Universality in the Full Counting Statistics of Trapped Fermions

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We study the distribution of particle number in extended subsystems of a one-dimensional non-interacting Fermi gas confined in a potential well at zero temperature. Universal features are identified in the scaled bulk and edge regions of the trapped gas where the full counting statistics are given by the corresponding limits of the eigenvalue statistics in Gaussian unitary random matrix ensembles. The universal limiting behavior is confirmed by the bulk and edge scaling of the particle number fluctuations and the entanglement entropy.

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The techniques for the manipulation of ultracold atomic gases have undergone a rapid development in the last decade and have provided experimental access to various interesting aspects of many-body quantum systems [1,2]. The common feature in the experiments is the presence of trapping potentials that can be tuned to confine particles into effective one-dimensional geometries [3–5], and theoretical predictions on various properties of 1D quantum gases [6–8] can directly be tested.

The strongly correlated phases of 1D quantum gases are characterized by the simultaneous presence of thermal and quantum noise. At ultralow temperatures, the dominating quantum noise reveals important information on the non-local character of correlations in the corresponding many-body states. In particular, 1D Bose liquids can be probed by measuring the full distribution of interference amplitude in experiments [9], showing remarkable agreement with the predictions of the theory [10]. In the case of a Fermi gas, an analogous concept is the full counting statistics (FCS) [11], which encodes the distribution of particle number in extended subsystems. The FCS shows interesting properties in the ground state of the Fermi gas and, in the noninteracting case, can also be used to extract the entanglement entropy of the subsystem [12,13].

The presence of trapping potentials leaves characteristic signatures on the FCS. For the ground state of the non-interacting Fermi gas, the FCS was studied in the presence of a periodic potential [14], and recently the effect of harmonic traps has been analyzed on the particle number fluctuations and entanglement [15]. However, the question whether some properties of the FCS hold irrespectively of the details of the potential has not yet been addressed.

Here, we point out a remarkable universality and show that for a broad class of trapping potentials, the proper scaling limits of the FCS in the bulk and edge regime of the trapped gas are given by the corresponding eigenvalue statistics of Gaussian unitary random matrix ensembles (GUE). Physically, the universality can be understood from the generic behavior of the trapping potential, being flat in the center and approximately linear around the edge of the high-density region. The FCS is derived by a semi-classical treatment of the single-particle wave functions and by finding scaling variables for the bulk and edge regimes through which the details of the potential can be completely eliminated in the thermodynamic limit.

The appearance of random matrix eigenvalue statistics in the FCS is rooted in the free fermion nature of the problem. However, using the Fermi-Bose mapping [16], the results immediately carry over to the bosonic Tonks-Girardeau gas. Since the latter one is accessible in cold-atom experiments [3,4], the measurement of the FCS might be feasible in the spirit of Ref. [9], where the full distribution function of an analogous observable could be extracted for trapped bosons.

The FCS is defined through the generating function

$$\chi(\lambda) = \langle \exp(i\lambda N_A) \rangle,$$
$$N_A = \int_A \rho(x) dx, \tag{1}$$

where N_A is the total number of particles in subsystem A, given by the integral of the density $\rho(x)$, and the expectation value is taken with the N-particle ground state of the system. For the spinless free Fermi gas, the FCS can be expressed as a Fredholm determinant [13,17]

$$\chi(\lambda) = \det[1 + (e^{i\lambda} - 1)K_A], \tag{2}$$

where K_A is an integral operator with the kernel

$$K_A(x, y) = \sum_{k=0}^{N-1} \varphi_k^*(x)\varphi_k(y) \tag{3}$$

given by the two-point correlation function restricted to the domain $x, y \in A$. It is constructed from the single-particle eigenfunctions of the Schrödinger equation

$$\frac{1}{2}\frac{d^2\varphi_k(x)}{dx^2} + [E_k - V(x)]\varphi_k(x) = 0, \tag{4}$$

where we have set $\hbar = m = 1$. For simplicity, we consider a symmetric $V(-x) = V(x)$ and monotonically increasing trapping potential such that for $x \to \infty$, one has $V(x) \to \infty$. The spectrum is thus discrete, and for each k, the solutions $\varphi_k(x)$ admit two classical turning points given by the condition $V(\pm x_{0k}) = E_k$. In the following, we will consider $x \geq 0$ since the real valued wave functions must obey the symmetry $\varphi_k(-x) = (-1)^k \varphi_k(x)$.

