Nonuniversal Power-Law Spectra in Turbulent Systems

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Turbulence is generally associated with universal power-law spectra in scale ranges without significant drive or damping. Although many examples of turbulent systems do not exhibit such an inertial range, power-law spectra may still be observed. As a simple model for such situations, a modified version of the Kuramoto-Sivashinsky equation is studied. By means of semianalytical and numerical studies, one finds power laws with nonuniversal exponents in the spectral range for which the ratio of nonlinear and linear time scales is (roughly) scale independent.

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Introduction.—Turbulence can generally be described as spatiotemporal chaos in open systems, brought about by the nonlinear interaction of many degrees of freedom under out-of-equilibrium conditions. As such, it is ubiquitous in nature and in the laboratory, and represents a fundamental challenge to theoretical physics. Power-law energy spectra constitute one of the most prominent features of such systems. A first prediction along those lines was provided for three-dimensional Navier-Stokes turbulence as early as 1941 by Kolmogorov [1]. The typical physical picture is that power laws emerge on scales where both energy injection and dissipation are negligible, i.e., in the so-called inertial range. Here, on the basis of dimensional analysis, the value of the spectral exponent is considered to be determined entirely by the nonlinear energy transfer rate, implying universality.

Interestingly, there exist numerous examples of turbulent systems which display (simple or broken) power laws even in the presence of multiscale drive and/or damping. These include, e.g., flows generated by space-filling fractal square grids [2], the mesoscale dynamics in dense bacterial suspensions [3], and turbulence in astrophysical [4] and laboratory [5] plasmas. At least in the latter case, numerical studies suggest that the observed power-law exponents are not universal, however [5]. Instead, they appear to depend on the underlying linear physics of the system. This finding clearly calls for a theoretical understanding that can also help to interpret and guide experimental as well as numerical investigations.

In a previous investigation [6], a simple model for density fluctuation spectra in magnetized laboratory plasmas was proposed, which is based on the notion of disparate-scale interactions between small-scale eddies and large-scale structures like mean or zonal flows, also taking into account effective linear drive and/or (eddy or Landau) damping. In this context, universal broken power laws with an exponential cutoff were predicted. In the present Letter, we consider an alternative scenario. It is shown that one may obtain nonuniversal power laws in a certain spectral range if the ratio of the relevant nonlinear and linear time scales is (roughly) scale independent there.

Modified Kuramoto-Sivashinsky model.—To enable a semianalytical treatment, we will employ a modified version of one of the simplest models for spatiotemporal chaos and turbulence, the Kuramoto-Sivashinsky equation (KSE), which was originally put forward to describe turbulence in magnetized plasmas [7,8], chemical reaction-diffusion processes [9], and flame front propagation [10]. In general, it can be used for the study of nonlinear, spatially extended systems driven far from thermodynamic equilibrium by long-wavelength instabilities in the presence of appropriate (translational, parity, and Galilean) symmetries, and subject to short-wavelength damping. In its one-dimensional form, it reads

$$u_t = -uu_x - \mu u_{xx} - \nu u_{xxxx} \tag{1}$$

for the velocity field $u(x, t)$ with the positive parameters μ and ν. The equation is supplemented by the periodic boundary condition $u(L, t) = u(0, t)$ for all $t \geq 0$ and the initial condition $u(x, t = 0) = u_0(x)$. Considering only functions that belong to $C^4(\Omega) \cap L^2(\Omega)$ ensures that the system has finite total kinetic energy. Equation (1) can be rewritten in dimensionless units by substituting $u \to \mu u/L$, $t \to tL^2/\mu$, $x \to Lx$, and $\nu \to L^2\mu\nu$. The nondimensionalized form of the equation is the same as before, with the modification $\mu = 1$. In the following, we keep the damping parameter ν undetermined, but all quantitative results are obtained with $\nu = 1$. The second- and fourth-order spatial derivatives on the right-hand side of Eq. (1) provide an energy source and sink, respectively. Similar to three-dimensional Navier-Stokes turbulence, energy is injected on large scales and dissipated on small scales, with the nonlinear term providing the interscale transfer.

