Kolmogorov-Zakharov Spectrum in AdS Gravitational Collapse

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We study black hole formation during the gravitational collapse of a massless scalar field in asymptotically D-dimensional anti–de Sitter AdS_D spacetimes for $D = 4, 5$. We conclude that spherically symmetric gravitational collapse in asymptotically AdS spaces is turbulent and characterized by a Kolmogorov-Zakharov spectrum. Namely, we find that after an initial period of weakly nonlinear evolution, there is a regime where the power spectrum of the Ricci scalar evolves as ω^{-s} with the frequency, ω, and $s \approx 1.7 \pm 0.1$.

Introduction.—Because of the anti–de Sitter/conformal field theory (AdS/CFT) correspondence the question of instability of AdS spacetimes sits in the intersection of mathematical and numerical relativity, string theory, field theory, and condensed matter physics. The stability of spacetimes is central to the study of general relativity as spacetime is not just a stage where interactions take place; rather, it is a dynamical participant. AdS is the maximally symmetric solution to the Einstein equations with a negative cosmological constant and one would expect it to be stable. It has recently been argued that AdS is nonlinearly unstable [1] to black hole formation. Namely, an arbitrarily small generic perturbation leads to the formation of a black hole. This is completely different from the situation in the other two maximally symmetric solutions to the vacuum Einstein equation—Minkowski (zero cosmological constant) and de Sitter (positive cosmological constant)—which are known to be stable. The feature responsible for this instability is the fact that AdS has a timelike boundary at spatial and null infinity. The mechanism proposed in [1], called weakly turbulent, as responsible for this instability was resonant mode mixing that gives rise to diffusion of energy from low to high frequencies.

The study of nonlinear instability of AdS initiated in [1] has been followed up in [2] for higher dimensions and in [3] for a different type of configuration in pure AdS but with angular dependence. Some discussions in the context of the relation to thermalization in the dual field theory have appeared in [4,5].

In this Letter we study in more detail the turbulent mechanism conjectured in [1]. We implement a suggestion put forward in [6] to apply the arsenal of methods of dynamical systems to the gravitational collapse in asymptotically AdS spacetimes. One of the most celebrated results in the theory of turbulent flows is Kolmogorov's theory of 1941 [7], a master piece of dimensional analysis (for modern accounts see the monographs [8,9]). In [6] we argued that the power spectrum of the Ricci scalar at the origin shows period doubling, a sign of turbulent evolution; here we establish that, in the appropriate regime, the power spectrum has a Kolmogorov-Zakharov scaling with the frequency, ω, as ω^{-s}, with $s \approx 1.7 \pm 0.1$. This spectrum is indicative of wave turbulence, a paradigm describing turbulence among strongly interacting waves [10,11].

Gravitational collapse in asymptotically D-dimensional AdS spacetimes.—We consider the dynamics of a massless scalar field, φ, in $D = d + 1$ dimensions minimally coupled to gravity with a negative cosmological constant Λ:

$$S = \int d^{d+1}x \sqrt{-g} \left(\frac{1}{16\pi G}(R - \Lambda) - \frac{1}{2}(\partial\varphi)^2 \right) \quad (1)$$

where G is Newton's constant. We focused on spherically symmetric configurations described by the following metric [1],

$$ds^2 = \sec^2\left(\frac{x}{\ell}\right)\left[-Ae^{-2\delta}dt^2 + A^{-1}dx^2 \right.$$
$$\left. + \ell^2\sin^2\left(\frac{x}{\ell}\right)d\Omega_{d-1}^2 \right], \quad (2)$$

where $\ell^2 = -d(d - 1)/2\Lambda$, $d\Omega^2$ is the metric on the unit $(d - 1)$-sphere. The functions A, δ and the scalar field, φ, depend on (t, x). Notice that the spatial domain is contained in the interval $0 < x < \pi/2$. The AdS spacetime, which is the maximally symmetric solution to the vacuum Einstein equations with a negative cosmological constant, Λ, corresponds to $A = 1$, $\delta = 0$ and $\varphi = 0$. In this context it is the analog of the Minkowski spacetime for zero cosmological constant and de Sitter spacetime for positive cosmological constant.

