Quantum Enhanced Multiple Phase Estimation

Peter C. Humphreys,* Marco Barbieri, Animesh Datta, and Ian A. Walmsley†
Department of Physics, Clarendon Laboratory, University of Oxford, Oxford OX1 3PU, United Kingdom
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We study the simultaneous estimation of multiple phases as a discretized model for the imaging of a phase object. We identify quantum probe states that provide an enhancement compared to the best quantum scheme for the estimation of each individual phase separately as well as improvements over classical strategies. Our strategy provides an advantage in the variance of the estimation over individual quantum estimation schemes that scales as $\mathcal{O}(d)$, where d is the number of phases. Finally, we study the attainability of this limit using realistic probes and photon-number-resolving detectors. This is a problem in which an intrinsic advantage is derived from the estimation of multiple parameters simultaneously.

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Introduction.—Recent developments in quantum metrology point to a new frontier of parameter estimation in which exploiting quantum states enables higher precision than can be achieved using only classical resources. Much of the work in this field to date has been directed towards the estimation of a single Hamiltonian parameter. This has been explored both theoretically [1–13] and experimentally, with the estimation of optical phase shifts by means of interferometry providing the dominant paradigm, in the setting of photonic systems as the leading platform [14–18].

One of the most important metrology problems to the wider research community is that of microscopy and imaging. Producing a quantum advantage in imaging would be of significant benefit in fields such as biology, particularly for the imaging of samples that are sensitive to the total illumination. Various approaches to quantum imaging have been proposed, typically exploring methods for increasing the diffraction limited resolution of optical imaging systems [19–25]. A recent classical investigation of quantum enhanced imaging made use of point estimation theory, quantifying differences between images by means of a single parameter [26]. However, imaging is inherently a multiparameter estimation problem, and deeper insights can be gained by studying it as such.

In this Letter, we consider a discretized model for phase imaging based on this approach. Phase imaging is a cornerstone of optical microscopy, typically realized using the related techniques of phase contrast and differential interference contrast imaging [27], that allows differences in refractive index to be detected in otherwise transparent media. So far, the potential for quantum enhancements to these techniques has yet to be explored. Our approach maps phase imaging onto the problem of multiple simultaneous phase estimation.

Our results provide a strategy for the estimation of multiple phases using correlated quantum states, in which the multiparameter nature of the problem leads to an intrinsic benefit when exploiting quantum resources. A surprising outcome of our analysis is that our quantum strategy provides an $\mathcal{O}(d)$ advantage, where d is the number of phases, over the optimal quantum individual estimation scheme of using $N00N$ states [7]. We further show that a resource advantage can be provided over the best classical phase estimation schemes.

Phase imaging.—We adopt a discretized model of phase imaging, in which we address the question of how to estimate d independent phases most efficiently with N photons. We note that earlier works have explored other aspects of multiple parameter estimation from a quantum information perspective. In the case of the estimation of parameters characterizing a set of noncommuting unitary operations, it was shown that entangled states and measurements can attain the Heisenberg limit in the number of photons used in each probe state [28–30]. In the commuting case, the problem of estimating d phases with an ensemble of single-photon probe states has been considered. A Bayesian approach showed that the cost of estimation increases with the number of parameters involved [31], and a Fisher information based approach showed that entangling two multilevel systems provides no advantage over using a single multilevel system [32]. More recently, the error associated with estimating two phases using three and four mode interferometers (and three and four photons, respectively) has been investigated [33].