In the classically allowed region $x < x_{0k}$, the wave functions are oscillatory with exactly k nodes, whereas for $x > x_{0k}$, they must vanish exponentially. The wave function which approximates the exact solution on both sides is known as the *uniform Airy approximation* and can be derived from a semiclassical treatment of the Schrödinger equation [18]. Up to the normalization factor C_k, it is given by

$$\psi_k(x) = \frac{C_k}{\sqrt{\xi_k'(x)}} \text{Ai}[\pm \xi_k(x)], \qquad (5)$$

where the $+$ $(-)$ sign applies in the classically forbidden (allowed) region and the argument of the Airy function is given through

$$\xi_k(x) = \left[\frac{3}{2} \int_{x_1}^{x_2} p_k(z) dz \right]^{2/3}, \qquad (6)$$

with $x_1 = \min(x, x_{0k})$ and $x_2 = \max(x, x_{0k})$. The momentum in the integrand of Eq. (6) is defined as $p_k(x) = \sqrt{2|E_k - V(x)|}$ and the approximate energy niveaus E_k can be obtained from the Bohr-Sommerfeld quantization formula [19]

$$\int_{-x_{0k}}^{x_{0k}} p_k(z) dz = (k + 1/2)\pi. \qquad (7)$$

In particular, we will be interested in power-law potentials of the form $V_p(x) = x^p/p$ with some even integer p. Note that we have set the characteristic length scale of the trap to one, which can easily be restored using the arguments of trap-size scaling [20]. Then, the integral in Eq. (7) yields

$$E_k \approx [\mathcal{N}_p(k + 1/2)]^{2\theta},$$
$$\mathcal{N}_p = \frac{\sqrt{\pi}\,\Gamma(3/2 + 1/p)}{\sqrt{2}\,p^{1/p}\Gamma(1 + 1/p)}, \qquad (8)$$

with the exponent given by $\theta = p/(p + 2)$.

In general, the approximate wave functions $\psi_k(x)$ given by Eq. (5) rely on a semiclassical argument and are thus expected to reproduce the exact ones $\varphi_k(x)$ only for $k \gg 1$. In fact, however, the uniform Airy approximation gives very good results even for the eigenfunctions of the low-lying levels. This is demonstrated on Fig. 1 for the quartic potential $V_4(x)$. The eigenfunctions $\varphi_k(x)$ are calculated to a high precision by Numerov's method [21] and compared to $\psi_k(x)$ where the integrals in Eq. (6) can be given through
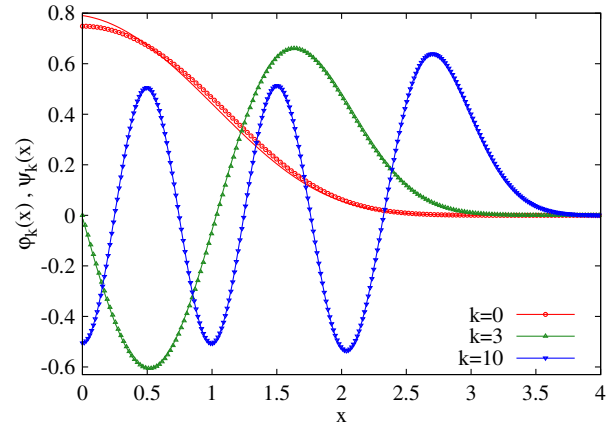

FIG. 1 (color online). Exact wave functions $\varphi_k(x)$ (symbols) and their uniform Airy approximations $\psi_k(x)$ (lines) for the quartic potential $V_4(x)$.

special (hypergeometric and incomplete beta) functions and evaluated numerically. While the overlap is reasonable for $k = 0$, the deviations are already very small for $k = 3$ and the two functions are essentially indistinguishable for $k = 10$.

