Medium-Induced QCD Cascade: Democratic Branching and Wave Turbulence

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We study the average properties of the gluon cascade generated by an energetic parton propagating through a quark-gluon plasma. We focus on the soft, medium-induced emissions which control the energy transport at large angles with respect to the leading parton. We show that the effect of multiple branchings is important. In contrast with what happens in a usual QCD cascade in vacuum, medium-induced branchings are quasidemocratic, with offspring gluons carrying sizable fractions of the energy of their parent gluon. This results in an efficient mechanism for the transport of energy toward the medium, which is akin to wave turbulence with a scaling spectrum $\sim 1/\sqrt{\omega}$. We argue that the turbulent flow may be responsible for the excess energy carried by very soft quanta, as revealed by the analysis of the dijet asymmetry observed in Pb-Pb collisions at the LHC.

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One important phenomenon discovered recently in heavy ion experiments at the LHC is that of *dijet asymmetry*, a strong imbalance between the energies of two back-to-back jets. This asymmetry is commonly attributed to the effect of the interactions of one of the two jets with the hot QCD matter that it traverses, while the other leaves the system unaffected. Originally identified [1,2] as missing energy, this phenomenon has been subsequently shown [3] to consist in the transport of a sizable part of the jet energy by soft particles toward large angles. Some of the features of in-medium jet propagation are well accounted for by the BDMPSZ mechanism for medium-induced radiation (from Baier, Dokshitzer, Mueller, Peigné, Schiff [4], and Zakharov [5]). However, most studies within this approach have focused on the energy lost by the leading particle, while the LHC data call for a more thorough analysis of the jet shape for which the effects of multiple branching at large angles are important. Within that context, an important step was achieved in Ref. [6], where it was shown that, in a leading order approximation, one could consider successive gluon emissions as independent of each other. This allows one to treat multiple emissions as a probabilistic branching process, in which the BDMPSZ spectrum plays the role of the elementary branching rate [7–9].

Specifically, the differential probability per unit time and per unit z for a gluon with energy ω to split into two gluons with energy fractions respectively z and $1-z$ is

$$\frac{d^2\mathcal{P}_{\rm br}}{dzdt} = \frac{\alpha_s}{2\pi}\frac{P_{g\to g}(z)}{\tau_{\rm br}(z,\omega)}, \qquad \tau_{\rm br} = \sqrt{\frac{z(1-z)\omega}{\hat{q}_{\rm eff}}}, \quad (1)$$

where $P_{g\to g}(z) = N_c[1-z(1-z)]^2/z(1-z)$ is the leading order gluon-gluon splitting function, N_c is the number of colors, $\hat{q}_{\rm eff} \equiv \hat{q}[1-z(1-z)]$, with $\hat{q}$ the jet quenching parameter (the rate for transverse momentum broadening via interactions in the medium), and $\tau_{\rm br}(z,\omega)$ is the time scale of the branching process. Note that we use light-cone (LC) coordinates and momenta, with the longitudinal axis defined by the direction of motion of the leading particle. Correspondingly, the "energy" ω truly refers to the LC longitudinal momentum p^+ and t to the LC "time" x^+. Equation (1) applies as long as $\ell \ll \tau_{\rm br}(z,\omega) < L$, where L is the length of the medium and ℓ is the mean free path between successive collisions. The second inequality above implies an upper limit on the average energy of the offspring gluons: $z(1-z)\omega \lesssim \omega_c$, where $\omega_c = \hat{q}L^2/2$ is the maximum energy that can be taken away by a single gluon. It follows from Eq. (1) that the probability for having just one emission throughout the medium is (for z not too close to 1) $\sim \bar{\alpha}\sqrt{\omega_c/z\omega}$, where $\bar{\alpha} \equiv \alpha_s N_c/\pi$. When this becomes of $\mathcal{O}(1)$, i.e., when $z\omega \lesssim \omega_s \equiv \bar{\alpha}^2\omega_c$, multiple branchings become important. Note the correlation between the energy ω of the emitted gluon and the emission angle $\theta_{\rm br}$ with respect to the jet axis: one has $\theta_{\rm br} \simeq (2\hat{q}/\omega^3)^{1/4}$, showing that soft gluons are emitted at large angles. This correlation will be important for the physical interpretation of our results.

