Stochasticity Effects in Quantum Radiation Reaction

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(Received 23 January 2013; published 2 August 2013)

When an ultrarelativistic electron beam collides with a sufficiently intense laser pulse, radiation-reaction effects can strongly alter the beam dynamics. In the realm of classical electrodynamics, radiation reaction has a beneficial effect on the electron beam as it tends to reduce its energy spread. Here we show that when quantum effects become important, radiation reaction induces the opposite effect; i.e., the energy distribution of the electron beam spreads out after interacting with the laser pulse. We identify the physical origin of this opposite tendency in the intrinsic stochasticity of photon emission, which becomes substantial in the quantum regime. Our numerical simulations indicate that the predicted effects of the stochasticity can be measured already with presently available lasers and electron accelerators.

DOI: 10.1103/PhysRevLett.111.054802 PACS numbers: 41.75.Ht, 12.20.Ds, 41.60.−m

A deep understanding of the dynamics of electric charges driven by electromagnetic fields is one of the most fundamental problems in physics, as it has implications in different fields, including accelerator, radiation, and high-energy physics. Apart from its impact on practical issues, as the construction of new experimental devices (e.g., quantum x-free electron lasers [1]), the investigation of the dynamics of electric charges (electrons, for definiteness) is also of pure theoretical interest, as it involves in general a coupled interplay between the electrons and their own electromagnetic field.

In the realm of classical electrodynamics, radiation-reaction (RR) effects stem from the backreaction on the electron dynamics of the electromagnetic field generated by the electron itself while being accelerated by a background electromagnetic field [2,3]. The Landau-Lifshitz (LL) equation has been recently identified as the classical equation of motion of an electron (mass m and charge $e<0$), including RR effects self-consistently [2–7], although alternative models have been suggested [8,9]. The analytical solution of the LL equation in a plane-wave field [10] shows that if an electron impinges with initial four-momentum p_0^μ onto a plane-wave field (electric-field amplitude E_0, central angular frequency ω_0, and propagating along the direction $\boldsymbol{n}$), RR effects substantially affect the electron dynamics, if $R_c=\alpha\chi_0\xi_0\gtrsim 1$ (see also [11]). Here, $\alpha = e^2$ is the fine-structure constant, $\chi_0 = [(np_0)/m]E_0/E_{\mathrm{cr}}$, with $n^\mu = (1, \boldsymbol{n})$ and $E_{\mathrm{cr}}=m^2/|e|=1.3\times 10^{16}\,\mathrm{V/cm}$, and $\xi_0 = |e|E_0/m\omega_0$ (units with $\hbar = c = 1$ are used throughout). The parameter R_c corresponds in order of magnitude to the average energy radiated by the electron in one laser period in units of the initial electron energy, and, although $\chi_0 \ll 1$ in the realm of classical electrodynamics [2], it can be of the order of unity [4,10,11]. For an ultrarelativistic electron initially counterpropagating with respect to the laser field with energy ε, it is $R_c=3.2\varepsilon[\mathrm{GeV}]I_0[10^{23}\,\mathrm{W/cm^2}]/\omega_0[\mathrm{eV}]$, with $I_0=E_0^2/4\pi$ being the laser pulse peak intensity. The numerical value of the parameter R_c shows the generally demanding requirements to observe large RR effects, and it explains why the LL equation still lacks an experimental confirmation (see [11–14] for recent experimental proposals). The expression of the parameter R_c is also in agreement with the well-known classical result that more energetic particles radiate more at given other conditions [15]. In turn, this explains physically the beneficial effect of RR when it is included, e.g., in the investigation of the production of electron [16] and ion [17–20] bunches in laser-plasma interaction. In fact, it is found that RR acts as a cooling mechanism and its effects render the energy spectra of the produced particle bunches more monochromatic than if RR is not included.

In this Letter, we show that when quantum effects become important RR induces exactly the opposite behavior and makes the energy distribution of an electron beam initially counterpropagating with respect to a strong laser field broader than it was before the interaction. We explain this striking difference between classical and quantum RR by relating it to the stochastic nature of the emission of radiation, which becomes substantial in the quantum regime and which can be described at small χ_0's via an additional stochastic term in the LL classical equation. By means of numerical simulations, we show that the broadening of the electron energy distribution in the quantum regime is measurable in principle with presently available technology also in an all-optical setup. Our results are relevant for future laser-based electron accelerators, indicating that one cannot rely on the beneficial effects of RR on the energy spread of the electron beam at sufficiently high electron energies that quantum effects become important. We note that the stochastic nature of photon emission has instead been shown to lower the laser intensity threshold at which electromagnetic cascades are generated [21] and to broaden the transverse spatial distribution of an electron bunch in the focusing magnetic fields of a synchrotron [22].

