)|^2\rangle$ will have a finite height which scales with L like $L^{2-n^2\eta}$.

In $d = 3$, the long-ranged nature of the smectic order implies sharp (i.e., δ-function) Bragg peaks, whose height scales linearly with system volume L^3.

We predict that with increasing noise the active smectic phase undergoes a dynamical phase transition into the fluid, polar ordered phase treated in a much earlier work [21]. This transition is in the equilibrium XY universality class [22], which in $d = 2$ is of the Kosterlitz-Thouless-type [23]. The phase diagram for $d = 2$ in the parameter space of our model is illustrated in Fig. 1.

Perhaps the most surprising feature of this phase diagram in the context of moving stripe patterns is the existence of a polar active ordered fluid phase at all in this case. After all, one might ask, if the stripes in such a system are destroyed, what remains to move?

The answer to this question was first given in [17], in which it was shown that, when a system of layers melts via dislocation proliferation, which is the mechanism for the transition here, the resultant phase retains orientational order, even though its translational order is destroyed. This is very similar to the mechanism that gives rise to the "hexatic" phase in two-dimensional melting (see the second and third references of [24]), and can be understood by recognizing that, after dislocation unbinding, the layers still *exist*, but simply now have only finite spatial extent and translational correlations. But they do have infinite range *orientational* correlations; hence, the phase reached by the melting of the polar active smectic, even if that active smectic is a moving stripe pattern, is a polar active fluid with long-ranged orientational order.

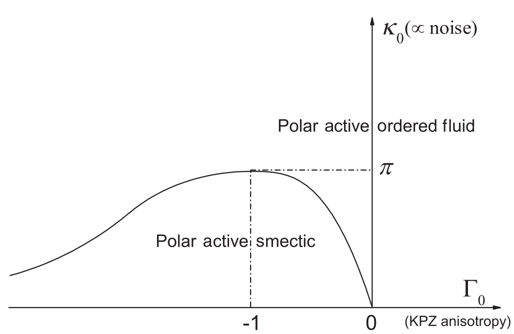

FIG. 1. Phase diagram in the κ_0-Γ_0 parameter space. Γ_0 is a measure of the anisotropy of the KPZ equation and is defined after Eq. (8), while κ_0 is essentially the dimensionless noise strength and is defined after Eq. (13). Note that the smectic phase only occurs for $\Gamma_0 < 0$. The polar active smectic to polar active ordered fluid transition is of the Kosterlitz-Thouless-type.

In the case where the number of the particles is conserved, we find long-ranged smectic order in $d=3$. In $d=2$, the linearized version of the full hydrodynamic theory predicts quasi-long-ranged smectic order; however, there are marginal nonlinearities which *may* invalidate this conclusion. We will investigate this in a future publication [10].

Unlike active nematics [25], in neither case are there giant number fluctuations in $d = 3$, nor are there any in $d = 2$ in the Malthusian case. The linearized hydrodynamic theory predicts none in $d = 2$ for the number-conserving case either, but the nonlinearities could change this.

We will now outline the derivation of these results, starting with the Malthusian case. In this case, the only important hydrodynamic variable is $u(\vec{r}, t)$. Symmetry considerations (specifically, translation and rotation invariance) require that u's equation of motion (EOM), to lowest order in a gradient expansion, takes the form

$$\partial_t u = v_0 - 2\lambda_\perp \partial_z u + (\nu_\perp \nabla_\perp^2 + \nu_z \partial_z^2)u + \lambda_z(\partial_z u)^2 + \lambda_\perp |\vec{\nabla}_\perp u|^2 + f, \quad (3)$$

where f is a Gaussian, zero-mean, white noise with variance $\langle f(\vec{r}, t)f(\vec{r}', t')\rangle = 2\Delta\delta^d(\vec{r} - \vec{r}')\delta(t - t')$ [26]. Rotation invariance forces the coefficient of the $\partial_z u$ term to be exactly -2 times that of the $|\vec{\nabla}_\perp u|^2$ term, because only the combination $\partial_z u - (1/2)|\vec{\nabla}_\perp u|^2$ is unchanged by a uniform rotation of the smectic layers [9].