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The periodic boundary conditions suggest a representation of $u(x, t)$ in terms of a Fourier series defined as

$$u(x, t) = \sum_{n\in\mathbb{Z}} \hat{u}(k_n, t)e^{ik_n x}, \quad (2)$$

where the wave numbers $k_n = n(2\pi/L)$ are discrete and $n \in \mathbb{Z}$. From the condition that $u(x, t)$ is real, it follows that $\overline{\hat{u}(k_n, t)} = \hat{u}(-k_n, t)$ where the overbar denotes complex conjugation. Expressing Eq. (1) in terms of Fourier coefficients gives

$$\hat{u}_t(k_n) = -\frac{ik_n}{2}\sum_{m\in\mathbb{Z}} \hat{u}(k_n - k_m)\hat{u}(k_m) + (k_n^2 - \nu k_n^4)\hat{u}(k_n), \quad (3)$$

where we have suppressed the time dependence for the ease of notation. Linearly, each mode is characterized by the drive or damping rate $\gamma = k_n^2 - \nu k_n^4$. The nonlinear term does not inject or dissipate energy (i.e., summed over n, it gives zero), but only redistributes it among the modes.

We now change the linear term according to $(k_n^2 - \nu k_n^4) \rightarrow (k_n^2 - \nu k_n^4)/(1 + bk_n^4)$, such that we obtain the modified KSE

$$\hat{u}_t(k_n) = -\frac{ik_n}{2}\sum_{m\in\mathbb{Z}} \hat{u}(k_n - k_m)\hat{u}(k_m) + \frac{k_n^2 - \nu k_n^4}{1 + bk_n^4}\hat{u}(k_n), \quad (4)$$

with a constant damping rate of ν/b in the high wave number limit (see Fig. 1). One motivation for such a modification comes from the (gyro-)kinetic theory of magnetized plasmas where the growth rates of linear instabilities tend to a negative constant for large perpendicular wave numbers [5]. Moreover, this is one of the simplest realizations of a controlled deviation from the classical inertial range. Note that the real-space representation of the modified linear term is well defined for all functions in the domain $C^4(\Omega) \cap L^2(\Omega)$.

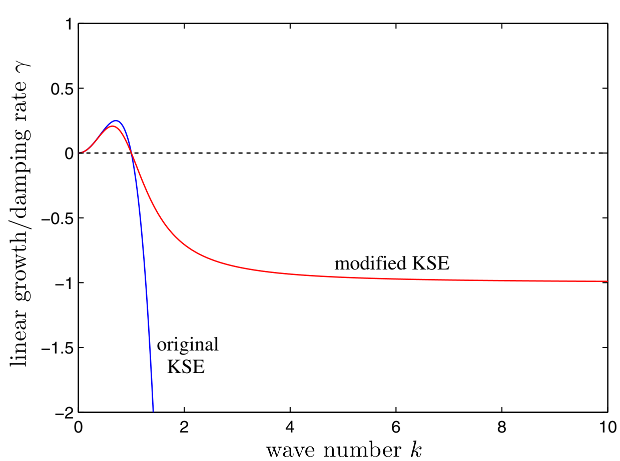

FIG. 1 (color online). Linear growth or damping rate γ as a function of wave number k for the original and modified Kuramoto-Sivashinski equation.

Energetics.—The energy budget equation corresponding to the modified version of Eq. (3) reads in Fourier space

$$\frac{\partial E(k_n, t)}{\partial t} = \sum_{m\in\mathbb{Z}} T(k_n, k_m, t) + 2\frac{k_n^2 - \nu k_n^4}{1 + bk_n^4}E(k_n, t), \quad (5)$$

where $T(k_n, k_m, t) = k_n \Im(\overline{\hat{u}(k_n, t)}\hat{u}(k_n - k_m, t)\hat{u}(k_m, t))$ and $E(k_n, t) = |\hat{u}(k_n, t)|^2$. We shall call the latter the energy of the k_n mode, while $T(k_n, k_m, t)$ will be referred to as the nonlinear energy transfer function. In contrast to incompressible fluid turbulence, it is not antisymmetric with respect to an interchange of k_n and k_m. Equation (5) reflects the fact that energy transfer takes place via three-wave interactions with $k_n + k_m + k_q = 0$. This transfer is conservative, i.e.,