Taking into account exactly RR in the full strong-field QED regime amounts to determining completely the S-matrix in the Furry picture [2], which describes the interaction of the electron-positron field with the radiation field in the presence of the strong background electromagnetic field. This is a formidable task and, thus, we limit here to the so-called "nonlinear moderately quantum" regime [23], where (i) $\xi_0 \gg 1$, such that nonlinear effects in the laser field amplitude are large, and (ii) $\chi_0 \lesssim 1$, such that nonlinear QED effects are already important, but electron-positron pair production is still negligible. In this regime, RR effects on the electron dynamics in a strong plane-wave field mainly stem from the sequential emission of many photons by the electron, and they can be investigated by means of a kinetic approach [24–26] (see [23], for an alternative, microscopic approach). In this approach, the electrons and the photons are described by distribution functions in phase space, which obey kinetic equations. Since electron-positron pair production is neglected, (i) the distribution function of positrons can be assumed to vanish identically, and (ii) the kinetic equation for the electron distribution function is not coupled to that of the photons [24–26]. Another realistic approximation, which allows us to avoid technical complications in favor of a clearer physical understanding, is to consider an electron bunch initially counterpropagating with respect to the laser field and with a typical energy $\varepsilon^* \gg m\xi_0$. This is the case, for example, in the realistic situation of an electron bunch with typical energy $\varepsilon^* = 1$ GeV colliding head-on with an optical ($\omega_0 = 1.55$ eV) laser field of intensity 10^{22} W/cm^2 [27] for which $m\xi_0 = 24$ MeV. The condition $\varepsilon^* \gg m\xi_0$ ensures that the transverse momentum of the electrons (with respect to the initial propagation direction) remains much smaller than the longitudinal one in passing through the plane wave [4], and this reduces the present problem to a one-dimensional one (see Supplemental Material [28]).

By assuming that the plane wave propagates along the positive y direction and that it is linearly polarized along the z direction, we can write its electric field as $\boldsymbol{E}(\varphi) = E_0 f(\varphi)\hat{\boldsymbol{z}}$, where $\varphi = \omega_0(t-y)$ is the laser phase and $f(\varphi)$ is the pulse-shape function such that $|f(\varphi)|_{\max} = 1$. If $p^\mu = (\varepsilon, \boldsymbol{p})$ is the four-momentum of an electron, it is convenient to introduce the "minus" momentum $p_- = \varepsilon - p_y$, which is a constant of motion in the plane-wave field under consideration [4]. However, if the electron emits a photon with four-momentum $k^\mu = (\omega, \boldsymbol{k})$, then its four-momentum changes to $p'^\mu = (\varepsilon', \boldsymbol{p}')$ and $p'_- = p_- - k_-$, with $p'_- = \varepsilon' - p'_y$ and $k_- = \omega - k_y$. The single-photon emission probability per unit phase φ and per unit $u = k_-/(p_- - k_-)$ in the ultrarelativistic regime $\xi_0 \gg 1$ reads [see Eq. (49) on p. 559 in [29]]

$$\frac{dP_{p_-}}{d\varphi du} = \frac{\alpha}{\sqrt{3}\pi}\frac{m^2}{\omega_0 p_-}\frac{1}{(1+u)^2}\left[\left(1+u+\frac{1}{1+u}\right)\times K_{2/3}\left(\frac{2u}{3\chi(\varphi,p_-)}\right) - \int_{2u/[3\chi(\varphi,p_-)]}^{\infty} dx K_{1/3}(x)\right], \tag{1}$$

where $K_\nu(\cdot)$ is the modified Bessel function of νth order and where $\chi(\varphi, p_-) = (p_-/m)|E(\varphi)|/E_{\rm cr}$, with $E(\varphi) = E_0 f(\varphi)$. Since the probability in Eq. (1) depends only on the phase-space variables φ and p_-, it is possible to describe the electron beam via an electron distribution $n_e(\varphi, p_-)$, which satisfies the kinetic equation (see [25] and Supplemental Material [28])