We now turn to the general case of determining multiple independent phases by distributing N photons across a probe state in an optimal manner. Our discretized phase imaging model consists of a $d+1$-mode interferometer with a preparation, an interaction, and a measurement stage as in Fig. 1. The preparation stage creates an arbitrary pure input state of the form

$$|\psi\rangle = \sum_{k=1}^{D} \alpha_k |N_{k,0}, N_{k,1}, ..N_{k,d}\rangle \equiv \sum_{k=1}^{D} \alpha_k |\mathbf{N}_k\rangle. \quad (1)$$

The distribution of photons in a given configuration k is expressed compactly by a vector $\mathbf{N}_k$, composed of

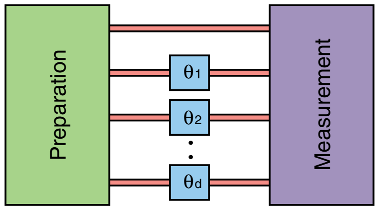

FIG. 1 (color online). Discretized phase imaging model. We consider the simultaneous estimation of d phases using a setup consisting of state preparation (green), independent phase application in each mode (blue), and state measurement (purple).

individual components $N_{k,m}$ that give the number of photons in mode m, such that $\sum_{m=0}^{d} N_{k,m} = N$. $D = (N+d)!/N!d!$ is the number of distinct configurations of distributing N photons across $d+1$ modes. Exploiting the global phase freedom of the problem, we choose the mode labeled 0 as our reference mode and therefore, the modes registering the phases are labeled $\{1, \ldots, d\}$. To each of these configurations, we associate a complex amplitude α_k. The state is normalized by requiring that $\sum_{k=1}^{D} |\alpha_k|^2 = 1$.

The input state acquires a phase through the unitary transformation $U_{\boldsymbol{\theta}} = e^{i\sum_{m=1}^{d} \hat{N}_m \theta_m}$, where θ_m is the phase accrued and $\hat{N}_m$ the number operator for mode m. Denoting $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_d\}$, the evolved state is given by

$$|\psi_{\boldsymbol{\theta}}\rangle = U_{\boldsymbol{\theta}}|\psi\rangle = \sum_{k=1}^{D} \alpha_k e^{i\mathbf{N}_k \cdot \boldsymbol{\theta}} |\mathbf{N}_k\rangle. \quad (2)$$

The precision of the estimate of $\boldsymbol{\theta}$, governed by its covariance matrix $\mathrm{Cov}(\boldsymbol{\theta})$, is lower bounded via the the quantum Cramér-Rao bound (QCRB) [34]

$$\mathrm{Cov}(\boldsymbol{\theta}) \geq (M \mathcal{I}_{\boldsymbol{\theta}})^{-1}, \quad (3)$$

where $\mathcal{I}_{\boldsymbol{\theta}}$ is the quantum Fisher information (QFI) matrix and M is the classical contribution from repeating the experiment [35]. This is a $d \times d$ sized matrix inequality which is satisfied when $\mathrm{Cov}(\boldsymbol{\theta}) - (M\mathcal{I}_{\boldsymbol{\theta}})^{-1}$ is a positive matrix. The QFI matrix is defined as [34,36]

$$[\mathcal{I}_{\boldsymbol{\theta}}]_{l,m} = \frac{1}{2} \langle \psi_{\boldsymbol{\theta}} | (L_l L_m + L_m L_l) | \psi_{\boldsymbol{\theta}} \rangle, \quad (4)$$

where the operators L_m are called symmetric logarithmic derivatives, defined for pure states by

$$L_{\theta_l} = 2(|\partial_{\theta_l} \psi_{\boldsymbol{\theta}}\rangle\langle \psi_{\boldsymbol{\theta}}| + |\psi_{\boldsymbol{\theta}}\rangle\langle \partial_{\theta_l} \psi_{\boldsymbol{\theta}}|). \quad (5)$$

We show in Supplemental Material Section I [37] that the QFI matrix associated with the estimation of the phases in our interferometer is

$$\mathcal{I}_{\boldsymbol{\theta}} = 4\sum_i |\alpha_i|^2 \mathbf{N}_i \mathbf{N}_i^T - 4\sum_{i,j} |\alpha_i|^2 |\alpha_j|^2 \mathbf{N}_i \mathbf{N}_j^T. \quad (6)$$

For this Letter, we consider only the ideal case of pure states. In this case, the bound is guaranteed to be saturated if the condition $\mathrm{Im}[\langle \psi_{\boldsymbol{\theta}} | L_l L_m | \psi_{\boldsymbol{\theta}} \rangle] = 0$ is satisfied, which is true in our case for all l, m, and $\boldsymbol{\theta}$ [38]. Thus, the QCRB can be saturated for the estimation of multiple phases simultaneously given the input states we study in Eq. (1).