$$\partial_t \mathcal{E}(k_n, t) + \partial_t \mathcal{E}(k_m, t) + \partial_t \mathcal{E}(k_q, t) = 0, \quad (6)$$

where $\mathcal{E}$ denotes the energy of a mode in a purely nonlinear subsystem that has been truncated to the three wave numbers k_n, k_m, and k_q. In a quasistationary turbulent state, the time average (denoted by $\langle\cdot\rangle_\tau$) of Eq. (5) reads

$$\sum_{m\in\mathbb{Z}} \langle T(k_n, k_m, t)\rangle_\tau + 2\frac{k_n^2 - \nu k_n^4}{1 + bk_n^4}E(k_n) = 0, \quad (7)$$

where $E(k_n)$ denotes $\langle E(k_n, t)\rangle_\tau$.

Energy transfer physics.—To gain insight into the turbulent dynamics of Eq. (4), it is solved numerically, employing the exponential time differencing fourth-order Runge-Kutta algorithm [11,12] and changing the normalized system size to 32π. We focus our investigations on the physics of the net nonlinear energy transfer. As it will turn out, the latter is dominated by nonlocal interactions in wave number space. Two neighboring high-k modes exchange energy via the coupling to a third mode with $k \sim 1$. This can be quantified by introducing the scale disparity parameter $S(k, p) = \max\{|k|, |p|, |k - p|\}/\min\{|k|, |p|, |k - p|\}$ defined in Refs. [13,14]. We shall follow the literature and refer to interactions with small (large) values of S as local (nonlocal). In Ref. [14], the observation was made that in Burgers turbulence, the net energy transfer in the inertial and dissipation ranges is dominated by local interactions, similar to Navier-Stokes turbulence. Our numerical simulations show that this type of behavior carries over to the original KSE. To our knowledge, this has not been shown before. The modified KSE exhibits a completely different scenario, however. The function $T(k_n, S)$, characterizing the energy transfer into mode k_n via triads with the scale disparity parameter S and defined over logarithmic S bands like in Ref. [14], is displayed in Fig. 2 as a function of S/k_n for three different values of k_n. In all three cases, one finds a strong peak at $S/k_n \sim 1$, implying that for Eq. (4), the net energy transfer at large wave numbers is dominated by nonlocal interactions, with a $k \sim 1$ mode acting as kind of a catalyst. Nevertheless, the energy

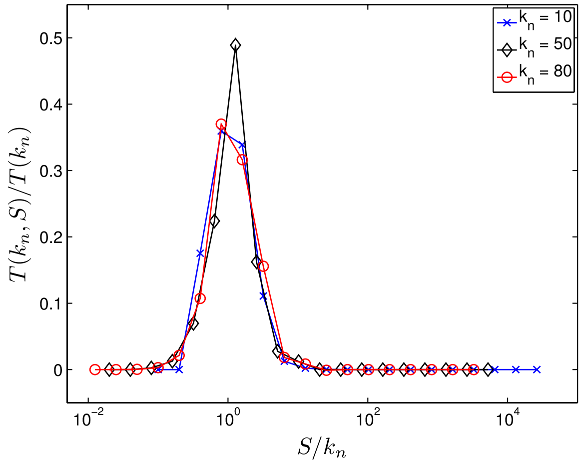

FIG. 2 (color online). Net energy transfer into mode k_n via triads with the scale disparity parameter S as a function of S normalized to k_n.

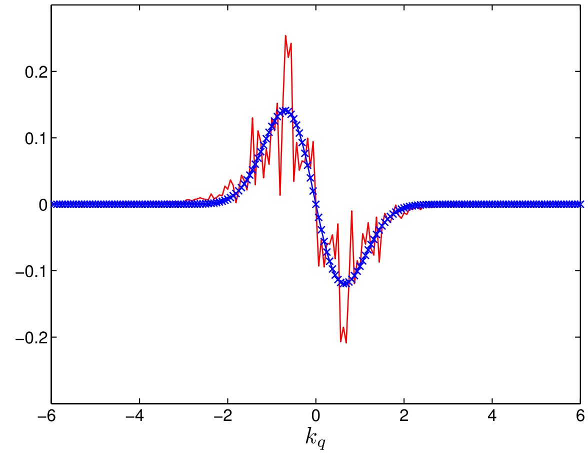

FIG. 3 (color online). Triple correlation normalized to $E(k_n + k_q^{\min})$ as a function of k_q for $k_n = 50$ and $b = 0.036$ compared to the model $f_P(k_q)/E(k_n + k_q^{\min})$ denoted by blue crosses.

