Highly Eccentric Kozai Mechanism and Gravitational-Wave Observation for Neutron-Star Binaries

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The Kozai mechanism for a hierarchical triple system could reduce the merger time of inner eccentric binary emitting gravitational waves (GWs) and has been qualitatively explained with the secular theory that is derived by averaging short-term orbital revolutions. However, with the secular theory, the minimum value of the inner pericenter distance could be excessively limited by the averaging operation. Compared with traditional predictions, the actual evolution of an eccentric inner binary could be accompanied by (i) a higher characteristic frequency of the pulselike GWs around its pericenter passages and (ii) a larger residual eccentricity at its final inspiral phase. These findings would be important for GW astronomy with the forthcoming advanced detectors.

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Introduction.—Today, large-scale laser interferometers are under development to attain a worldwide network of second-generation gravitational-wave (GW) detectors [1]. Their overall sensitivities will be improved by a factor of ~ 10, with drastic noise reduction at the lower frequency regime down to ~ 10 Hz [1]. Accordingly, the understanding of the basic properties of potential astrophysical sources has become significant, more than ever.

One of the most promising targets of these detectors is the inspiral of a neutron-star binary (NSB), and in this Letter, we focus our attention to GW observation for NSBs. From identified samples in our Galaxy, NSBs are expected to have very small residual eccentricities [$O(10^{-5})$] around 10 Hz [2,3].

Meanwhile, it has been pointed out that the Kozai mechanism might play important roles for compact binary mergers [4–6]. This mechanism works for hierarchical triple systems and oscillates pericenter distances of inner binaries due to the exchange of angular momenta between the inner and outer orbits [7]. This characteristic feature can be qualitatively understood with the secular theory for which, following a perturbative method in analytical mechanics, we effectively average out short-term fluctuations associated with both the inner and outer orbital revolutions [8,9]. Since energy loss due to GW emission depends strongly on pericenter distance, the Kozai mechanism can largely reduce the merger time of an inner NSB of a triple system. This interesting possibility has been actively discussed mostly with the secular theory, including the averaging operations [4–6] (see also Ref. [10]).

In this Letter, we show that for a highly eccentric inner binary emitting GWs, there is a breakdown of the secular theory or orbital-averaged approximation, in comparison to the full numerical integration. To handle evolution of such a binary, we need to properly resolve the two orbital revolutions without taking their averages. For an inner NSB, this could result in (i) a higher characteristic frequency of the pulselike GWs around its pericenter passages, (ii) a higher residual eccentricity at its final inspiral phase, and (iii) a shorter merger time. All of these changes could be more than 1 order of magnitude. Our findings (i) and (ii) are significant for the advanced detectors and their data analyses. While quantitative evaluation for the merger rate requires detailed astronomical assumptions and is beyond scope of this Letter, the last one (iii) indicates a higher merger rate for NSBs of triples in star clusters [5]. This is because the outer third body would be frequently perturbed there.

In this Letter, we only discuss relativistic effects for hierarchical triples, but tidal effects around planets also depend strongly on orbital distance [11] (see also Ref. [12] for collisions of white dwarf binaries). For extrasolar planetary systems (e.g., hot Jupiters [13,14]), an investigation similar to this work would be worth considering.

Secular theory.—We study evolution of a hierarchical triple system of point masses m_0, m_1, and m_2. We basically use the geometrical units with $G=c=(m_0+m_1+m_2)=1$. The inner binary is composed of m_0 and m_1, and we denote its semimajor axis by a_1 and its instantaneous orbital separation by d_1. In the next section, we also introduce astrophysical units, considering m_0-m_1 as a NSB. For the outer third body m_2, we define its semimajor axis a_2, relative to the mass center of the inner binary (total mass $M_1 \equiv m_0 + m_1$). Likewise, we use the labels $j = 1$ and 2 for the inner and outer orbital elements (e.g., e_1 for the inner eccentricity) and assume hierarchical orbital configurations with $\alpha \equiv a_1/a_2 \ll 1$.