Since we are interested in purely quantum enhancements, we henceforth set $M = 1$ in Eq. (3). Then taking the trace of both sides gives a lower bound on the total variance of all the phases estimated

$$|\Delta\boldsymbol{\theta}|^2 \equiv \sum_{m=1}^{d} \delta\theta_m^2 \equiv \mathrm{Tr}[\mathrm{Cov}(\boldsymbol{\theta})] \geq \mathrm{Tr}[\mathcal{I}_{\boldsymbol{\theta}}^{-1}]. \quad (7)$$

The saturation of the matrix QCRB implies a saturation of the above inequality, and in the rest of this Letter, we will be concerned with minimizing $|\Delta\boldsymbol{\theta}|^2$.

Optimal probe states.—It is well-known that the best quantum probe of N photons for estimating a single phase is the $N00N$ state which saturates the corresponding QCRB and attains the Heisenberg limit of $|\Delta\boldsymbol{\theta}|^2 = 1/N^2$ [7]. The origin of this scaling is the number variance for the two modes, which scales as N^2. Based on this intuition, we consider a generalization in which our quantum probe is a coherent superposition of N photons in one of the modes and none in any of the other d modes. Due to the symmetry of our problem over the d modes in which we choose to estimate the phases, we consider the quantum probe

$$|\psi\rangle = \alpha(|0, N, \ldots, 0, 0\rangle + |0, 0, \ldots, N, 0\rangle + \cdots + |0, 0, \ldots, 0, N\rangle) + \beta|N, 0, \ldots, 0, 0\rangle, \quad (8)$$

such that $d\alpha^2 + \beta^2 = 1$. For these states, the QFI matrix can be found using Eq. (6). As the QFI only depends on the amplitude of α and β, we assume that they are real without loss of generality. Under this assumption, β is uniquely determined by the normalization condition and is, therefore, no longer an independent variable,

$$[\mathcal{I}_{\boldsymbol{\theta}}]_{l,m} = 4N^2(\delta_{l,m}\alpha^2 - \alpha^4). \quad (9)$$

The minimum total variance in Eq. (7) can be found by minimizing $\mathrm{Tr}[\mathcal{I}_{\boldsymbol{\theta}}^{-1}]$ via differentiation with respect to α,

$$|\Delta\boldsymbol{\theta}_s|^2 = \frac{(1+\sqrt{d})^2 d/4}{N^2}, \quad (10)$$

for $\alpha = 1/\sqrt{d + \sqrt{d}}$. We label this state as $|\psi_s\rangle$. This bound should now be compared to the variance of estimating the d phases $\boldsymbol{\theta}$ using d separate interferometers independently. Assuming for simplicity that d is a factor of N [39], the best quantum strategy uses $N00N$ states with a maximum of N/d photons, with a variance of d^2/N^2 for each phase. Then, the total variance for this approach is $|\Delta\boldsymbol{\theta}_{\mathrm{ind}}|^2 = d^3/N^2$. In a classical strategy where the probe is restricted to uncorrelated coherent states of the form $\otimes_{i=1}^{d} |\alpha_i\rangle$, such that $\sum_{i=1}^{d} \langle \alpha_i | \hat{N}_i | \alpha_i \rangle = N$, the total variance is $|\Delta\boldsymbol{\theta}_{\mathrm{clas}}|^2 = d^2/N$.