By introducing the auxiliary variables $\Phi = \varphi'$ and $\Pi = A^{-1}e^{\delta}\dot{\varphi}$, where the overdots and primes denote derivatives with respect to t, x, respectively, the field equations read:

$$\delta' = -\frac{8\pi G\ell}{d - 1} \cos\left(\frac{x}{\ell}\right)\sin\left(\frac{x}{\ell}\right)(\Pi^2 + \Phi^2), \quad (3)$$

$$A' = -\frac{8\pi G A \ell}{d-1}\cos\left(\frac{x}{\ell}\right)\sin\left(\frac{x}{\ell}\right)(\Pi^2 + \Phi^2),$$
$$+ \frac{1-A}{\ell\cos(\frac{x}{\ell})\sin(\frac{x}{\ell})}\left[d - 2 + 2\sin^2\left(\frac{x}{\ell}\right)\right] \quad (4)$$

$$\dot{\Phi} = (Ae^{-\delta}\Pi)', \quad (5)$$

$$\dot{\Pi} = \frac{1}{\tan^{d-1}(\frac{x}{\ell})}\left[\tan^{d-1}\left(\frac{x}{\ell}\right)Ae^{-\delta}\Phi\right]'. \quad (6)$$

The first two equations are constraint equations, the third equation is a consequence of the definition of the auxiliary variables, and the last is the Klein-Gordon equation $g^{\mu\nu}\nabla_\mu(\partial_\nu\varphi) = 0$. Hereafter, we assume units where $8\pi G = d - 1$.

There is a natural mass function, $m(x, t)$, in AdS spacetimes given by

$$1 - \frac{2m}{r^{d-2}} + \frac{r^2}{\ell^2} = g^{\alpha\beta}\partial_\alpha r \partial_\beta r, \quad (7)$$

where the standard spherical coordinate r is related to x as $r = \ell\tan(x/\ell)$, and one can show that,

$$m(x, t) = (1 - A)\frac{\ell^{d-2}\sin^{d-2}(\frac{x}{\ell})}{2\cos^d(\frac{x}{\ell})}. \quad (8)$$

This expression gives the total mass energy inside a radius x at the instant t. The Arnowitt-Deser-Misner (ADM) mass of the system is obtained by evaluating the mass function asymptotically, or $M_{ADM} = \lim_{x\to\pi\ell/2}m(x, t)$. As a typical feature of various spacetimes, including AdS, the ADM mass is a conserved quantity.

Toward a dynamical system via spectral methods.—We integrate numerically the field equations (2)–(5) using the Galerkin-Collocation method [12]. The central notion of any spectral method is to approximate a system of partial differential equations by a finite set of ordinary differential equations associated to unknown coefficients or to the grid values of certain quantities. This dynamical system approach proves very useful in describing the complex dynamics of the scalar field collapse in D-dimensional AdS spacetimes.

The starting point is to provide an approximation for the metric functions in the form of series of appropriate basis functions:

$$\Pi_a = \sum_{k=0}^{N} a_k(t)\psi_k(y), \qquad \Phi_a = \sum_{k=0}^{N-1} b_k(t)\chi_k(y), \quad (9)$$

$$A_a = 1 + \sum_{k=0}^{M} c_k(t)\psi_k^{(A)}(y), \qquad \delta_a = \sum_{k=0}^{M} \delta_k(t)\psi_k^{(\delta)}(y). \quad (10)$$

In the above expressions M, N are the truncation orders; $a_k(t)$, $b_k(t)$, $c_k(t)$, $\delta_k(t)$ are the unknown modes or coefficients that constitute the spectral representation of the metric functions. The basis functions $\psi_k(y)$, $\chi_k(y)$, $\psi_k^{(A)}(y)$, $\psi_k^{(\delta)}(y)$ depend on the spatial coordinate $y = 4x/\pi\ell - 1$ which varies in the interval $-1 \le y < 1$. These basis functions are linear combinations of Chebyshev's polynomials which satisfy the condition of regularity of the field equations at the origin ($x = 0$ or $y = -1$) and at the spatial infinity ($x = \ell\pi/2$ or $y = 1$) [1].

The substitution of the approximations given by Eqs. (9) and (10) into the field equations (3)–(6) results in the residual equations. We impose that the residual equations vanish at certain points known as the collocation or grid points. As a consequence, the field equations are transformed into sets of ordinary differential equations for the modes ($a_k(t)$, $b_j(t)$), along with algebraic equations $c_k = c_k(a_i, b_j, \Pi_l, \Phi_m)$ and $\delta_k = \delta_k(a_i, b_j, \Pi_l, \Phi_m)$, where Π_l, Φ_m are the grid values of the functions Π, Φ, respectively.