It will be useful to express the energy ω of a radiated gluon in terms of the energy fraction $x \equiv \omega/E$ of the initial energy E and to replace the light-cone time t by the dimensionless variable

$$\tau \equiv \bar{\alpha}\sqrt{\frac{\hat{q}}{E}}t = \bar{\alpha}\sqrt{2x_c}\frac{t}{L}, \quad (2)$$

where $x_c \equiv \omega_c/E$. We restrict ourselves here to the case $E < \omega_c$, i.e., $x_c > 1$, leaving the discussion of the $E > \omega_c$ case to a forthcoming publication. Note that the maximal value of τ is $\tau_{\rm max} = \bar{\alpha}\sqrt{2x_c}$, corresponding to $t = L$. Then, the branching probability (1) can be written as

$$\frac{d\mathcal{P}_{\rm br}}{dzd\tau} = \frac{1}{2}\frac{\mathcal{K}(z)}{\sqrt{x}}, \quad (3)$$

where $\mathcal{K}(z) \equiv f(z)/[z(1-z)]^{3/2} = \mathcal{K}(1-z)$ and $f(z) \equiv [1 - z(1-z)]^{5/2}$.

In this Letter, we focus on one observable that characterizes the average properties of the in-medium cascade: the gluon spectrum $D(x,\tau) \equiv x(dN/dx)$, with N the number of gluons. By exploiting the fact that successive branchings are independent [6] and using standard techniques for classical branching processes [10], one can show that $D(x,\tau)$ obeys the following evolution equation:

$$\frac{\partial D(x,\tau)}{\partial \tau} = \int dz \mathcal{K}(z)\left[\sqrt{\frac{z}{x}} D\left(\frac{x}{z},\tau\right) - \frac{z}{\sqrt{x}} D(x,\tau)\right]. \tag{4}$$

The initial condition corresponds to a single gluon carrying all the energy, that is, $D(x,\tau=0) = \delta(x-1)$. We shall refer to the right-hand side of Eq. (4) as the "collision term" and denote it as $I[D]$. Its physical interpretation is clear: The first contribution, which is nonlocal in x (except when x is close to 1), is a *gain term*: it describes the rise in the number of gluons at x due to emissions from gluons at larger x. Note that the function $D(x,\tau)$ has support only for $0 \le x \le 1$, which limits the first z integral in Eq. (4) to $x < z < 1$. The second contribution to the collision term, local in x, represents a *loss term*, describing the reduction in the number of gluons at x due to their decay into gluons with smaller x. Taken separately, the gain term and the loss term in Eq. (4) have end point singularities at $z = 1$, but these singularities exactly cancel between the two terms and the overall equation is well defined.

For $\tau \ll 1$, we may attempt to solve Eq. (4) in perturbation theory, i.e., by iterations. Thus, by substituting, in the collision term, $D(x,\tau)$ by its initial value $D^{(0)}(x) = \delta(x-1)$, one obtains (for $x < 1$)

$$D^{(1)}(x,\tau) = \frac{\tau f(x)}{\sqrt{x}(1-x)^{3/2}}. \tag{5}$$

For $\tau = \tau_{max}$, this is just the BDMPSZ spectrum. For reasons that will become clear shortly, we refer to the small-x part of this spectrum as the "scaling spectrum," i.e., $D_{sc}(x) \sim 1/\sqrt{x}$. *A priori*, because one expects the small-x region of the spectrum to be populated by multiple branchings, leading to a breakdown of perturbation theory when $x \lesssim \tau^2$, one could also expect the spectrum to be strongly modified in this region. As we shall see, this is not at all the case: the scaling spectrum remains remarkably stable (see also Ref. [7] for a similar observation).

In order to go beyond perturbation theory and get insight into the nonperturbative features of Eq. (4), we have considered a simpler version of this equation, obtained by modifying the kernel to $\mathcal{K}_0(z) = 1/[z(1-z)]^{3/2}$ [i.e., replacing the smooth function $f(z)$ by 1 in Eq. (3)]. This simplification does not affect the singular behavior of the kernel near $z = 0$ and $z = 1$, which determines the qualitative features of the solution, but it allows us to solve Eq. (4) exactly, via a Laplace transform. The solution reads

$$D_0(x,\tau) = \frac{\tau}{\sqrt{x}(1-x)^{3/2}} e^{-\pi[\tau^2/(1-x)]}. \tag{6}$$

The essential singularity at $x = 1$ is a nonperturbative effect that can be understood as a Sudakov suppression factor [8] (i.e., the vanishing of the probability to emit no gluon in any finite time). Aside from this exponential factor, one recognizes the scaling spectrum which $D_0(x,\tau)$ is proportional to at small x. This is illustrated in Fig. 1: we see that the scaling spectrum is established early on and remains stable as time progresses. For small times, its amplitude grows linearly with τ: the system can then be viewed as a radiating source located at $x \lesssim 1$ and feeding all the small-x modes. As time passes, the source weakens and eventually disappears into the left moving "shock wave" visible in Fig. 1.