cascade itself is local. The relevant triadic interactions can be realized in two different ways: $k_m \approx k_n$ and $k_n - k_m$ small or k_m small and $k_n - k_m \approx k_n$. Defining for convenience $k_q = k_m - k_n$, the nonlinearity becomes $k_n \sum_{q\in\mathbb{Z}} P(k_n, k_q)$ where the summand represents the triple correlation $\Im(\langle \overline{\hat{u}(k_n,t)}\,\overline{\hat{u}(k_q,t)}\,\hat{u}(k_n+k_q,t)\rangle_\tau)$. Considering the numerical results mentioned before, we have the following picture of the energy transfer in Fourier space. A large mode k_n receives energy (on average) mainly from the mode $k_n - k_d$, where k_d is a relatively small wave number in the drive range that mediates the transfer. Part of this energy is dissipated and the rest is forwarded primarily to the mode $k_n + k_d$ again via k_d. The first term in Eq. (7) has to balance the energy dissipated by the k_n mode which is the difference between the energy received by k_n and the one given by k_n.

Closure model and resulting energy spectra.—To find a closure model for Eq. (7), we search for an approximation of $P(k_n, k_q)$ at large wave numbers. The form of P produced by direct numerical simulations is shown in Fig. 3. It confirms the above picture of nonlinear energy transfer. The most dominant coupling is indeed with modes in the drive range, and from the minimum and maximum of the curve one sees that $k_d \approx 1/\sqrt{2}$, which is nearly the linearly most unstable mode for small b. The curve is approximately antisymmetric about $k_q = 0$. However, it is important that the antisymmetry is not exact: the maximum (at $k_q^{\max} \approx -k_d$) is slightly higher than the absolute value of the minimum (at $k_q^{\min} \approx -k_q^{\max}$). This discrepancy is the reason that, at high wave numbers, the spectrum decreases when k_n increases. Hence, summing over k_q will lead to a positive contribution that cancels the linear term in Eq. (7) which is negative for high k_n. For an approximation of the triple correlation function P, we model the form of the curve in Fig. 3 by $f_P(k_q) = -k_q E(k_n + k_d)\psi_\xi(k_q) - k_q E(k_n - k_d)\psi_{-\xi}(k_q)$ where $\psi_\xi(k_q)$ is a localized function centered at and symmetric around $k_q = \xi$ where the value of ξ can depend on k_d. The small asymmetry of f_P is provided by the slightly different prefactors $E(k_n - k_d)$ and $E(k_n + k_d)$ and k_q ensures the change in sign.

This model allows for an analytically tractable closure of the spectral energy budget equation at high k_n as

$$\sum_{q\in\mathbb{Z}} f_P(k_q) \approx \frac{1}{\Delta k}\int_{-\infty}^{+\infty} f_P(k_q)dk_q$$
$$= -\frac{\Phi(\xi)}{\Delta k}(E(k_n + k_d) - E(k_n - k_d)), \quad (8)$$

where $\Phi(\xi) = \int k_q \psi_\xi(k_q)dk_q = -\Phi(-\xi)$. Considering that $k_d \approx 1/\sqrt{2} \ll k_n$, we have $E(k_n - k_d) - E(k_n + k_d) \approx -\sqrt{2}dE/dk$ where a continuum of wave numbers is assumed. Hence,

$$-\frac{1}{\lambda}k\frac{dE}{dk} + 2\frac{k^2 - \nu k^4}{1 + bk^4}E(k) = 0, \quad (9)$$

where $\lambda = \Delta k/(2\sqrt{2}\Phi(\xi))$. In physical units, λ has the dimension of time, and at high k, $1/\lambda$ can be interpreted as a typical nonlinear frequency. The factor 2 takes into account that for high k_n, the nonlinear energy transfer function shows the same structure also at small k_m and large $k_n - k_m$ as we discussed previously. The solution of the above differential equation is readily obtained as