Far away from the turning point x_{0k}, the uniform Airy approximation (5) reproduces the WKB approximation [18] which can be obtained from asymptotic expansions of $\text{Ai}[\pm \xi_k(x)]$. In particular, in the classically allowed region, one has

$$\psi_k(x) = \frac{C_k}{\sqrt{\pi p_k(x)}} \cos\left(\int_x^{x_{0k}} p_k(z) dz - \frac{\pi}{4} \right). \qquad (9)$$

For simple potential wells with two classical turning points, the normalization constant can be fixed by imposing [22]

$$C_k^2 \int_{-x_{0k}}^{x_{0k}} p_k^{-1}(z) dz = 2\pi. \qquad (10)$$

Differentiating Eq. (7) with respect to k, one arrives to the simple formula $C_k^2 = 2(dE_k/dk)$ and thus the normalization factor accounts for the spectral density.

The WKB form of the wave functions (9) gives a good approximation in the entire classically allowed regime not too close to the turning points but depends on the details of the potential $V(x)$. To find universal features of the FCS, we first focus on the *bulk* of the trapped gas and we choose the subsystem as the interval $A = [-\ell, \ell]$ deep in the high-density region $\ell \ll x_{0N}$. Expanding the functions $p_k(x)$ around $x = 0$, one obtains

$$\psi_k(x) \approx \frac{C_k}{\sqrt{\pi \sqrt{2E_k}}} \cos\left(k\frac{\pi}{2} - \sqrt{2E_k}\, x \right), \qquad (11)$$

which is valid up to linear terms in x. Substituting into Eq. (3), one arrives to the following sum

$$K_A(x, y) \approx \sum_{k=0}^{N-1} \frac{C_k^2}{2\pi\sqrt{2E_k}} \cos\sqrt{2E_k}(x - y)$$

$$+ \sum_{k=0}^{N-1} \frac{C_k^2(-1)^k}{2\pi\sqrt{2E_k}} \cos\sqrt{2E_k}(x + y). \qquad (12)$$

We are interested in the $N \to \infty$ limit of the FCS, where the first sum in Eq. (12) diverges. Therefore, we introduce new variables and define the scaling limit of the kernel as

$$K_r(u, v) = \lim_{N\to\infty} \frac{1}{\sqrt{2E_N}} K_A\left(\frac{u}{\sqrt{2E_N}}, \frac{v}{\sqrt{2E_N}}\right), \qquad (13)$$

where the subscript refers to the domain $[-r, r]$ of the kernel in the scaled variables with the effective length $r = \ell\sqrt{2E_N}$. Introducing the variable $z = \sqrt{E_k/E_N}$, the first sum in Eq. (12) can be converted into an integral while the second, alternating sum vanishes in the scaling limit. Writing $dz = (dE_k/dk)/\sqrt{4E_k E_N}$ and using the expression of C_k^2 in terms of the spectral density, one finds

$$K_r(u, v) = \frac{1}{\pi} \int_0^1 dz \cos z(u - v) = \frac{\sin(u - v)}{\pi(u - v)}. \qquad (14)$$

Hence, in the bulk scaling limit, we recover the sine kernel which appears in the theory of GUE random matrices. Indeed, the probability $E(n, t)$ of finding n eigenvalues in an interval $[-r, r]$ in the bulk of the GUE spectrum is given by [23]

$$E(n, r) = \frac{(-1)^n}{n!} \frac{d^n}{dz^n} \det(1 - zK_r)\bigg|_{z=1}. \qquad (15)$$

Fourier transforming Eq. (15) with respect to n yields the determinant in Eq. (2) with $K_A = K_r$, and thus the bulk FCS of the Fermi gas is identical to the bulk GUE eigenvalue statistics.