Another important feature of the branching dynamics illustrated in Fig. 1 is the fact that the total energy which is stored in the spectrum (i.e., in the gluon modes with $0 < x < 1$) decreases with time: $\mathcal{E}_0(\tau) \equiv \int_0^1 dx D_0(x,\tau) = e^{-\pi\tau^2}$. This is related to the existence of a scaling solution, as alluded to above: the fact that the spectrum keeps the same shape at small x when increasing τ implies that the energy flows from higher to lower values of x without accumulating at any value $x > 0$. This should be contrasted to what happens in standard parton cascades, like that described by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [11]. In that case, the spectrum becomes steeper and steeper at small x with increasing evolution time, and the energy sum rule $\int_0^1 dx D(x,\tau) = 1$ is satisfied at any τ—"the energy remains in the spectrum." Returning to the medium-induced branching process, we note that the energy is conserved in that case, too, since it is so at each elementary branching. Formally, what happens is that a "condensate" develops at $x = 0$, playing the role of a sink where the excess energy coming from the large-x region gets stored. With increasing time, a substantial fraction of the total energy can thus flow "outside the spectrum."

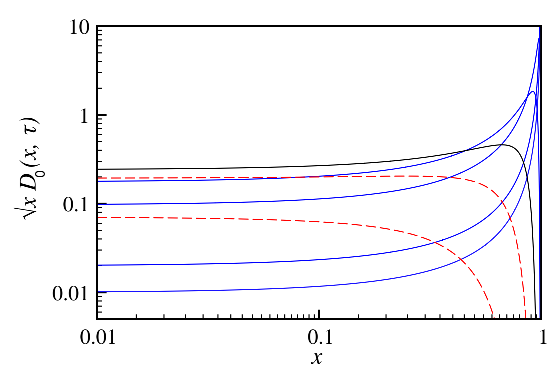

FIG. 1 (color online). Plot (in log-log scale) of $\sqrt{x}D_0(x,\tau)$, with $D_0(x,\tau)$ given by Eq. (6), as a function of x for various values of τ (full lines from bottom to top, $\tau = 0.01$, 0.02, 0.1, 0.2, 0.4; dashed lines from the top down, $\tau = 0.6$, 0.9).

To shed more light on this flow phenomenon, it is instructive to analyze an auxiliary problem—that of a system driven by a permanent source of energy localized at $x = 1$. Consider then the equation

$$\frac{\partial D(x,\tau)}{\partial \tau} = A\delta(1-x) + \mathcal{I}[D]. \tag{7}$$

For the simplified kernel $\mathcal{K}_0$, one readily verifies that the "turbulent spectrum" (see below)

$$D_{\text{tb}}(x,\tau) = \frac{A}{2\pi\sqrt{x(1-x)}}(1 - e^{-\pi[\tau^2/(1-x)]}) \tag{8}$$

solves this equation with initial condition $D_{\text{tb}}(x, \tau = 0) = 0$ [observe that the derivative of $D_{\text{tb}}(x,\tau)$ is equal to $D_0(x,\tau)$, to within the multiplicative constant A]. By comparing Figs. 1 and 2, one sees that the behaviors of $D_0(x,\tau)$ and $D_{\text{tb}}(x,\tau)$ are remarkably similar at small τ. However, the most remarkable property of $D_{\text{tb}}(x,\tau)$ is that it converges to a steady function $D_{\text{st}}(x) = (A/2\pi)/\sqrt{x(1-x)}$. To understand this, we note that $D_{\text{st}}(x)$ annihilates (exactly) the collision term, i.e., $\mathcal{I}[D_{\text{st}}] = 0$, as can be verified by an explicit calculation. As time goes on, the solution $D_{\text{tb}}(x,\tau)$ is gradually driven to $D_{\text{st}}(x)$, but since this fixed-point solution reduces to the scaling spectrum at $x \ll 1$, one observes no change in the shape at small x but just an overall time-dependent scaling.