In an equilibrium smectic, the first three terms ($\propto v_0$, $2\lambda_\perp$, and $\nu_\perp$) and the nonlinear terms ($\propto \lambda_z$ and $\lambda_\perp$) in Eq. (3) are all forbidden by rotation invariance of the free energy. They are, however, permitted here [28] simply because rotation invariance at the level of the EOM, which is all one can demand in an active system, does not rule them out. Because the average speed v_0 of the layers is uniform, it does not lead to relative motion of the layers and, hence, does not affect the degree of smectic order. The $2\lambda_\perp$ and $\lambda_\perp$ terms imply that u fluctuations propagate at a speed $2\lambda_\perp$ along the local layer normal. The physical content of the $\nu_\perp$ term is that layer curvature produces a local vectorial asymmetry which must modify the directed motion of the layers as this is a driven system. A similar term occurs in single membranes with "pumps" [28]. The λ_z term arises because the active speed of the layers depends on the layer spacing.

In the EOM of *apolar* active smectics [8], the first two terms and the nonlinear terms in Eq. (3) are all forbidden. This is because apolar active smectics are invariant under the *simultaneous* transformation $u \to -u$, $z \to -z$, and hence their EOM must be likewise invariant. However, these terms are allowed here where this up-down symmetry is absent. These two nonlinear terms radically affect the phase boundary (see Fig. 1) in $d = 2$.

To simplify Eq. (3), we introduce another field variable $u' \equiv u - v_0 t$ and new coordinates: $z' = z - 2\lambda_\perp t$, $\vec{r}'_\perp = \vec{r}_\perp$. In terms of these, Eq. (3) becomes

$$\partial_t u' = \nu_\perp \nabla^2_{\perp'} u' + \nu_z \partial^2_{z'} u' + \lambda_\perp |\vec{\nabla}_{\perp'} u'|^2 + \lambda_z (\partial_{z'} u')^2 + f. \tag{4}$$

Because the moving frame z' moves with respect to the old frame z at the same velocity $2\lambda_\perp \hat{z}$ as the u fluctuations in frame z, u fluctuations appear stationary in the frame z', which explains the absence of a term $\propto (\partial_{z'} u')$ in Eq. (4).

Stability requires $\nu_{\perp,z} > 0$, but imposes no constraints on the signs of $\lambda_{\perp,z}$. Indeed, their signs need not be the same, which proves to be crucial for the existence of the polar active smectic phase.

Equation (4) has exactly the same form as the anisotropic KPZ equation [15]. However, there is a crucial difference. The original KPZ equation [16] describes the hydrodynamics of crystal growth, and the hydrodynamic variable is h, the height of a surface. Clearly, states with different heights h are always physically distinguishable. However, for smectics, the state is periodic in u' with period a, the spacing between neighboring smectic layers. This allows for the existence of topologically stable dislocations, which can unbind, thereby "melting" (i.e., disordering) the smectic, in analogy to such "dislocation mediated melting" in a variety of translationally ordered equilibrium systems [17,24]. In $d = 2$, this is the aforementioned Kosterlitz-Thouless phase transition, which is absent in the "noncompact" anisotropic KPZ equation.

Simple power counting shows that the nonlinear terms in Eq. (4) are irrelevant in $d = 3$; hence, the linear theory is valid. A straightforward calculation then shows that $\langle |u'(\vec{q}, t)|^2 \rangle \propto 1/q^2$ for all directions of wave vector $\vec{q}$, where $u'(\vec{q}, t)$ is the spatial Fourier transform of $u'(\vec{r}, t)$. This in turn implies that the real space fluctuation $\langle |u'(\vec{r}, t)|^2 \rangle$ is finite as system size $L \to \infty$, which implies long-ranged smectic order [29].

In $d = 2$ the nonlinear terms in Eq. (4) become marginal, and a dynamical renormalization group (RG) analysis is needed. This has already been done for the $d = 2$ crystal growth problem [15], the resulting RG recursion relations are

$$\frac{d\nu_\perp}{d\ell} = \left[\beta - 2 + \frac{g}{32\pi}(1 - \Gamma)\right]\nu_\perp, \tag{5}$$

$$\frac{d\lambda_{\perp,z}}{d\ell} = (\chi + \beta - 2)\lambda_{\perp,z}, \tag{6}$$

$$\frac{d\Delta}{d\ell} = \left[-2\chi + \beta - 2 + \frac{g}{64\pi}(3\Gamma^2 + 2\Gamma + 3)\right]\Delta, \tag{7}$$

$$\frac{d(\nu_z/\nu_\perp)}{d\ell} = -\frac{g}{32\pi}\frac{\nu_z}{\nu_\perp}(1 - \Gamma^2), \tag{8}$$