The bulk scaling limit can be tested through the cumulants $\kappa_m = (-i\partial_\lambda)^m \ln\chi(\lambda)|_{\lambda=0}$ of the particle number. In particular, we calculated the fluctuations κ_2 as a function of ℓ for the potential $V_4(x)$. The numerics is simplified by considering, instead of the integral operator K_A, the overlap matrix $\mathbf{C}_A$ with elements

$$C_{A,kl} = \int_A dx \varphi_k(x)\varphi_l(x) \qquad (16)$$

and using $\text{Tr}K_A^n = \text{Tr}\,\mathbf{C}_A^n$ [24]. Then, the FCS of Eq. (2) can be treated as a regular determinant and one has $\kappa_2(\ell) = \text{Tr}\,\mathbf{C}_A(\mathbf{1} - \mathbf{C}_A)$. This is then compared to existing results derived using the asymptotics of Fredholm determinants with the sine kernel [17]

$$\kappa_2(r) = \text{Tr}K_r(1 - K_r) = \frac{1}{\pi^2}(\log 4r + \gamma + 1), \qquad (17)$$

where γ is the Euler constant and we neglected terms vanishing for $N \to \infty$.

The results are shown in Fig. 2 for various N with the dashed lines representing the curves $\kappa_2(r)$. One can see a
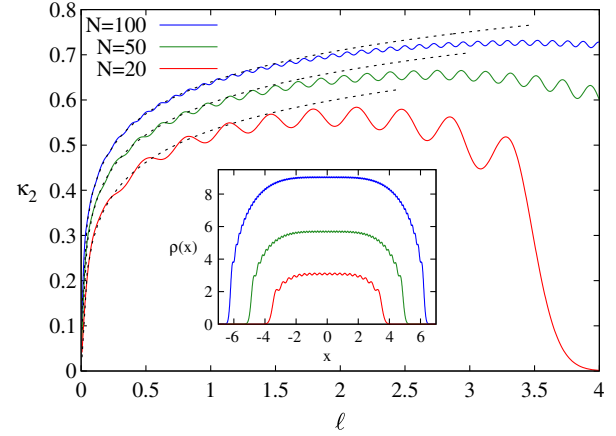

FIG. 2 (color online).   Particle number fluctuations for the quartic potential $V_4(x)$ in the interval $[-\ell, \ell]$ (solid lines) compared to the prediction of Eq. (17) in the bulk scaling limit (dashed lines) for various N. The inset shows the corresponding density profiles $\rho(x)$.

good agreement for small ℓ, but the solid curves $\kappa_2(\ell)$ deviate from the scaling prediction as soon as the segment size exceeds the size of the flat region in the densities $\rho(x) = K_A(x, x)$, shown in the inset. However, the amplitude of the $\mathcal{O}(x^2)$ corrections to Eq. (12) is proportional to $V''(0)/E_k$, which vanishes for $V_p(x)$ with $p \geq 4$, and thus the flat density region extends for higher p. Note also the strong oscillations in $\kappa_2(\ell)$ as well as in $\rho(x)$ that are results of the alternating sum in Eq. (12) and diminish for higher N. The bulk scaling was further tested by calculating the entanglement entropy $S(\ell) = -\text{Tr}[\mathbf{C}_A \ln \mathbf{C}_A + (1 - \mathbf{C}_A)\ln(1 - \mathbf{C}_A)]$ and comparing it to the scaling prediction $S(r)$ [24,25] with similarly looking results as in Fig. 2.