First, we briefly discuss the long-term secular evolution of the triple system in Newtonian dynamics, following the approach developed by von Zeipel [8]. By suitably using canonical transformations, we effectively average the short-term fluctuations associated with both the inner and outer mean anomalies l_1 and l_2 (the instantaneous angular positions of the inner and outer point masses [11]).

The relevant Hamiltonian after the averaging operations can be evaluated perturbatively with the expansion parameter $\alpha \ll 1$. The leading-order (quadrupole) term $H_{qd} = O(\alpha^2)$ is given by [4–6,9]

$$H_{qd} = C_{qd}[(2 + 3e_1^2)(1 - 3\theta^2) - 15e_1^2(1 - \theta^2)\cos 2\omega_1], \tag{1}$$

with $C_{qd} \equiv m_0 m_1 m_2 \alpha^2/[16M_1 a_2(1 - e_2^2)^{3/2}]$ and the argument of the inner pericenter ω_1 [11]. Here, we define $\theta \equiv \cos I$ with the opening angle I between the inner and outer orbital angular momentum vectors (identical to the angle i in Ref. [9]). We denote the next-order (octupole) term by $H_{oc}[= O(\alpha^3)]$ [9]. For our secular analysis of the inner binary, we keep up to this term for the gravitational perturbation externally induced by m_2. But, there exists a relation $H_{oc} \propto (m_0 - m_1)$, resulting in $H_{oc} = 0$ for $m_0 = m_1$ [9]. Later, we use this property to examine possible effects of the subleading terms.

Next, we mention general relativistic corrections to the system, using the post-Newtonian (PN) expansion. The lowest-order (1PN) term H_{1PN} for our hierarchical configuration is obtained after averaging the inner mean anomaly l_1 as [4–6,8]

$$H_{1PN} = -\frac{3m_0 m_1 M_1}{a_1^2(1 - e_1^2)^{1/2}}. \tag{2}$$

At this stage, our effective Hamiltonian H_c for the secular evolution is given by

$$H_c = H_{qd} + H_{oc} + H_{1PN}, \tag{3}$$

and the system is conservative (thus putting the subscript c above) [4–6]. Using canonical equations and transformations of variables, we have, e.g.,

$$\left(\frac{d\omega_1}{dt}\right)_c = 6C_{qd}\left(\frac{4\theta^2}{G_1} + \cdots\right) + \text{O.T.} + \frac{3}{a_1(1 - e_1^2)}\left(\frac{M_1}{a_1}\right)^{3/2}, \tag{4}$$

$$\left(\frac{de_1}{dt}\right)_c = 30C_{qd}\frac{e_1(1 - e_1^2)}{G_1}(1 - \theta^2)\sin 2\omega_1 + \text{O.T.}, \tag{5}$$

$(da_1/dt)_c = (da_2/dt)_c = dH_c/dt = 0$, and the scaling relations $(de_2/dt)_c = \text{O.T.}$ and $(d\omega_2/dt)_c = O(\alpha^2)$. Here, we defined $G_1 = m_0 m_1[a_1(1 - e_1^2)/M_1]^{1/2}$ and put O.T. for terms of $O(\alpha^3)$ originating from H_{oc} [4,6]. The total angular momentum is conserved with $(d/dt)\times\sqrt{a_2(1 - e_2^2)} = O(\alpha^3)$ for the magnitude of the outer one.

The triple system becomes dissipative at the 2.5 PN order, due to emission of GWs. Given our hierarchical configuration, the dissipation predominantly works for the inner binary, and we include its effects only for a_1 and e_1, using standard formulas for isolated eccentric binaries [15]. Combining these with the conservative contributions, we can write down the final expressions for the secular evolution such as $d\omega_1/dt = (d\omega_1/dt)_c$,

$$\frac{da_1}{dt} = -\frac{64m_0 m_1 M_1}{5a_1^3(1 - e_1^2)^{7/2}}\left(1 + \frac{73}{24}e_1^2 + \frac{37}{96}e_1^4\right), \tag{6}$$

$$\frac{de_1}{dt} = -\frac{304m_0 m_1 M_1 e_1}{15a_1^4(1 - e_1^2)^{5/2}}\left(1 + \frac{121}{304}e_1^2\right) + \left(\frac{de_1}{dt}\right)_c, \tag{7}$$

which have strong dependencies on $1 - e_1$. We also have $a_2 = \text{const}$. These secular equations have been widely used for analyzing long-term evolutions of relativistic hierarchical triple systems [4,6].