where χ and β are the rescaling exponents of u' and t (i.e., $u' \to u' e^{\chi\ell}$, $t \to t e^{\beta\ell}$), $\Gamma \equiv (\lambda_z \nu_\perp)/(\nu_z \lambda_\perp)$, $g \equiv (\Delta \lambda_\perp^2)/(\nu_\perp^{5/2} \nu_z^{1/2})$, and we have chosen to rescale lengths isotropically. These recursion relations can be solved exactly for $g(\ell)$, $\Gamma(\ell)$, and flow to a stable fixed point if $\Gamma_0 \equiv \Gamma(\ell = 0) < 0$: as $\ell \to \infty$, $\Gamma(\ell) \to -1$, $1 + \Gamma(\ell) \propto 1/\sqrt{\ell}$, and $g(\ell) \propto 1/\ell$. The RG flows in Γ-g space are illustrated in Fig. 2.

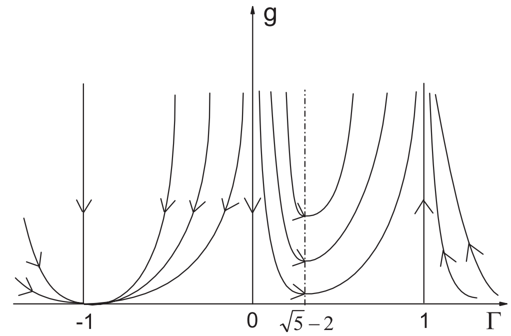

FIG. 2. The RG flow in the Γ-g parameter space for active smectics in $d = 2$. For $\Gamma < 0$ and $g > 0$, all flow lines go to a stable fixed point $(-1, 0)$; for $\Gamma > 0$ and $g > 0$, all flow lines go to infinity.

The vanishing of the effective strength $g(\ell)$ of the nonlinearities when $\Gamma_0 < 0$ suggests that for negative Γ's an effective linearized theory is sufficient. This proves to be the case; to establish it, we will now use the trajectory integral matching method [30] to compute $\langle |u'(\vec{q}, t)|^2 \rangle$, which determines the presence or absence of smectic order [29]. We will restrict this calculation to the case $\Gamma_0 < 0$, since, as we will see, only then is a stable smectic phase possible. Performing this standard procedure, we obtain

$$\langle |u'(\vec{q}, t)|^2 \rangle = \frac{\Delta(\ell^*)}{\nu_\perp(\ell^*)} \frac{e^{2\chi\ell^*}}{\left[q_\perp^2 + \frac{\nu_z(\ell^*)}{\nu_\perp(\ell^*)} q_z^2\right]}, \tag{9}$$

where $\ell^* \equiv \ln \Lambda/q$, with Λ the ultraviolet cutoff. For small q ($\ll \Lambda$), $\ell^* \gg 1$; in this limit, we find

$$\frac{\Delta}{\nu_\perp}(\ell^*) \approx \exp\left\{-2\chi(\ell^* - \ell_1) - \frac{1}{32\pi}\int_{\ell_1}^{\ell^*} d\ell' g(\ell') \times [1 + \Gamma(\ell')]\right\}\left[\frac{\Delta}{\nu_\perp}(\ell_1)\right], \tag{10}$$

where ℓ_1 is some fixed value of the renormalization group "time" ℓ at which the large ℓ approximations become valid.

Note that the integral over ℓ' in this expression converges as $\ell^* \to \infty$ since $g(\ell')[1 + \Gamma(\ell')] \propto \ell'^{-3/2}$ as $\ell' \to \infty$; hence, $(\Delta/\nu_\perp)(\ell^* \to \infty) \to C' e^{-2\chi\ell^*}$, where C' is a finite, nonzero constant. This convergence makes the scaling of u' correlations the same as that predicted by the linear theory, as we will now show.

Also note that, since $\Gamma(\ell^*) \to -1$ as $\ell^* \to \infty$,

$$\frac{\nu_z(\ell^*)}{\nu_\perp(\ell^*)} \to -\frac{\lambda_z(\ell^*)}{\lambda_\perp(\ell^*)} = \left|\frac{\lambda_z^0}{\lambda_\perp^0}\right|$$

where $\lambda_{\perp,z}^0$ are the bare values, the last equality following because the ratio $(\lambda_z(\ell^*))/(\lambda_\perp(\ell^*))$ does not renormalize, as can be seen from the recursion relation, Eq. (6), for the λ's. Thus, using Eq. (10) in Eq. (9) gives

$$\langle |u'(\vec{q}, t)|^2\rangle = \frac{C'}{\left[q_\perp^2 + \left|\frac{\lambda_z^0}{\lambda_\perp^0}\right| q_z^2\right]}, \tag{11}$$

which clearly scales as $1/q^2$ for all directions of wave vector $\vec{q}$, as in the linear theory. This scaling implies "logarithmic roughness" of the smectic layers in $d = 2$: specifically Eq. (1) with $\gamma = |\lambda_\perp^0/\lambda_z^0|$ and $C = C'\sqrt{|\lambda_\perp^0/\lambda_z^0|}/2\pi$, i.e., quasi-long-ranged smectic order [23].