As expected, both the quantum strategies follow the Heisenberg scaling in the total number of photons for the total variance. However, the quantum simultaneous strategy has an additional advantage over the others. Comparing the three bounds, we find

$$|\Delta\boldsymbol{\theta}_s|^2 \leq |\Delta\boldsymbol{\theta}_{\rm ind}|^2 \leq |\Delta\boldsymbol{\theta}_{\rm clas}|^2, \quad (11)$$

where the first inequality is strict for $d > 1$, and the second for $d < N$. As typical instances would consist of many more photons than the number of parameters to be estimated, we are guaranteed that our strategy of simultaneous quantum estimation is better than individual estimation. Furthermore, the advantage, shown in Fig. 2, over the best quantum strategy of independent estimation improves linearly with the number of phases, scaling as $1/4d$. This is our main result.

The advantage of simultaneous quantum phase estimation is $\mathcal{O}(d)$ [40], and one might wonder if this is the maximum possible advantage that can be obtained by quantum probes of the form Eq. (1). We do not have an analytic proof that this is the case, and numerical searches are hampered by the unfavorable scaling in the number of state configurations D, since in the limit of large N, d, this is $D \sim 2^{(N+d)S}$, where S is the binary Shannon entropy of $d/(N+d)$. We have, however, performed a numerical optimization to find the states with the minimal total variance in the parameter ranges $d = 1{:}6$, $n = 1{:}6$ and found that the optimal states always have the form in Eq. (8).

The definition of a trial is central for a proper accounting of resources and therefore for identifying any quantum advantages. We have defined a trial to consist of a complete characterization of all d phases using N photons. Alternative definitions can be considered, such as when a trial simply consists of a single illumination of the sample with N photons, with freedom to use these photons differently in each trial. In the latter case, an alternative strategy of using all N photons to estimate a single phase in a given trial, switching through the phases to be

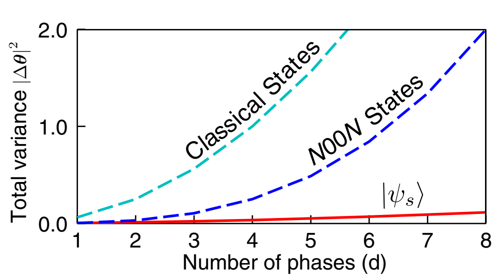

FIG. 2 (color online). Strategies for multiple phase estimation using $N = 16$ photons. The red line gives the total variance $|\Delta\boldsymbol{\theta}_s|^2$ for the quantum simultaneous strategy using the states $|\psi_s\rangle$, the blue dashed line gives the variance $|\Delta\boldsymbol{\theta}_{\rm ind}|^2$ achievable using $N00N$ states, and the cyan dashed line gives the variance $|\Delta\boldsymbol{\theta}_{\rm clas}|^2$ for an equivalent classical state.

estimated in each trial, can also produce an $\mathcal{O}(d^2)$ scaling in the variance. Now, however, Nd photons are required in order to provide one set of estimates for the phases, this will lead to d fewer trials per phase, and therefore, a factor $1/d$ slower convergence to the Cramér-Rao bound.

Optimal measurements.—We now turn to the problem of identifying measurements that can realize the quantum advantages in multiphase estimation. Although we know that the QRCB can be saturated in principle, it is important to identify the measurements that allow us to do so in practice. In Supplemental Material Section II [37], we consider positive-operator-valued measurement (POVM) sets in which one element is a projection onto the probe state after transformation by the interferometer with phases $\boldsymbol{\theta}_s$. We show that these sets saturate the QCRB at this specific point in parameter space, and that the associated classical Fisher information matrix is equal to the QFI matrix.

One such construction, for the probe $|\psi_w\rangle$, is given by $\boldsymbol{\Pi} = \{|\Upsilon_l\rangle\langle\Upsilon_l|\}$, where $|\Upsilon_l\rangle = \sum_m \Upsilon_{l,m}|\mathbf{N}'_m\rangle$ and $|\mathbf{N}'_m\rangle$ is the configuration with N photons in mode m and no photons in any other mode. The component amplitudes are given by

$$\Upsilon_{l,m} = \begin{cases} \sqrt{\frac{(l-1)!}{(l+1)!}}, & m \leq l-1; \\ -\sqrt{\frac{l}{l+1}}, & m = l; \\ 0, & m > l, \end{cases} \quad (12)$$

for $l = 1, \ldots, d$ and $m = 0, \ldots, d$. The additional $l = 0$ state is given by $\Upsilon_{0,m} = 1/\sqrt{d+1}$. This set saturates the QFI for $\boldsymbol{\theta}_s = 0$. An explicit construction for $d = 3$ is shown in Table 1 in the Supplemental Material Section II [37]. A similar set of projectors can also be obtained for the optimal state given by Eq. (8).