$$\frac{\partial n_e}{\partial\varphi} = \int_{p_-}^{\infty} dp_{i,-}\frac{dP_{p_{i,-}}}{d\varphi dp_-}n_{i,e} - n_e\int_0^{p_-} dk\frac{dP_{p_-}}{d\varphi dk_-}, \tag{2}$$

with $n_e = n_e(\varphi, p_-)$, $n_{i,e} = n_e(\varphi, p_{i,-})$, and

$$\frac{dP_{p_{i,-}}}{d\varphi dp_-} = \frac{p_{i,-}}{p_-^2}\frac{dP_{p_{i,-}}}{d\varphi du}\bigg|_{u=(p_{i,-}-p_-)/p_-}, \tag{3}$$

$$\frac{dP_{p_-}}{d\varphi dk_-} = \frac{p_-}{(p_- - k_-)^2}\frac{dP_{p_-}}{d\varphi du}\bigg|_{u=k_-/(p_- - k_-)}. \tag{4}$$

Equation (2) is an integro-differential equation; i.e., it is nonlocal in the momentum p_-. This is intimately connected to the quantum nature of the emission of radiation. In fact, the latter is described quantum mechanically as the emission of photons, which carry energy and momentum, such that, if an electron emits a photon with momentum k_-, its initial state with a given momentum $p_{0,-}$ will be coupled to that with momentum $p_{0,-} - k_-$, with k_- ranging from 0 to $p_{0,-}$. Note that the complete distribution function of the electron bunch also contains a dependence on the variable $T = (t+y)/2$ (see Supplemental Material [28], where we show that, in the present regime, the center of the electrons' spatial distribution moves along the negative y axis according to the equation $y \approx -t$ and without changing its shape).

In order to investigate the classical limit of Eq. (2) for $\chi(\varphi, p_-) \ll 1$, it is convenient to perform the change of variable $v = (p_{i,-} - p_-)/p_-\chi(\varphi, p_-)$ [$v = k_-/(p_- - k_-)\chi(\varphi, p_-)$] in the first (second) integral in Eq. (2). By expanding the resulting equation in $\chi(\varphi, p_-)$ and by keeping terms up to the order $\chi^3(\varphi, p_-)$, one obtains the Fokker-Planck-like equation [30] (see also [26,31])

$$\frac{\partial n_e}{\partial\varphi} = -\frac{\partial}{\partial p_-}[A(\varphi,p_-)n_e] + \frac{1}{2}\frac{\partial^2}{\partial p_-^2}[B(\varphi,p_-)n_e], \tag{5}$$

with a "drift" coefficient $A(\varphi, p_-)$ and a "diffusion" coefficient $B(\varphi, p_-)$ given by

$$A(\varphi,p_-) = -\frac{2\alpha m^2}{3\omega_0}\chi^2(\varphi,p_-)\left[1 - \frac{55\sqrt{3}}{16}\chi(\varphi,p_-)\right], \tag{6}$$

$$B(\varphi, p_-) = \frac{\alpha m^2}{3\omega_0}\frac{55}{8\sqrt{3}}p_-\chi^3(\varphi, p_-), \tag{7}$$

respectively. Note that Eq. (5) is no longer an integro-differential equation but rather a partial differential equation. In other words, at small quantum photon-recoil effects, the distribution function of electrons with momentum p_- depends essentially only on its values close to p_- and its dynamics is local. Higher-order corrections in $\chi(\varphi, p_-)$ would result in the appearance of terms proportional to higher derivative's order of $n_e(\varphi, p_-)$ with respect to p_-.

If we first consider only the terms proportional to $\chi^2(\varphi, p_-)$ in Eq. (5), this equation has the form of a Liouville equation:

$$\frac{\partial n_e}{\partial \varphi} = -\frac{\partial}{\partial p_-}\left(n_e\frac{dp_-}{d\varphi}\right), \qquad \frac{dp_-}{d\varphi} = -\frac{I_{cl}(\varphi, p_-)}{\omega_0} \tag{8}$$