Numerical results.—In this section, we numerically discuss the Kozai mechanism for relativistic hierarchical triples, first using the secular equations and then directly integrating the PN equations for three-body systems. While a triple system has many parameters, we fix most of them to concisely explain our new findings.

In our geometrical units, we fix the masses at $M_1 = 0.2$, $m_2 = 0.8$, and the initial orbital parameters at $a_1 = 3.57 \times 10^5$, $a_2 = 60a_1 = 2.14 \times 10^7 \equiv a_{2i}$ (i.e., initially $\alpha = 1/60$), $e_1 = 0.2$, and $e_2 = 0.6$. We also set the initial angular variables at $\omega_1 = \pi/2$ and $\Omega_1 = \omega_2 = 0$ (Ω_1 is the longitude of the inner ascending node [11]). For our study, the remaining important parameter is the initial inclination I_i. We explore the regime $I_i \sim 90°$ for which an inner binary can merge in a short time (also preferable for costly direct calculations).

For an actual astrophysical system, we presume that the inner binary is a NSB with their total mass $M_1 = 2.8M_\odot$. Then, the initial axes correspond to $a_1 = 0.05$ AU and $a_2 = a_{2,i} \equiv 3$ AU. Below, instead of the direct time variable t, we use the effective outer revolution cycles $N_2 \equiv t/P_{2i}$ defined with the initial orbital period $P_{2i} = 2\pi a_{2i}^{3/2}$ (corresponding to 1.38 yr). The primary GW frequency of a quasicircular inner binary becomes 10 Hz ($\sim$ the lower end of the advanced detectors) at the critical separation $a_1 = a_{1cr} \equiv 34.6$.

Since observed NSBs have nearly equal masses (with a relative difference of $\lesssim 7\%$ [2]), we mainly set $m_0 = m_1 = 0.1$ in geometrical units. For an isolated binary with a semimajor axis $a = 0.05$ AU and masses $m_0 = m_1 = 1.4M_\odot$, the merger time due to GW emission becomes 1.0×10^{10} yr even for $e = 0.7$.

As mentioned earlier, the octupole term H_{oc} vanishes for $m_0 = m_1$. In order to safely estimate its potential effects, we also examine the case $(m_0, m_1) = (0.11, 0.09)$.

Results with the secular theory.—As an example for predictions of the secular theory, in Fig. 1, we provide the inner semimajor axis a_1 and pericenter distance $r_{p1} \equiv a_1(1 - e_1)$ as functions of the outer cycles N_2.

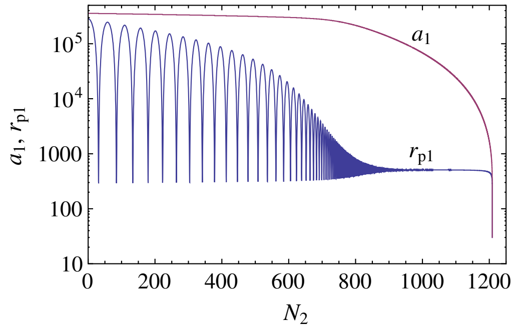

FIG. 1 (color online). Evolution of the inner semimajor axis a_1 and pericenter distance $r_{p1} = (1 - e_1)a_1$. These results are obtained with the traditional secular theory. We set $m_1 = m_2 = 0.1$, $m_2 = 0.8$ with initial inclination $I_i = 91°$ and initial eccentricities $e_1 = 0.2$ and $e_2 = 0.6$. The inner binary merges at the outer cycles $N_{2m} = 1209$.

Their ratio r_{p1}/a_1 is identical to $(1 - e_1)$. The basic parameters for this calculation are given in the caption.