To evolve the spacetime we consider the initial data [1]: $\Phi(0, x) = 0$, $\Pi(0, x) = \epsilon_0\exp(-\tan^2 x/\sigma^2)$, with σ and ϵ_0 as free parameters. In the spectral space these initial data are expressed as $b_j(0) = 0$ and $\Pi(0, x_j) = \sum_{k=0}^{N} a_k(0)\psi_k(x_j)$, where x_j denotes the collocation points. From this last expression, the initial modes $a_k(0)$ are obtained. With these initial conditions, the dynamical system is numerically evolved and the relevant functions reconstructed afterwards. We will show elsewhere the full details of this procedure and the numerical tests that guarantee the accuracy of the integration. Here, as an example of numerical accuracy, we show in Fig. 1 the behavior of the relative error in the ADM mass $|M_{ADM}(t) - M_{ADM}(t = 0)|/M_{ADM}(t = 0)$, where M_{ADM} is defined below Eq. (8). The truncation orders are from top to bottom: 10 (gray), 20 (cyan), 30 (red), 40 (blue), and 50 (gold). Note that the periodic structure in time (with period approximately π) is the result of the timelike boundary at spatial infinity typical of AdS and has been pointed out in numerical simulations in [1,4,6].
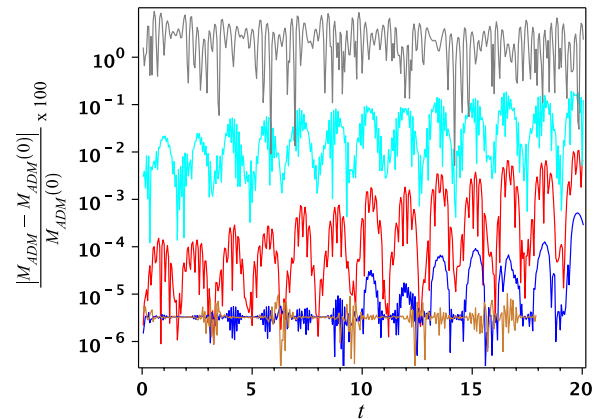

FIG. 1 (color online). Relative error of the ADM mass as a function of time.

The weakly nonlinear regime.—By decomposing the scalar field in eigenmodes of the AdS Laplacian, the analysis of [1] gained important insight into the interactions among these modes during the gravitational evolution. It was shown analytically that for generic initial data there is no way to remove all the resonances; some secular terms persist at higher order in perturbation theory, even after accounting for modulation of phase. Inspired by this analytical result, [1] also showed numerically that the time scale after which the weakly nonlinear approximation breaks down is of the order $t_{NL} \sim 1/\epsilon_0^2$, where ϵ_0 is the amplitude of the initial profile.

Let us discuss, motivated by the results of [1], a convenient way of viewing the dynamics of a scalar field in AdS spacetimes [6]. In the weak field regime, $\delta, 1 - A \ll 1$, the Einstein-Klein-Gordon equation in spectral space can be written as,

$$\ddot{a}_k(t) - \frac{1}{\tan^2(x(y))}[\tan^2(x(y))]'a_k(t)$$
$$+ \mathcal{F}(\delta, 1 - A, a_k, \dot{a}_k, \ddot{a}_k, \ldots) = 0. \quad (11)$$

The first two terms represent the evolution of the linearized modes (see Wald [13]) of the scalar field that oscillate in well-defined frequencies $\omega_j^2 = (d + 2j)^2$, $j = 0, 1, 2, \ldots$ The term denoted by $\mathcal{F}$ contains the interaction with the gravitational sector. Therefore, we imagine our system as a set of nonlinearly interacting oscillators, and consequently the periodic motion can be understood as taken place on a torus in the abstract phase space spanned by the modes $a_k(t)$. Further numerical experiments indicated that if the term $\mathcal{F}$ is initially small, its influence on the long term dynamics is to alter the natural frequencies of oscillation producing quasiperiodic motion. Then, the motion becomes unstable and the torus is eventually destroyed forming a strange attractor, which is a signature of chaos in the dynamical system. This means that the evolution of the scalar field in AdS is turbulent. This route to turbulence resembles the Ruelle-Takens scenario [14] where the initial amplitude ϵ_0 plays the role of the control parameter. We presented evidence for such a picture in [6].