The other regime where universal features are expected to emerge is near the edge of the high-density region. Close to the classical turning point, the argument $\xi_k(x)$ of the Airy function in Eq. (5) can be expanded around x_{0k} and yields [18]

$$\psi_k(x) \approx \frac{C_k}{\sqrt{\alpha_k}} \text{Ai}[\alpha_k(x - x_{0k})], \qquad (18)$$

with $\alpha_k = [2V'(x_{0k})]^{1/3}$ giving the inverse of the typical length scale. The subsystem is now fixed as the interval $A = (x_{0N} + s/\alpha_N, \infty)$ starting close to the edge of the high-density region and extending to infinity. Note that this choice of the interval A strongly limits the terms contributing to the sum in Eq. (3) since the Airy functions in Eq. (18) are shifted gradually to the left for decreasing k and for $|x_{0k} - x_{0N}| \gg |s|/\alpha_N$ they become exponentially small in A. The edge scaling limit of the kernel is then defined as

$$K_s(u, v) = \lim_{N\to\infty} \frac{1}{\alpha_N} K_A\left(x_{0N} + \frac{u}{\alpha_N}, x_{0N} + \frac{v}{\alpha_N}\right), \qquad (19)$$
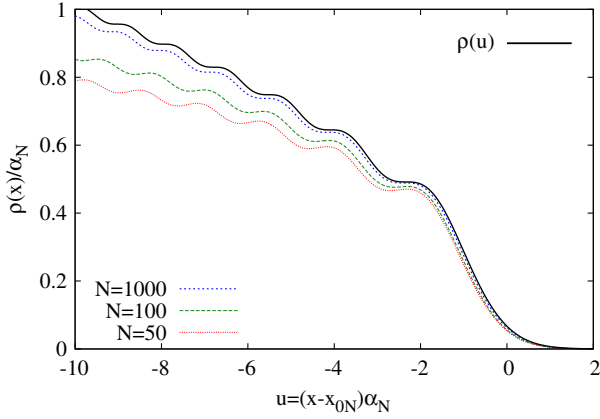
FIG. 3 (color online).   Rescaled density profiles (dashed lines) for $V_4(x)$ near the edge of the high-density region for various N. The uppermost (solid) line shows the $N \to \infty$ scaling function $\rho(u)$; see the text.
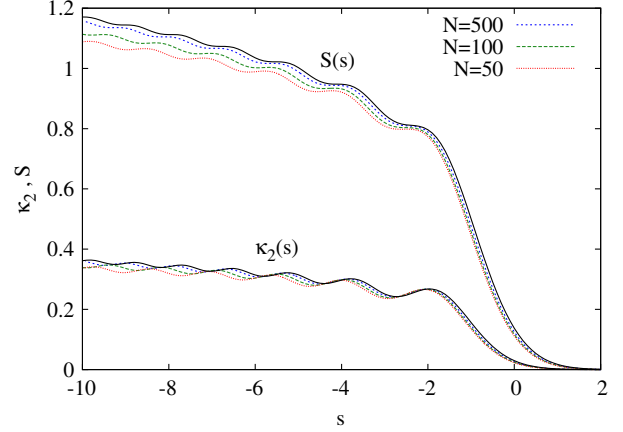

FIG. 4 (color online).   Particle number fluctuations κ_2 and entropy S in the scaled edge region for various N. The solid lines show the respective scaling functions $\kappa_2(s)$ and $S(s)$.

where the subscript refers to the domain $u, v \in (s, \infty)$ in the new variables. The factors α_k/α_N appearing in the arguments of the Airy functions can be approximated as

$$\frac{\alpha_k}{\alpha_N} \approx 1 + \frac{V''(x_{0N})}{3V'(x_{0N})}(x_{0k} - x_{0N}), \qquad (20)$$

and since $|x_{0k} - x_{0N}| \sim |s|/\alpha_N$, the second term vanishes in the limit $N \to \infty$ for any well behaved potential. We thus set $\alpha_k = \alpha_N$ in evaluating Eq. (19) and introduce $z = \alpha_N(x_{0N} - x_{0k})$. Using $x_{0k} = V^{-1}(E_k)$, one finds $dz \approx -C_k^2/\alpha_N^2$ and consequently

$$\begin{aligned} K_s(u, v) &= \int_0^\infty dz \mathrm{Ai}(u + z)\mathrm{Ai}(v + z) \\ &= \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)}{u - v}. \end{aligned} \qquad (21)$$