$$E(k) = \tilde{E}_0 \exp\left(\frac{\lambda}{\sqrt{b}}\arctan(\sqrt{b}k^2) - \frac{\lambda\nu}{2b}\ln(1 + bk^4)\right), \quad (10)$$

with $\tilde{E}_0$ a constant of integration. In the limit of large wave numbers, the second term in the exponent dominates and leads to

$$E(k) = E_0 k^{-2\lambda\nu/b}, \quad (11)$$

where E_0 is a constant. This is a power-law spectrum with a nonuniversal scaling exponent. The latter is set by the ratio

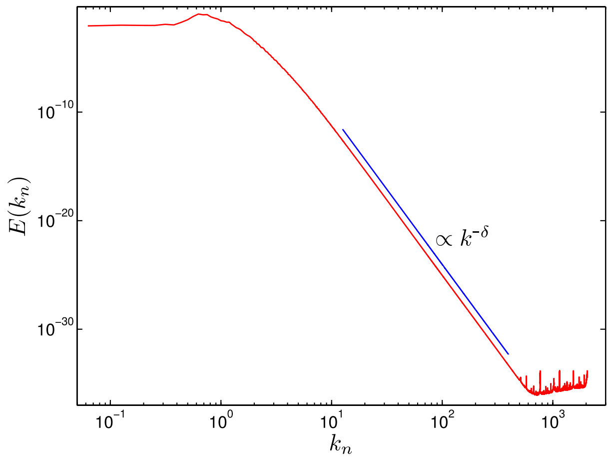

FIG. 4 (color online). Fit of a power law to the high-k end of the energy spectrum for $b = 0.036$.

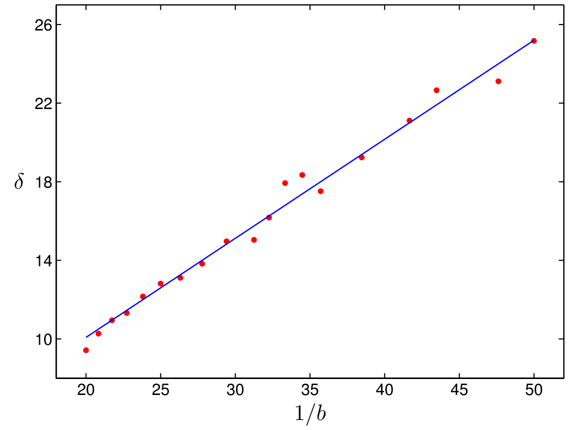

FIG. 5 (color online). The exponent $\delta = 2\lambda\nu/b$ in the power law $E(k) \propto k^{-\delta}$ as a function of the damping rate $1/b$ for $\nu = 1$.

of the linear damping rate ν/b and the nonlinear frequency $1/\lambda$. An analytically convenient form for ψ is $\psi(k_q) = a_1 e^{(k_q - a_2)^2/a_3}$ where a_1, a_2, and a_3 are free parameters. Their values may be determined by a fit to the numerical data which is shown with blue crosses in Fig. 3. One can easily check that this particular choice for ψ gives for the ratio between the maximum and the absolute value of the minimum

$$\frac{f_P(-k_d)}{|f_P(k_d)|} \approx \frac{(1 + re^{4k_d a_2/a_3})}{(1 + re^{-4k_d a_2/a_3})} e^{-4k_d a_2/a_3}, \qquad (12)$$

where $r = E(k_n - k_d)/E(k_n + k_d)$. A least squares fit gives $a_1 \approx 0.1403$, $a_2 \approx 0.2578$, and $a_3 \approx 0.7564$ which leads to $f_P(-k_d)/|f_P(k_d)| \approx 1.217$. The corresponding numerical value is 1.184 and the good agreement signifies that the particular form of f_P chosen captures well the important asymmetry of the triple correlation.