As can be seen, the probability $p_l = |\langle\psi_w|\Upsilon_l\rangle|^2$ associated with each outcome is transparently related to the phases, with p_1 involving only θ_1, p_2 only θ_1, θ_2, and so on. This suggests that an estimator could be easily created that would allow one to determine the probability distribution for the phases given a set of experimental outcomes.

Realistic probes and measurements.—The optimal probe states and measurements involve quantum correlated states that may be challenging to implement in practice. In this section, we present examples of probe states that may be relatively easier to prepare, and show the enhancements predicted earlier are achievable using realistic measurements.

For single parameter estimation, it was shown that the Holland-Burnett (HB) state [1,10], generated by interfering two pure N photon states on a 50/50 beam splitter, can also lead to a $1/N^2$ Heisenberg scaling in estimation. This state is significantly easier to generate than the ideal $N00N$ state since it does not rely on the use of optical nonlinear interactions or quantum gates. Further, these states are also

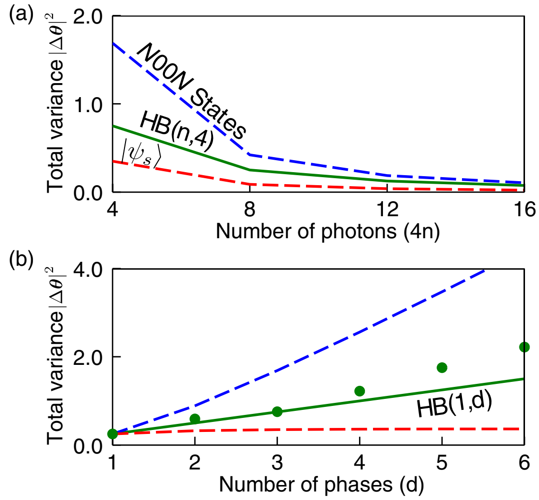

FIG. 3 (color online). (a) *Realistic probes:* The green line gives numerical calculations of the total variance from the QCRB for the simultaneous estimation of 4 phases using HB$(n, 4)$ states as a function of n. For comparison, the blue and red dashed lines give the QCRB for equivalent $N00N$ and $|\psi_s\rangle$ states, respectively. (b) *Realistic measurements:* The green dots show the total variance for the simultaneous estimation of d phases using a HB$(1, d)$ state and a measurement apparatus consisting of a Fourier multiport followed by PNRDs. The green line gives the QCRB variance error for the same HB$(1, d)$ state, while the blue and red dashed lines again give the QCRB for equivalent $N00N$ and $|\psi_s\rangle$ states, respectively.

known to be close to optimal with respect to losses in the quantum sensor.

We consider a multimode generalization of these states, generated by means of Fourier multiport devices that implement a quantum Fourier transform (QFT) [18,33,41], for two modes this is equivalent to a 50/50 beam splitter. As in the creation of HB states, n photons are input into each mode of the QFT device, leading to an $N = n(d + 1)$ photon state output, that we denote HB(n, d). This state is then used for phase estimation. Our results include as a special case, recent work by Spagnolo *et al.* [33] which explored the QFI associated with this device for the specific case of $d = 2, 3$ with $n = 1$.