Nonlinear regime: A Kolmogorov-Zakharov spectrum.— For a diffeomorphism invariant theory physical information is contained in coordinate-independent quantities; this is one of the most challenging aspects of applying the standard theory of dynamical systems to gravitational problems. We follow up on one of the suggestions presented in [6]: to study the power spectrum of the scalar curvature. For the backgrounds we are considering the scalar curvature can be written as:

$$R = 1/l^2[A\cos(x)^2(-\Pi^2 + \Phi^2) - d(d - 1)]. \quad (12)$$

We have found similar results using the power spectrum of the mass defined in Eq. (8). Mindful that results about the power spectrum might be affected by coordinate redefinitions but hopeful that the general theorems proving that

relativistic chaos is coordinate invariant [15] extend to our context, we proceed to a detailed study of properties of the power spectrum of the Ricci scalar.

Given the notion that for a given amplitude ϵ_0 the weakly nonlinear regime lasts for a time interval proportional to $1/\epsilon_0^2$, by suitably choosing the amplitudes we can reliably separate the regimes of the scalar field evolution during collapse. In Fig. 2 we present the power spectrum for two systems with amplitudes 10^{-3} (red straight line) and 2.5 (black line), respectively. The small amplitude evolution is still in the weakly nonlinear regime where the system is well described by a set of weakly coupled oscillators yielding well-defined peaks at frequencies corresponding to those of the eigenvalues of the Laplace operator in AdS. This is precisely what we see in the power spectrum of this signal. Clearly, for higher frequencies we already witness a more complicated picture signalling that the oscillators are not free and, as discussed in [1], interact by transferring energy to higher frequency modes. In the same figure we have plotted the power spectrum for a higher amplitude simulation in which the nonlinear regime sets in earlier. The spectrum of the of the nonlinear evolution seems to follow a precise scaling and we have also plotted the $-5/3$ slope line that corresponds to the Kolmogorov spectrum.

It is well known that Kolmogorov scaling is a property of a steady-state system, that is, of systems subjected to an injection of energy at some scale and allowing a sink or energy at some other scale [7–9]. Therefore, the only way for us to view the appearance of this type of spectrum during collapse is as a regime in the full evolution of the black hole formation. Since the time scales of the different regimes are governed by the initial amplitude ϵ_0 we have *de facto* created a steady-state situation by suitably choosing the values of the amplitude.
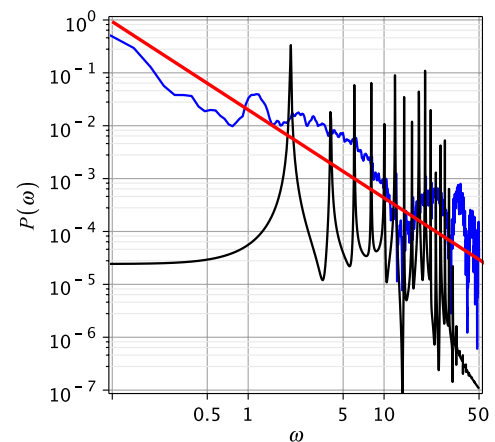

FIG. 2 (color online). Power spectrum for different amplitudes in gravitational collapse in AdS$_5$. The linear regime ($\epsilon_0 = 10^{-3}$) with well-defined frequencies (black line); a nonlinear regime ($\epsilon_0 = 2.5$) that approximately follows a power law and a straight line that is the Kolmogorov 5/3 scaling (red line).
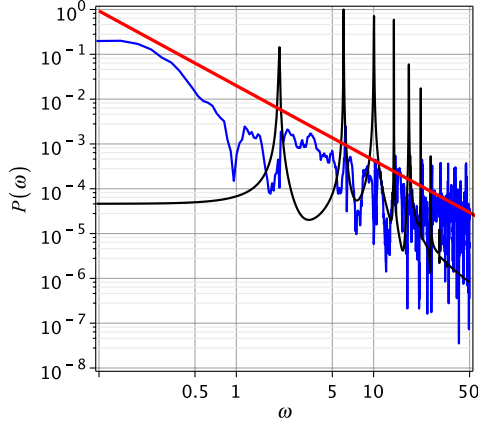
FIG. 3 (color online). Power spectrum for different amplitudes in gravitational collapse in AdS_4. The linear regime ($\epsilon_0 = 10^{-2}$) with well-defined frequencies (black line): a nonlinear regime ($\epsilon_0 = 2.5$) that approximately follows a power law and a straight line that is the Kolmogorov 5/3 scaling (red line).