Earlier arguments [15] for logarithmic roughness in the anisotropic KPZ equation incorrectly asserted that since g vanishes upon renormalization, the problem *must* become linear. Our analysis here shows instead that this requires that *both* g and $1 + \Gamma$ vanish *fast enough*. Had $g(1 + \Gamma)$ vanished slower as $\ell' \to \infty$, e.g., like $\ell'^{-1/2}$ rather than $\ell'^{-3/2}$, the integral in Eq. (10) would fail to converge as $\ell^* \to \infty$, thereby invalidating the linearized theory. Indeed, precisely such an excessively slow vanishing of a nonlinearity upon RG invalidates linear elastic theory in equilibrium smectics in $d = 3$ [31].

We will show later that, in the Malthusian case, in both $d = 2$ and $d = 3$, the spatial Fourier transform of the number density ρ at small q is given by

$$\langle |\rho(\vec{q}, t)|^2\rangle = C_1 q_z^2 \langle |u(\vec{q}, t)|^2\rangle + C_2, \tag{12}$$

where C_1 and C_2 are constants. Since this remains finite as $q \to 0$, there are no giant number fluctuations.

Our analysis above has ignored topological defects. To see how these affect the stability of polar active smectics in $d = 2$, we begin by analyzing the linearized theory [i.e., ignoring the $\lambda_{\perp,z}$ terms in Eq. (4)]. Anisotropically rescaling $r_\perp'' = r_\perp'$, $z'' = \sqrt{\nu_\perp/\nu_z} z'$, and expressing u' in terms of $\theta = 2\pi u'/a$, this linearized theory becomes

$$\partial_t \theta = \nu_\perp \nabla''^2 \theta + f', \tag{13}$$

where $\langle f'(\vec{r}'', t) f'(\vec{0}, 0)\rangle = \kappa \nu_\perp \delta(\vec{r}'')\delta(t)$, with $\kappa \equiv \Delta(2\pi/a)^2/\sqrt{\nu_\perp \nu_z}$. Equation (13) is identical to the simplest relaxational EOM for an equilibrium XY model [22], with θ being the angle of the magnetization. This mapping implies a dislocation unbinding phase transition [24] in polar active smectics in $d = 2$, when $\kappa = \pi$, with smectic order for smaller κ, and none for larger.

So far we have ignored the nonlinear terms in Eq. (4). Their effect can be included simply by replacing the bare value κ_0 of κ with the $\ell \to \infty$ limit of its renormalized value (which is finite and nonzero for $\Gamma_0 < 0$). Equating this limit to π gives the phase boundary in terms of κ_0 and Γ_0 for $\Gamma_0 < 0$:

$$\kappa_0 = -\frac{4\pi\Gamma_0}{(1 - \Gamma_0)^2}, \tag{14}$$

which is illustrated in Fig. 1.

Now we turn to the case where the number of particles is conserved. In this case, the fluctuation $\delta\rho \equiv \rho - \rho_0$ of the density ρ about its mean value ρ_0 becomes another important hydrodynamical variable. Number conservation implies $\partial_t \delta\rho = -\vec{\nabla}\cdot\vec{j}$, where $\vec{j}$ is the number density current. Symmetry arguments and a gradient expansion imply $\vec{j} = \vec{j}_L + \vec{j}_{NL}$, where the linear piece

$$\begin{aligned}\vec{j}_L = &-[j_0 + v_\rho \delta\rho + D_z \partial_z \delta\rho + v_{\rho u}^z \partial_z u \\ &+ ((c_\perp - w)\nabla_\perp^2 + c_z \partial_z^2) u]\hat{z} - D_\perp \vec{\nabla}_\perp \delta\rho \\ &- v_{\rho u}^\perp \vec{\nabla}_\perp u - w \vec{\nabla}_\perp \partial_z u - \vec{f}_\rho,\end{aligned} \tag{15}$$