Figure 3 shows numerical calculations of the expected variance of estimation for these states, calculated from the QCRB [Eq. (7)]. Our calculations suggest that the HB$(1, d)$ states give the closest performance to the probe $|\psi_s\rangle$ previously considered. As the number of photons input into each mode is increased, the variance of estimation moves away from that achievable using $|\psi_s\rangle$, and approaches the error for simultaneous phase estimation using $N00N$ states. The observed decrease in performance of the HB(n, d) state is because the probability amplitude associated with the terms in which the photons are highly bunched in one mode decreases significantly with n and d [42], and it is these terms that are most sensitive to the phases in the interferometer. It is also this property, however, that ensures that these states are robust against loss in the single phase case [10], something that is not a property of the $N00N$ states. The degree to which multiphase estimation can be loss tolerant is not yet known.

Although HB(n, d) states do not perform as well as comparable $|\psi_s\rangle$ probe states, they do at least as well as $N00N$ states, which are just as challenging to prepare as $|\psi_s\rangle$ states. The ease of experimental generation of multimode HB states may make them an attractive candidate for multiple phase estimation protocols. This is particularly the case for $n = 1$ states, which could be produced using heralded single photons, and demonstrate the best comparative performance over $N00N$ states of the same photon number.

In addition to the challenges of optimal state preparation, the optimal measurements involve projections onto complex multiphoton states, and thus, they they may not be experimentally feasible. It is therefore important to show that an experimentally realistic measurement scheme exists that can achieve or approach the QCRB. We calculated numerically the variance of the phase estimation given by the classical Fisher information for HB(n, d) states using a detection scheme in which the different modes are combined using a balanced Fourier multiport device, followed by ideal photon-number-resolving detectors (PNRD). Since the probability of different combinations of detector outcomes depends on the phases, a maximum likelihood scheme could in principle be used to estimate the phases given a set of measured detector outcomes. As the accuracy of estimation is dependent on the value of $\boldsymbol{\theta}$, numerical optimization over the phases was used determine the minimum possible error. Calculations were carried out for the multimode HB$(1, d)$ states [the class of HB(n, d) states that exhibited the best performance], and are shown in Fig. 3(b). The calculated variance is comparable to the QFI, and below that achievable using $N00N$ states.

Conclusions.—Our analysis of imaging as a multiparameter estimation problem presents an alternative approach to the typical methods based on enhancing diffraction limits, and may be of interest for other quantum enhanced imaging problems. In addition, our results should be of wide interest as many problems, such as strain sensing, range finding, and gravitational wave detection can be recast as optical phase estimation [43]. They should also motivate an investigation into the nature of the quantum resources at the root of the enhancement shown.

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*Corresponding author.
peter.humphreys@physics.ox.ac.uk
†Corresponding author.
i.walmsley1@physics.ox.ac.uk