with $I_{\rm cl}(\varphi, p_-) = (2/3)\alpha m^2\chi^2(\varphi, p_-)$ being the classical intensity of radiation [4]. The equation for p_- in Eq. (8) is exactly the classical single-particle equation resulting from the LL equation [10] (see also [32]). In other words, the terms in Eq. (5) proportional to $\chi^2(\varphi, p_-)$ describe the classical dynamics of the electron distribution including RR. The fact that Eq. (8) has the form of a Liouville equation implies, as it must be, that the classical dynamics of the electron distribution is deterministic [30]. Also, since the single-particle equation in Eq. (8) admits the analytical solution [10] $p_-^{(c)}(\varphi, p_{0,-}) = p_{0,-}/h(\varphi, p_{0,-})$, with $h(\varphi, p_{0,-}) = 1 + (2/3)\alpha(p_{0,-}/\omega_0)(E_0^2/E_{\rm cr}^2) \times \int_0^\varphi d\varphi' f^2(\varphi')$ for an electron with initial momentum $p^\mu(0) = p_0^\mu = (\epsilon_0, \boldsymbol{p}_0)$ $(p_{0,-} = \epsilon_0 - p_{0,y})$, one can write explicitly the exact analytical solution of Eq. (8) by means of the method of characteristics (see also [33–35] and [36] for the solution in a constant electromagnetic field of different configurations and in a monochromatic plane wave, respectively). If the distribution $n_e(0, p_-)$ at the initial phase $\varphi = 0$ is given, for example, by the Gaussian distribution $n_e(0, p_-) = N\exp[-(p_- - p_-^*)^2/2\sigma_{p_-}^2]$, where N is a normalization factor, p_-^* is the average value of p_-, and σ_{p_-} is the standard deviation [37], then the solution of Eq. (8) reads

$$n_e(\varphi, p_-) = \frac{N}{g^2(\varphi, p_-)}\exp\left\{-\frac{1}{2\sigma_{p_-}^2}\left[\frac{p_-}{g(\varphi, p_-)} - p_-^*\right]^2\right\}, \tag{9}$$

with $g(\varphi, p_-) = h(\varphi, -p_-)$. Since $p_{0,-}$ in $h(\varphi, p_{0,-})$ is positive for finite values of $p_{0,y}$ and $p_{0,-} \to 0$ only at $p_y \to +\infty$, and since $p_{0,-} = p_-^{(c)}(\varphi, p_{0,-})g(\varphi, p_-^{(c)}(\varphi, p_{0,-}))$, the function $g(\varphi, p_-)$ must be non-negative for all values of φ, and the equation $g(\varphi, p_{-,\rm max}) = 0$ fixes the maximum value $p_{-,\rm max} = p_{-,\rm max}(\varphi)$ allowed for the variable p_- at each φ. Moreover, we observe that $0 < \partial p_-^{(c)}(\varphi, p_{0,-})/\partial p_{0,-} < 1$ for $\varphi > 0$, such that, due to RR effects, the difference $\Delta p_-(\varphi)$ between the momenta of two electrons decreases for increasing values of φ. This implies that classical RR effects tend to decrease the energy width of the electron distribution in agreement with previous results obtained in studying the production of particle beams via laser-plasma interaction [16–20]. Also, if $\sigma_{p_-} \ll p_-^*$ in Eq. (9), it can be seen that the distribution $n_e(\varphi, p_-)$ is approximately a Gaussian centered at $p_-^{(c)}(\varphi, p_-^*)$ and with effective width $\sigma_{p_-}^{(c)}(\varphi, p_-^*) \approx \sigma_{p_-}/h^2(\varphi, p_-^*)$ decreasing at increasing φ's.

The quantum corrections in Eq. (5) to the classical kinetic equation (8) stem from two different contributions. The first one affects the drift coefficient $A(\varphi, p_-)$ [see Eq. (6)], and it coincides with the leading quantum correction to the total intensity of radiation found in Refs. [25,29]. This correction does not change the structure of the classical kinetic equation (8) but only the "effective" momentum change per unit phase. Since this leading quantum correction is negative, we expect that it tends to decrease the reduction of the width with respect to the classical prediction. However, by replacing the classical intensity of radiation $I_{\rm cl}(\varphi, p_-)$ with the corresponding quantum one $I_q(\varphi, p_-)$ [see, e.g., Eq. (83) on p. 522 in [29]], the resulting Liouville equation would still predict a reduction of the width of the electron distribution function. This statement can be proven mathematically, but it can also be understood intuitively as quantum mechanically more energetic electrons *on average* emit more radiation. On the other hand, however, the second leading quantum correction corresponds to the diffusion coefficient $B(\varphi, p_-)$ in Eq. (7), and it alters the structure of the classical kinetic equation. The appearance of a diffusionlike term in the kinetic equation of the electron distribution is intimately connected to the stochastic nature of the quantum emission of photons. According to the theory of stochastic differential equations, in fact, the Fokker-Planck-like equation (5) is equivalent to the single-particle stochastic equation $dp_- = -A(\varphi, p_-)d\varphi + \sqrt{B(\varphi, p_-)}dW$, where dW represents an infinitesimal stochastic function [30]. The diffusion term in Eq. (5) also tends to increase the width of the distribution function [30], and, as we will see numerically, it is responsible of the broadening of the distribution function. It can be shown, for example, that a Gaussian distribution centered at p_-^* and with width $\sigma_{p_-} \ll p_-^*$ at $\varphi = 0$ remains approximately Gaussian at $\varphi > 0$, centered at $p_-^{(q)}(\varphi, p_-^*) \approx p_-^{(c)}(\varphi, p_-^*)[1 + \delta h(\varphi, p_-^*)]$ and with width