Thus, we recover the Airy kernel in the edge scaling limit. The FCS is then identical to the GUE edge eigenvalue statistics [26], which follows immediately from a formula analogous to Eq. (15) by replacing r with s.

As a first test of the scaling limit, we calculated the edge density profiles $\rho(x)$ in the potential $V_4(x)$ for various N and compared them to the density scaling function $\rho(u) = K_s(u, u) = [\mathrm{Ai}'(u)]^2 - u\mathrm{Ai}^2(u)$, with the result shown in Fig. 3. To reach larger particle numbers N, we used, instead of the exact wave functions $\varphi_k(x)$, the $\psi_k(x)$ in Eq. (5), which gives excellent results for the profile in the edge region. The finite-size scaling of the data can be inferred from the correction term in Eq. (20). For power-law potentials, one has $V_p''(x_{0N})/V_p'(x_{0N}) = (p - 1)/x_{0N}$, and multiplying by $(x_{0k} - x_{0N}) \sim \alpha_N^{-1}$, we obtain a $N^{-2/3}$ scaling of the finite-size corrections, which is consistent with the data in Fig. 3. Note that the exponent of N is independent of p; however, the prefactor $p - 1$ implies that the scaling collapse gets worse for larger p which we indeed observed in the numerics for $p = 6$.

The edge limit of the FCS is verified by calculating the particle number fluctuations κ_2 and entanglement entropy S of the interval (s, ∞) in the scaled coordinates. They are shown in Fig. 4 for various N and compared to the scaling prediction, calculated using a powerful numerical toolbox for the evaluation of Fredholm determinants arising in random matrix theory [27]. Both κ_2 and S converge slowly to their respective scaling functions.

In conclusion, we have shown the universality of the FCS for a noninteracting trapped Fermi gas in the bulk and edge regions, given by the respective eigenvalue statistics of GUE random matrices. Interestingly, the same universal limits emerge for *non-Gaussian* unitary random matrix ensembles, where the potential $V(x)$ appears in the exponential weight function and the semiclassical asymptotics of the corresponding orthogonal polynomials has to be analyzed [28,29]. This leads to results that are surprisingly similar to Eq. (5), even though the two problems coincide only in the trivial Gaussian case, $V(x)$ being the harmonic potential.

One expects that the same universality would emerge for trapped fermions on a lattice and might even generalize to other potentials. In fact, the connection between FCS and GUE statistics has recently been pointed out for the time evolution of lattice fermions from a step-initial condition [30]. However, in this case, the correlation matrix is unitarily equivalent to the one describing the ground state of a chain with a *linear* potential [31], and hence the universality of the FCS carries over to the gradient problem. Note that without the lattice, the Airy functions in Eq. (18) are the *exact* eigenstates and the spectrum is continuous; thus, the edge limit (21) describes the FCS in the entire high-density region while the bulk regime is missing.

It would also be interesting to consider the dynamical FCS after releasing the gas from the trap. In the harmonic case, the corresponding time-dependent Schrödinger equation can be solved exactly and the kernel has, up to an irrelevant phase factor, the equilibrium form in

appropriately rescaled variables [32]. Thus, the limiting scaling forms of the FCS are also unchanged. However, for general $V(x)$, the situation is more complicated and requires a careful analysis. Finally, in this nonequilibrium context, one could study the waiting time distribution between counting events where connections to random matrix theory have recently been revealed [33].

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