[1] M. J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[2] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).
[3] V. Giovannetti, S. Lloyd, and L. Maccone, Nature (London) **412**, 417 (2001).
[4] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[5] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[6] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[7] H. Lee, P. Kok, and J. P. Dowling, J. Mod. Opt. **49**, 2325 (2002).
[8] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. Lett. **102**, 040403 (2009).
[9] M. G. Genoni, S. Olivares, D. Brivio, S. Cialdi, D. Cipriani, A. Santamato, S. Vezzoli, and M. G. A. Paris, Phys. Rev. A **85**, 043817 (2012).
[10] A. Datta, L. Zhang, N. Thomas-Peter, U. Dorner, B. J. Smith, and I. A. Walmsley, Phys. Rev. A **83**, 063836 (2011).
[11] G. M. D'Ariano and M. G. A. Paris, Phys. Rev. A **55**, 2267 (1997).
[12] G. D'Ariano, C. Macchiavello, and M. Sacchi, Phys. Lett. A **248**, 103 (1998).
[13] J. Söderholm, G. Björk, B. Hessmo, and S. Inoue, Phys. Rev. A **67**, 053803 (2003).
[14] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Nature (London) **429**, 161 (2004).
[15] T. Nagata, R. Okamoto, J. L. O'brien, K. Sasaki, and S. Takeuchi, Science **316**, 726 (2007).
[16] I. Afek, O. Ambar, and Y. Silberberg, Science **328**, 879 (2010).
[17] H. Yonezawa, D. Nakane, T. A. Wheatley, K. Iwasawa, S. Takeda, H. Arao, K. Ohki, K. Tsumura, D. W. Berry, T. C. Ralph, H. M. Wiseman, E. H. Huntington, and A. Furusawa, Science **337**, 1514 (2012).
[18] A. Vourdas and J. A. Dunningham, Phys. Rev. A **71**, 013809 (2005).
[19] V. Giovannetti, S. Lloyd, L. Maccone, and J. Shapiro, Phys. Rev. A **79**, 013827 (2009).
[20] E. Brainis, Opt. Express **19**, 24 228 (2011).
[21] O. Schwartz and D. Oron, Phys. Rev. A **85**, 033812 (2012).
[22] F. Guerrieri, L. Maccone, F. N. C. Wong, J. H. Shapiro, S. Tisa, and F. Zappa, Phys. Rev. Lett. **105**, 163602 (2010).
[23] G. Brida, M. Genovese, and I. R. Berchera, Nat. Photonics **4**, 227 (2010).
[24] M. Tsang, Phys. Rev. Lett. **102**, 253601 (2009).
[25] H. Shin, K. W. C. Chan, H. J. Chang, and R. W. Boyd, Phys. Rev. Lett. **107**, 083603 (2011).
[26] C. A. Pérez-Delgado, M. E. Pearce, and P. Kok, Phys. Rev. Lett. **109**, 123601 (2012).
[27] C. Preza, D. L. Snyder, and J. A. Conchello, J. Opt. Soc. Am. A **16**, 2185 (1999).
[28] H. Imai and A. Fujiwara, J. Phys. A **40**, 4391 (2007).
[29] A. Fujiwara, Phys. Rev. A **65**, 012316 (2001).
[30] M. A. Ballester, Phys. Rev. A **69**, 022303 (2004).
[31] C. Macchiavello, Phys. Rev. A **67**, 062302 (2003).
[32] M. A. Ballester, Phys. Rev. A **70**, 032310 (2004).
[33] N. Spagnolo, L. Aparo, C. Vitelli, A. Crespi, R. Ramponi, R. Osellame, P. Mataloni, and F. Sciarrino, Sci. Rep. **2**, 862 (2012).
[34] C. Helstrom, *Quantum Detection and Estimation Theory, Mathematics in Science and Engineering* (Academic Press, Massachusetts, 1976).
[35] In this multiparameter scenario, a single trial has to necessarily consist of an estimate of all d phases.
[36] M. G. A. Paris, Int. J. Quantum. Inform. **07**, 125 (2009).
[37] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.111.070403 for additional details on calculations presented in the manuscript, including a derivation of the QFI for simultaneous phase estimation and a proof of the saturation of the QFI by the optimal measurement construction.
[38] K. Matsumoto, J. Phys. A **35**, 3111 (2002).
[39] For cases in which d is not a factor of N, we show in the Supplemental Material Section III [37] that this is a conservative estimate, since it underestimates the minimum error achievable using N photons.
[40] The linear advantage in joint estimation provided by this quantum probe state is not unique. Setting $\alpha = \beta = 1/\sqrt{d+1}$ in Eq. (8) provides a slightly different probe state $|\psi_w\rangle$ that has a total variance of $|\Delta\boldsymbol{\theta}|^2 = d(d+1)/2N^2$, providing a scaling enhancement of $1/2d$. In an actual experiment, it may be easier to produce this balanced probe state, in which the phase and reference mode amplitudes are the same. This state preserves the $\mathcal{O}(d)$ scaling improvement over individual phase estimation and provides a total variance within a factor of 2 of that of the optimal state for large d.
[41] Y. L. Lim and A. Beige, Phys. Rev. A **71**, 062311 (2005).
[42] M. C. Tichy, M. Tiersch, F. Mintert, and A. Buchleitner, New J. Phys. **14**, 093015 (2012).
[43] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photonics **5**, 222 (2011).