The inner binary merges at $N_{2m} = 1209$ that is considerably smaller than the cycles $N_{2m} = O(10^{10-11})$ for isolated binaries with moderate initial eccentricities [5,6]. Because of the Kozai mechanism, the inner eccentricity e_1 oscillates in the rangle $0.2 \lesssim e_1 \lesssim 0.9992$, and the minimum pericenter distance becomes $r_{p1} \simeq 300$.

When we switch off the radiation reaction and also drop the octupole and higher terms, we have conserved quantities in the secular theory, as mentioned after Eq. (5) [in particular, $\sqrt{a_2(1 - e_2^2)}$]. These conserved quantities actually allow us to set a lower limit $r_{p1} \sim 300$ close to Fig. 1 (see, e.g., Ref. [6] for the role of the 1PN effect).

In Fig. 1, the energy of the inner binary is radiated mostly around the close approaches $d_1 \sim 300$. As discussed in the literature [5,6], the oscillation amplitude of e_1 decreases gradually due to the 1PN apsidal precession [the last term in Eq. (4)], and, at $N_2 \gtrsim 1000$, the inner elements evolve, as if an isolated binary. The binary becomes nearly circular at the final phase close to the merger. At the critical separation $a_1 = a_{1cr}$, the residual eccentricity becomes $e_{1cr} = 5.3 \times 10^{-3}$.

In Fig. 2, using the symbols on the solid lines, we show the duration N_{2m} and the residual eccentricity e_{1cr} at $a_1 = a_{1cr}$ for $I_i \sim 90°$. The results (circles) for $(m_0, m_1) = (0.1, 0.1)$ are similar to those (triangles) for $(m_0, m_1) = (0.11, 0.09)$. Therefore, for the present parameters, the octupole term plays a minor role, and the perturbative expansion itself is effective for the secular theory (see also Refs. [14,16]).

Direct three-body calculations.—Now, we move to direct three-body calculations. We use PN equations of motion for spinless three-body systems and handle the three particles equivalently. In addition to the conservative terms at the Newtonian, 1PN, and 2PN orders (given, e.g., in Ref. [17]), we included the dissipative 2.5 PN terms by

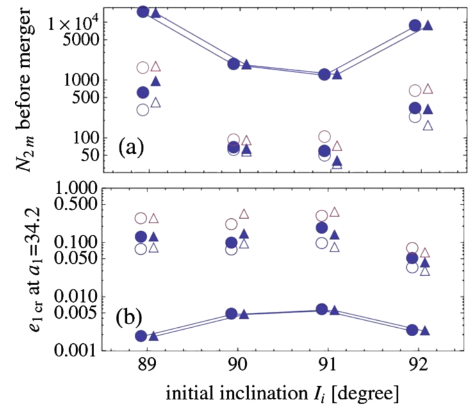

FIG. 2 (color online). (a) The circles ($m_0 = m_1 = 0.1$) and triangles ($m_0 = 0.11$, $m_1 = 0.09$) represent the outer revolution cycles N_{2m} before the mergers of the inner binaries (slightly displaced horizontally to prevent overlaps of symbols). The symbols with lines are obtained from the traditional secular theory. These without lines are from direct three-body calculations. For each inclination I_i, in total 50 runs with random initial mean anomalies are analyzed, and we show the median values (filled symbols) and the first (25%) and third (75%) quantiles (open symbols). (b) The residual eccentricity e_{1cr} of the inner binaries at the semimajor axes $a_1 = a_{1cr} = 34.6$ (corresponding to the primary GW frequency of 10 Hz for a NSB).

using Eq. (41) in Ref. [18]. Unless otherwise stated, we excluded the time-consuming 2 PN terms that would be briefly discussed later.

For numerical integration, we apply a fourth-order Runge-Kutta scheme with an adaptive time-step control [19]. We terminate our runs, when the inner semimajor axis decreases to $a_1 = a_{1cr}$ or when the instantaneous separation d_1 becomes less than $10M_1$. The later condition reflects our perturbative (PN) treatment of nonlinear gravity, but no run encountered this condition. For numerical evaluation of the orbital elements a_j and e_j ($j = 1, 2$), we use the consecutive maximum $[(1 + e_j)a_j]$ and minimum $[(1 - e_j)a_j]$ of the instantaneous orbital separations d_j.