Consistency checks.—To check for consistency, we also computed numerically the energy spectra for different values of the damping rate ν/b. As can be seen in Fig. 4, one can thus confirm that a constant high-k damping rate leads to an energy spectrum in the form of a power law (in contrast to the standard KSE, which displays an exponential falloff), and that the associated spectral exponents are indeed proportional to the damping rate. According to a linear fit to the data in Fig. 5, one obtains $\lambda \approx 0.25$, whereas the fitting procedure in the context of Fig. 3 yields a slightly larger value of $\lambda \approx 0.4$. The reason for this is that the area enclosed by the ragged curve (which is essential for computing the precise value of the energy transfer) is nearly 1.6 times larger than the area under the blue curve in Fig. 3. Taking this correction into account, the two approaches agree very well, providing a consistent overall picture.

Conclusions.—Motivated by the fact that many turbulent systems in nature as well as in the laboratory exhibit power-law spectra even in the absence of a clean inertial range, we studied as a simple model system a modified version of the Kuramoto-Sivashinsky equation, with a constant high-k damping rate. Via semianalytical and numerical studies, we demonstrated the existence of power laws with nonuniversal scaling exponents in the spectral range for which the ratio of nonlinear and linear time scales is (roughly) scale independent. Such situations may arise in various physical systems with multiscale drive and/or damping, including, in particular, magnetized laboratory plasmas [15]. In this context, the present work provides a plausible explanation for the observation of nonuniversal power laws in numerical studies [5].

Another possible application of these findings is kinetic Alfvén wave turbulence, as it is thought to occur, e.g., in the solar wind. In this case, one has to compare the nonlinear energy transfer rates (which scale like $k_\perp^{4/3}$ at sub-ion-gyroradius scales) with the Landau damping rates of kinetic Alfvén waves. The latter may have rather complex k dependencies, with details depending on the ion-to-electron temperature ratio and the plasma β [16]. There seem to exist parameter regimes and k ranges for which the ratio of linear and nonlinear frequencies is roughly scale independent, such that nonuniversal power-law spectra may emerge.

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[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
[2] R. Stresing, J. Peinke, R. E. Seoud, and J. C. Vassilicos, Phys. Rev. Lett. **104**, 194501 (2010).
[3] H. H. Wensink, J. Dunkel, S. Heidenreich, K. Drescher, R. E. Goldstein, H. Löwen, and J. M. Yeomans, Proc. Natl. Acad. Sci. U.S.A. **109**, 14308 (2012).

[4] B. G. Elmegreen and J. Scalo, Annu. Rev. Astron. Astrophys. **42**, 211 (2004).
[5] T. Görler and F. Jenko, Phys. Rev. Lett. **100**, 185002 (2008); Phys. Plasmas **15**, 102508 (2008).
[6] Ö. D. Gürcan, X. Garbet, P. Hennequin, P. H. Diamond, A. Casati, and G. L. Falchetto, Phys. Rev. Lett. **102**, 255002 (2009).
[7] R. E. LaQuey, S. M. Mahajan, P. H. Rutherford, and W. M. Tang, Phys. Rev. Lett. **34**, 391 (1975).
[8] B. I. Cohen, J. A. Krommes, W. M. Tang, and M. N. Rosenbluth, Nucl. Fusion **16**, 971 (1976).
[9] Y. Kuramoto and T. Tsusuki, Prog. Theor. Phys. **52**, 1399 (1974); Prog. Theor. Phys. Suppl. **64**, 346 (1978).
[10] G. I. Sivashinsky, Acta Astronaut. **4**, 1177 (1977); **6**, 569 (1979).
[11] S. M. Cox and P. C. Matthews, J. Comput. Phys. **176**, 430 (2002).
[12] A.-K. Kassam and L. N. Trefethen, SIAM J. Sci. Comput. **26**, 1214 (2005).
[13] Y. Zhou, Phys. Fluids A **5**, 2511 (1993).
[14] S. S. Girimaji and Y. Zhou, Phys. Lett. A **202**, 279 (1995).
[15] B. Teaca, A. B. Navarro, F. Jenko, S. Brunner, and L. Villard, Phys. Rev. Lett. **109**, 235003 (2012).
[16] G. G. Howes, S. C. Cowley, W. Dorland, G. W. Hammett, E. Quataert, and A. A. Schekochihin, Astrophys. J. **651**, 590 (2006).