$$\sigma_{p_-}^{(q)}(\varphi, p_-^*) \approx \sigma_{p_-}^{(c)}(\varphi, p_-^*)\left[1 + 2\delta h(\varphi, p_-^*) + \frac{1}{2\sigma_{p_-}^2}\int_0^\varphi d\varphi' B(\varphi', p_-^*)\right], \tag{10}$$

where the correction $\delta h(\varphi, p_-^*) = [55\sqrt{3}/16h(\varphi, p_-^*)] \int_0^\varphi d\varphi' \chi(\varphi', p_-^*)\partial \log h(\varphi', p_-^*)/\partial\varphi' > 0$ is due to the quantum correction to the drift coefficient. Equation (10) shows that both mentioned quantum effects tend to increase the width of the distribution function. Moreover, the correction induced by the diffusion term is roughly

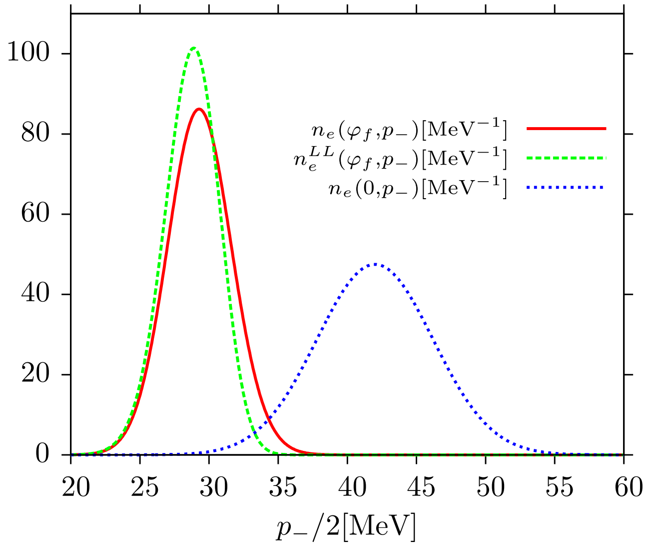

FIG. 1 (color online). Comparison of the initial electron distribution (dotted, blue line) and the final electron distribution according to Eq. (2) (solid, red line) and to Eq. (9) (dashed, green line), as functions of $p_-/2 \approx \varepsilon$. The laser and the electron distribution parameters are given in the text.

$\eta = p_-^{*2}/\sigma_p^2 \gg 1$ times larger than $\delta h(\varphi, p_-^*)$, and the approximated approach based on the Fokker-Planck equation is valid at $\alpha\xi_0\eta\Phi_L\chi^{*2} \ll 1$, with Φ_L being the total laser phase. Finally, we mention that the Fokker-Planck equation in some situations, like in the case of an initial δ-like momentum distribution and vanishing drift term, predicts the appearance of spurious particles with momentum larger than the initial one. This indicates that a completely consistent treatment requires the solution of the full equation (2), which will be carried out below numerically.

In Supplemental Material [28], it is shown that the effect on the broadening of the electron momentum distribution can also be interpreted in terms of the entropy of the distribution itself.

In order to show that the effects discussed above can be in principle measured with presently available technology, we consider below two numerical examples. In both cases we assume a laser pulse with $f(\varphi) = \sin^2(\varphi/2N_L)\sin(\varphi)$ for $\varphi \in [0, \varphi_f] = [0, 2N_L\pi]$ and zero elsewhere, where N_L is the number of laser cycles and with $\omega_0 = 1.55$ eV, and an initial Gaussian electron distribution with a total number of 1000 electrons.