For the direct calculations, we need to specify the initial mean anomalies l_j. Since the three-body problem depends strongly on initial conditions, we randomly distribute the initial mean anomalies to examine statistical trends of evolutions. For each initial inclination I_i and mass combination in Fig. 2, we made 50 runs and evaluated their median values and first or third quantiles of the durations N_{2m} and the residual eccentricities e_{1cr}. For $m_0 = m_1 = 0.1$ and $I_i = 90°$, we additionally made 50 runs, including the 2PN terms, and obtained the median values $N_m = 78.3$ and $e_{1cr} = 0.136$ that are close to the corresponding ones in Fig. 2. Therefore, for our analyses, the 2PN effect would not be important.

We found that in the direct calculations, the outer parameters a_2 and e_2 stay nearly at their initial values, in agreement with the secular theory. However, Fig. 2

shows that the duration N_{2m} and residual $e_{1\rm cr}$ are totally different [20].

To closely look at these discrepancies, we select an illustrative sample among the 50 runs for $I_i = 91°$ and $m_0 = m_1 = 0.1$. This run ended at $N_{2m} = 52.5$ with the residual $e_{1\rm cr} = 0.313$ (close to the upper quantile in Fig. 2). If we simply use the outer cycle N_2 (as in Fig. 1), the semimajor axis a_1 comes to appear merely as a step function, and we cannot resolve its rapid final evolution. Therefore, for Fig. 3, we plot, on a logarithm scale, the remaining cycles $\Delta N \equiv N_{2m} - N_2$ before the merger.

We can see that up to $\Delta N = O(0.1)$, the axis a_1 is nearly a constant, but the pericenter distance r_{p1} has a modulation period $\sim P_2$, the orbital period of the outer binary. This reflects the eccentric motion of the outer point mass m_2 characterized by l_2, rather than an effective ring in the secular theory. Temporally neglecting the radiation reaction, we follow Ref. [12] and briefly discuss the impacts of this discreetness for evolution of the inner specific angular momentum vector $\boldsymbol{j}_1$ [closely related to e_1 and r_{p1} as $|\boldsymbol{j}_1| = \sqrt{a_1(1 - e_1^2)}$]. Its variation $\Delta \boldsymbol{j}_1$ due to m_2 in one inner orbital revolution depends strongly on the exact position of m_2 and thus has a stochastic character (denoting its rms value by δj_1).

For $\delta j_1 \ll |\boldsymbol{j}_1|$, the total variation of $\boldsymbol{j}_1$ after a few outer orbital cycles could be close to that caused by the corresponding outer ring, and the orbital averaging could be efficient. However, for a highly eccentric case with $\delta j_1 \gtrsim |\boldsymbol{j}_1|$, the averaging method would break down, and consequently, the associated lower limit for r_{p1} (mentioned in the previous section) would be no longer valid. In the direct three-body integral, the discreetness of m_2 is naturally included, and we have possibilities to realize r_{p1} smaller than the limit obtained with the secular theory. While we temporally neglected the radiation reaction for simplicity, we can expect similar differences for our dissipative systems.

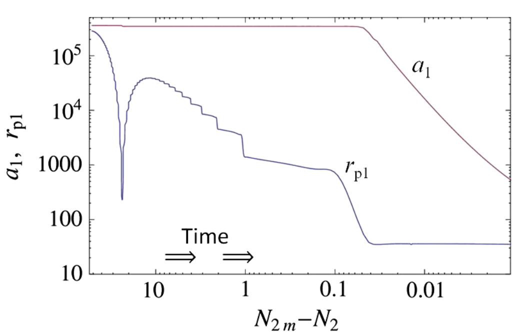

FIG. 3 (color online). Similar to Fig. 1, but given from a direct three-body calculation. The inner binary takes $e_{1\rm cr} = 0.313$ at $a_1 = a_{1\rm cr}$ and merges at the outer cycle $N_{2m} = 52.5$. For the horizontal axis, we use the remaining outer cycle $\Delta N \equiv N_{2m} - N_2$ before the inner merger.