with $\vec{f}_\rho$ a Gaussian noise with statistics

$$\langle f_{\rho i}(\vec{r}, t) f_{\rho j}(\vec{0}, 0)\rangle = (\Delta_z \delta_{ij}^z + \Delta_\perp \delta_{ij}^\perp)\delta(\vec{r})\delta(t), \tag{16}$$

while the nonlinear piece is given by

$$\begin{aligned}\vec{j}_{NL} = &-[\lambda_{\perp\rho}|\vec{\nabla}_\perp u|^2 + \lambda_{z\rho}(\partial_z u)^2 + v_\rho \delta\rho \partial_z u \\ &+ g_\rho \delta\rho^2]\hat{z} + g_u \partial_z u \vec{\nabla}_\perp u + v_\rho \delta\rho \vec{\nabla}_\perp u.\end{aligned} \tag{17}$$

Similar symmetry arguments and gradient expansions give the EOM for u,

$$\begin{aligned}\partial_t u = &\, v_0 + v_u \partial_z u + v_{u\rho}\delta\rho + \nu_z \partial_z^2 u + \nu_\perp \partial_\perp^2 u + \nu_\rho \partial_z \delta\rho \\ &+ \lambda_\perp |\vec{\nabla}_\perp u|^2 + \lambda_z (\partial_z u)^2 + g\delta\rho^2 + g_c \delta\rho \partial_z u + f_u,\end{aligned} \tag{18}$$

where the noise f_u has the same statistics as f in Eq. (4).

If we neglect the nonlinear terms in $\vec{j}$ and Eq. (18), a straightforward calculation shows that $\langle |u(\vec{q}, t)|^2\rangle \sim 1/q^2$, which implies quasi-long-ranged smectic order in $d = 2$ and long-ranged order in $d = 3$. We also find that $\langle |\delta\rho(\vec{q}, t)|^2\rangle$ goes to a finite value as $q \to 0$, which implies no giant number fluctuations in either $d = 2$ or $d = 3$.

Simple power counting shows that the nonlinear terms in $\vec{j}$ and Eq. (18) are irrelevant in $d = 3$ in the RG sense. Hence, these linear results should apply in $d = 3$, at least in systems with sufficiently small nonlinearities. In $d = 2$, similar power counting shows that all of the nonlinear terms in $\vec{j}$ and Eq. (18) become marginal and, hence, could potentially change the behavior at long wavelengths.

We can analyze number fluctuations in the Malthusian case by modifying the continuity equation for $\delta\rho$ to include birth and death, as has been done [14] for polar ordered fluid flocks. Dropping irrelevant terms, this gives

$$\partial_t \delta\rho = \alpha \partial_z u + v_{\rho u}^\perp \nabla_\perp^2 u + v_{\rho u}^z \partial_z^2 u - \delta\rho/\tau + f_{b-d}, \tag{19}$$

where τ and f_{b-d} are, respectively, the characteristic time and noise in the birth and death rate [14], and the α term reflects the dependence of those rates on the local layer spacing. We take f_{b-d} to be zero-mean Gaussian white noise, with statistics $\langle f_{b-d}(\vec{r}, t) f_{b-d}(\vec{r}', t')\rangle = 2\Delta_{b-d}\delta^d(\vec{r} - \vec{r}')\delta(t - t')$.

Fourier transforming Eq. (19) in space, and solving the resultant linear stochastic ordinary differential equation for the correlations of $\delta\rho$ gives [32], to leading order in q, Eq. (12), with $C_1 = \tau^2\alpha^2$ and $C_2 = \tau\Delta_{b-d}$.

In conclusion, we have developed the hydrodynamic theories of both number-conserving and non-number-conserving ("Malthusian") polar active smectics, with no momentum conservation in either case, in both $d = 2$ and $d = 3$. In the Malthusian case, we've shown that polar active systems are described by the anisotropic KPZ equation, and can exhibit a two-dimensional smectic phase, which cannot exist in equilibrium. This phase is stable only when the two relevant nonlinearities in that equation have opposite signs, and disorders via a Kosterlitz-Thouless transition. We also show that a three-dimensional (3D) smectic phase, which has only quasi-long-ranged order in equilibrium, has true long-ranged order in polar active systems. In neither $d = 2$ nor $d = 3$ are there giant number fluctuations. In the number-conserving case, the linearized hydrodynamic theory predicts the existence of 2D and 3D polar active smectic phases. For the number-conserving case in $d = 3$, the linearized theory is adequate, while in $d = 2$, there are marginal nonlinearities which await a full dynamical RG treatment.

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