In the first numerical example, we choose the laser and electron parameters such that quantum effects are negligible, whereas RR effects are relatively large. We set $I_0 = 4.3 \times 10^{20}$ W/cm^2 ($\xi_0 = 10$), $p_-^* = 84$ MeV ($\varepsilon^* \approx p_-^*/2 = 42$ MeV) such that $\chi^* = (p_-^*/m)(E_0/E_{\mathrm{cr}}) \approx 5 \times 10^{-3}$, $\sigma_{p_-} = 8.4$ MeV, and $N_L = 1600$ (pulse duration τ of about 4 ps). The results for the initial and final distribution are shown in Fig. 1. As expected, the final distribution $n_e(\varphi_f, p_-)$, calculated by solving numerically Eq. (2) (solid, red line), and the classical analytical solution $n_e^{\mathrm{LL}}(\varphi_f, p_-)$ [see Eq. (9)] are very similar and both show a reduction of the width from 8.4 to 4.7 MeV. Note that the average energy $\varepsilon_f^* = 30$ MeV of the final distribution also fulfills fairly well the condition $\varepsilon_f^* \gg m\xi_0 = 5$ MeV [see the discussion around Eq. (3) of Supplemental Material [28]]. In the second numerical example, instead, we want to probe the quantum regime, and we set $I_0 = 2 \times 10^{22}$ W/cm^2 ($\xi_0 = 68$) [27], $p_-^* = 2$ GeV ($\varepsilon^* \approx 1$ GeV) such that $\chi^* = 0.8$, $\sigma_{p_-} = 0.2$ GeV, and $N_L = 10$ ($\tau \approx 30$ fs). Electron beams with such energies are nowadays available not only in conventional accelerators but also by employing plasma-based electron accelerators [38,39], allowing in principle for an all-optical setup. The results of our numerical simulations are shown in Fig. 2. The figure shows that the full quantum calculations based on Eq. (2) predict a broadening of the electron distribution [Fig. 2(b)], according to our analysis above, whereas the classical calculations based on the exact solution in Eq. (9) [see Fig. 2(c)] predict a strong narrowing of the distribution. Moreover, according to the discussion above Eq. (10), if we consider the classical equation (8) and we perform the substitution $I_{\mathrm{cl}}(\varphi, p_-) \to I_q(\varphi, p_-)$, the corresponding results [see Fig. 2(d)] still predict a narrowing of the distribution function. This clearly supports the idea that the broadening of the electron distribution is an effect of the importance of the stochasticity of the emission of radiation, which becomes substantial in the quantum regime. Finally, we notice that the average energy ε_f^* of the final electron distribution, also according to the most unfavorable classical treatment ($\varepsilon_f^* = 173$ MeV), fairly well fulfills the condition $\varepsilon_f^* \gg m\xi_0 = 35$ MeV [see the discussion around Eq. (3) of Supplemental Material [28]].

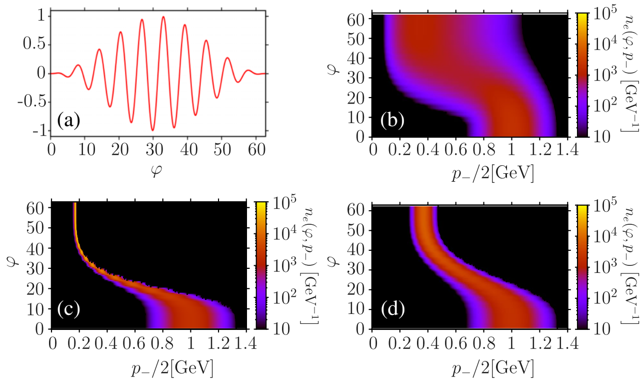

FIG. 2 (color online). Phase evolution of the electron distribution as a function of $p_-/2 \approx \varepsilon$ for a 10-cycle $\sin^2$-like laser pulse [part (a)] according to Eq. (2) [part (b)], to Eq. (9) [part (c)], and to Eq. (8) with the replacement $I_{\mathrm{cl}}(\varphi, p_-) \to I_q(\varphi, p_-)$ [part (d)]. The laser and the electron distribution parameters are given in the text.

The authors gratefully acknowledge useful discussions with J. G. Kirk, N. V. Elkina, and T. Blackburn.

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