Indeed, in Fig. 3, at the turning point $\Delta N \sim 4 \times 10^{-2}$, the quantity $1 - e_1$ takes a minimum value, corresponding to $r_{p1} = 38$ (much smaller than Fig. 1). Then, the inner binary evolves almost independently of the outer body m_2 with rapidly decreasing a_1 from $a_1 = 3.0 \times 10^5$ but nearly conserving r_{p1} for a while.

For an orbit with $1 - e_1 \ll 1$, GW emission is dominated at the pericenter passages, and the radiated energy there is given as $\delta E \sim -85\pi(m_0 m_1)^2 M_1^{1/2}/(12\sqrt{2} r_{p1}^{7/2})$, depending strongly on r_{p1} [21]. At the turning point in Fig. 3, this amounts to a fraction

$$Y \sim 0.19\left(\frac{a_1}{3.0 \times 10^5}\right)\left(\frac{r_{p1}}{38}\right)^{-7/2} \quad (8)$$

of the inner orbital energy $-m_0 m_1/2a_1$.

Meanwhile, for the secular theory, we can simply estimate the local minimum of $1 - e_1$ from Eq. (7) (with $de_1/dt = 0$) [6]. This is determined by the balance between the two effects, the dissipative radiation reaction working only around $d_1 = O(r_{p1}) \ll a_1$ and the tidal effect (by m_2) operating mainly during $d_1 = O(a_1)$. Neglecting the octupole terms, we obtain the minimum pericenter distance $r_{p1,\rm min} = a_1(1 - e_1)_{\rm min}$ as

$$r_{p1,\rm min} \simeq 60\left(\frac{a_2/a_1}{71}\right)\left(\frac{a_1}{3.0 \times 10^5}\right)^{1/6}\left(\frac{X}{1}\right)^{-1/3} \quad (9)$$

with the factor $X \equiv \sin^2 I |\sin 2\omega_1| \le 1$. Thus, even with the highly conservative setting $X = 1$, the distance $r_{p1} = 38$ at the turning point in Fig. 3 is not allowed in the secular theory, and the radiated fraction becomes at most $Y = 0.02$, in contrast to Eq. (8). Equation (9) has been used in previous studies, with additionally evaluating X [6]. But, along with the insufficient treatment of the discreteness effect (mentioned earlier), the two temporally separated effects are directly compared in Eq. (9) without resolving the inner orbital phase. Roughly speaking, even at $d_1 \gg r_{p1}$, a nearly radial inner orbit could be prohibited by the radiation reaction that intrinsically has no effect there.

Discussions.—Finally, we comment on the implications of our results for GW astronomy. In Fig. 3, after the turning point, the inner binary emits pulselike GWs around the pericenter passages [22]. This waveform has a characteristic frequency $(M_1/r_{p1}^3)^{1/2}/\pi \sim 10$ Hz that is $\sim$30 times higher than the counterpart in Fig. 1. While Figs. 1 and 3 are given for a specific set of parameters, this shift would be encouraging for ground-based GW observation, given the formidable noise walls below $\sim$10 Hz [1].

Figure 2 shows that we could have larger residual eccentricities $e_{1\rm cr}$ and also shorter merger times than the estimations by the secular theory. These differences are closely related to the decrease of the pericenter distances and suggest a higher merger rate of NSBs in star clusters, as discussed earlier. For a quasicircular binary, the residual eccentricity could be probed through the associated phase modulation of inspiral GWs [3]. For a NSB detectable with advanced detectors at SNR ~ 15, the resolution of the residual value e_{1cr} (at 10 Hz) would be $\Delta e_{1cr} \simeq 0.01$ [3]. Interestingly, this is just between the two predictions in Fig. 2, and we might discriminate the origins of NSB mergers with the upcoming GW detectors.

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