We complete our analysis by calculating the flow of energy that gets transmitted per unit time from the region $x > x_0$ to the region $x < x_0$. If we denote by $\mathcal{E}(x_0,\tau) = \int_{x_0}^1 dx D(x,\tau)$ the total energy that is contained in the modes with $x > x_0$ and recognize that the rate of change of $\mathcal{E}(x_0,\tau)$ is due both to a possible source of strength A localized at $x = 1$ and to the flow $\mathcal{P}(x_0,\tau)$ at x_0, we get the general expression

$$\mathcal{P}(x_0,\tau) \equiv A - \frac{\partial \mathcal{E}(x_0,\tau)}{\partial \tau} = -\int_{x_0}^1 dx \mathcal{I}[D]. \tag{9}$$

An explicit calculation for $D = D_{\text{tb}}$ yields

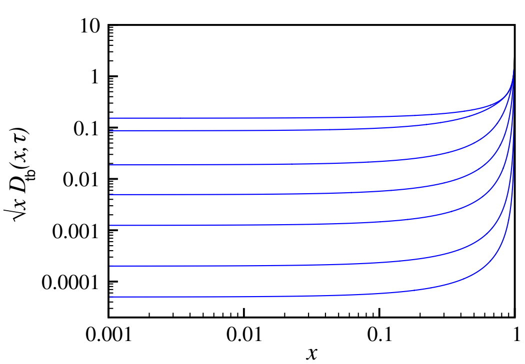

FIG. 2 (color online). The function $\sqrt{x}D_{\text{tb}}(x,\tau)$ [Eq. (8) for $A = 1$] at various times—from early time, where it resembles Fig. 1, until late time, when it approaches the steady state and saturates at the value $A/2\pi$ at small x. The values of τ are, from bottom to top, 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, and 1.

$$\mathcal{P}(x_0,\tau) = A\left[1 - e^{-\pi\tau^2}\text{erfc}\left(\sqrt{\frac{\pi x_0}{1-x_0}}\tau\right)\right], \tag{10}$$

where erfc(x) denotes the complementary error function. In order to analyze the physical content of this expression, it is actually useful to rewrite the integral of the collision term in Eq. (9) in the following form:

$$\mathcal{P}(x_0,\tau) = \int_0^1 dz z\mathcal{K}(z) \int_{x_0}^{\min(1,x_0/z)} dx \frac{D(x,\tau)}{\sqrt{x}}. \tag{11}$$

At small times, $\pi\tau^2 \ll 1$, and for x_0 not too close to either 0 or 1, one can use the expansion erfc$(x) \simeq 1 - 2x/\sqrt{\pi}$ in Eq. (10) and get $\mathcal{P}(x_0,\tau) \simeq 2A\tau\sqrt{x_0/(1-x_0)}$. This result can also be obtained from Eq. (11) by substituting $D(x,\tau)$ with $D^{(1)}(x,\tau) = A\tau\delta(1-x)$, as appropriate at small time. Thus, we can interpret this early-time contribution to $\mathcal{P}(x_0,\tau)$ as due to direct radiation from the source at $x = 1$ toward the various modes at $x_0 < 1$. Note that this involves branchings with $z \le x_0$ which, for $x_0 \ll 1$, are strongly asymmetric ($z \ll 1$).

As time goes on, however, the distribution of energy among the various modes is such that gain and loss terms equilibrate locally, at which point a steady state is reached with all the energy provided by the source flowing throughout the entire system and leaving the population of the various modes unchanged. In the steady regime reached for $\tau \gtrsim 1/\sqrt{\pi}$, the energy flux $\mathcal{P}(x_0,\tau)$ is both stationary (τ independent) and uniform (x_0 independent), and equal to A (the flux inserted by the source). Actually, this uniform component of the flow develops already at earlier times. It can be obtained by evaluating Eq. (10) at $x_0 = 0$ and reads $\mathcal{P}(x_0 = 0,\tau) = A(1 - e^{-\pi\tau^2})$. This result can be recovered from Eq. (11) with $x_0 \ll 1$ by approximating $D(x,\tau)$ with the scaling part of the spectrum (8), $D(x,\tau) \simeq A(1 - e^{-\pi\tau^2})/(2\pi\sqrt{x})$, and noting that

$$v_0 \equiv \int_0^1 dz \frac{1}{\sqrt{z}(1-z)^{3/2}} \ln\frac{1}{z} = 2\pi. \tag{12}$$

What this second calculation demonstrates is that, in contrast to what happens for the direct radiation, here the typical branchings involve the whole range of z values [about half of the value of the integral (12) comes from the range $0.15 \lesssim z \lesssim 0.85$]. We refer to this property as "quasidemocratic branching."

The properties that we have just discussed, namely, the existence of a steady scaling solution when the system is coupled to a source and, related to it, the presence of a component of the flow that is independent of the energy, are distinctive signatures of what is known as (weak) wave turbulence [13]. A crucial ingredient of this phenomenon is the locality of the interactions in momentum space, a property which in the present case is only marginally satisfied, as quasidemocratic branching.

We wish to stress that democratic branching is not common in standard parton cascades, like the one described by the DGLAP equation, which are rather controlled by very asymmetric branchings (with z near 0 or 1). In particular, one can verify that for the DGLAP cascade, the energy flow vanishes when $x_0 \to 0$ [roughly like $\mathcal{P}(x_0) \sim x_0 \ln(1/x_0)$]: the total energy of the cascade remains in the spectrum, as already mentioned. Furthermore, the total energy carried by the soft modes at $x \leq x_0$ with $x_0 \ll 1$ is relatively small, as can be inferred from phase-space considerations. This is very different from the turbulent cascade studied here, in which a significant fraction of the total energy is transported below any given value $x_0 > 0$, meaning at very large angles.

Many of the features that we have uncovered by studying the source problem and for the simplified kernel remain valid without the source, and for the general kernel, as we have verified via an explicit numerical solution. We return now to this initial setting and limit ourselves to small times, for which we can obtain analytical estimates. The flow, calculated from Eq. (11), takes the form

$$\mathcal{P}(x_0, \tau) \simeq 2\sqrt{x_0} + \upsilon\tau, \tag{13}$$

where $\upsilon = 4.96$ is given by an integral similar to that in Eq. (12) but with the full $f(z)$ in the integrand. One recognizes in Eq. (13) the two components that we discussed earlier, that is, the direct radiation ($2\sqrt{x_0}$) and the turbulent flow ($\upsilon\tau$). Although formally subleading at small times, the turbulent flow dominates over direct radiation when $x_0 \lesssim \tau^2$, that is, in the region where multiple branchings are known to be important.

The total energy transported by the turbulent flow can be estimated by integrating the second term of Eq. (13) over time. Returning to physical units, one gets

$$\mathcal{E}_{\text{flow}} = E\frac{\upsilon\tau_{\max}^2}{2} = \upsilon\bar{\alpha}^2\omega_c, \tag{14}$$

a result which, remarkably, is independent of the energy E of the leading particle. This turbulent flow is a part of the jet energy that is not carried by the particles present in the spectrum. It corresponds to what we identified earlier as the energy stored in a condensate at $x = 0$. In more physical terms, we may associate this energy with that transferred to the medium in the form of very soft quanta emitted at large angles.

It is beyond the scope of this Letter to present a detailed comparison with the data. However, the following order-of-magnitude estimates should confirm the relevance of the present discussion for the dijet asymmetry observed at the LHC. Using the conservative estimate $\omega_c = 40$ GeV (corresponding to $\hat{q} = 1\ \text{GeV}^2/\text{fm}$ and $L \simeq 4$ fm), together with $\bar{\alpha}^2 \simeq 0.1$, one finds $\mathcal{E}_{\text{flow}} \simeq 20$ GeV, a value that compares well with the observations. Indeed, the detailed analysis by CMS [3] shows that the energy imbalance between the leading and the subleading jets is compensated by an excess of semihard ($p_T < 8$ GeV) quanta propagating at large angles, outside the cone defining the subleading jet. For the most asymmetric events, the total energy in excess is about 25 GeV. Remarkably, most of this energy (about 80%) is carried by very soft quanta with $p_T \leq 2$ GeV [14]. This observation would be difficult to reconcile with the hypothesis that these particles come from gluons in a BDMPSZ-like spectrum (which would imply that most of the excess energy would be carried by the hardest gluons with energies $\lesssim 8$ GeV). But, it could be naturally explained by associating these soft particles with those transported by the turbulent flow that we have discussed in this Letter.

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