We are, therefore, not describing a steady-state situation but a phase in the evolution of scalar gravitational collapse in AdS. We know that the ultimate fate in the evolution of the system is the formation of a black hole. However, we uncover Kolmogorov scaling by judiciously choosing the amplitude and the time interval from our numerical simulations of gravitational collapse. We have, at least two important scales in the collapse problem: the nonlinear scale $t_{NL} \sim 1/\epsilon_0^2$ and a new scale that we call the Kolmogorov scale, t_K; evolution beyond t_K involves the violent collapse into a black hole.

Turbulence is different depending on the dimensionality of the space. Below we provide results for collapse in asymptotically AdS_4 spacetimes. However, in the spherically symmetric approximation to gravitational collapse all functions depend on only two coordinates—time and radius. As can be seen from Eq. (6), different powers of the radius dominate the evolution near the center. We verified explicitly a claim advanced in [2] that the same turbulent mechanism is at work in spherically symmetric collapse in AdS_4 and AdS_5. Below we present the power spectrum for gravitational collapse in AdS_4 and verify the existence of a power scaling in the appropriate regime see Fig 3.

Another important evidence for the universality of our claim is its independence of the initial scalar field profile. Figure 4 shows a power spectrum for an initial profile given as the sum of eigenfunctions of AdS. Let us briefly explain the reasoning for these initial conditions. The AdS limit of the metric Eq. (2) corresponds to $A = 1$, $\delta = 0$. Consider expanding the scalar field as $\phi(t, x) = \sum_j \phi_j(t) e_j(x)$, where the basis $e_j(x)$ are eigenfunctions of the following operator:

$$\frac{1}{\tan^2 x} \frac{d}{dx}\left(\tan^2 x \frac{d}{dx} e_j(x)\right) + (3 + 2j)^2 e_j = 0. \quad (13)$$
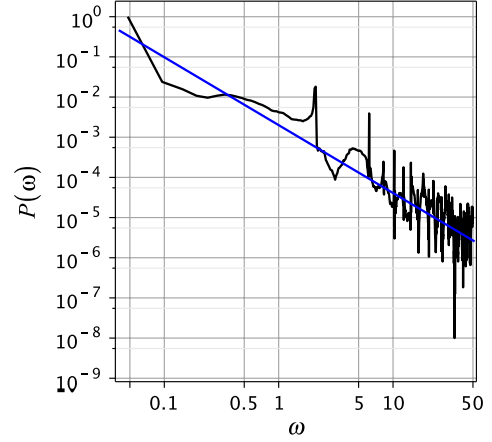

FIG. 4 (color online). Power spectrum for an initial condition $\phi(t = 0, x) = \sum_{i=1}^{20} e_i(x)$, built out of 20 eigenmodes of the Laplacian in AdS.

The functions $e_j(x)$ can be orthonormalized as [1]:

$$e_j(x) = d_j \cos^3 x\, {}_2F_1\left(-j, 3 + j, \frac{3}{2}; \sin^2 x\right),$$

$$d_j = \sqrt{\frac{16(j+1)(j+2)}{\pi}}, \quad \int_0^{\pi/2} e_j(x) e_k(x) \tan^2 x\, dx = \delta_{jk}. \quad (14)$$

Upon substitution in the Klein-Gordon equation one obtains

$$\ddot{\phi}_j(t) + \omega_j^2 \phi_j(t) = 0, \quad (15)$$

corresponding to free harmonic oscillators with frequencies $\omega_j = 3 + 2j$. An initial condition of this type is particularly well-suited to the problem and it gives us the ability to study the linearized regime very carefully and understand its transition into the nonlinear regime.

We have further performed simulations with a varying number of modes and the results are consistent with those obtained for an initial Gaussian profile. The slope of the power spectrum corresponds to $s = 1.69$, compatible with those obtained for the Gaussian profile.

Conclusions.—We have provided a more complete picture of the gravitational collapse in AdS. In particular, we have gone beyond previous descriptions which focused on the weakly nonlinear regime characterized by transfer of energy to high frequency modes [1]. We have uncovered that after this weakly nonlinear regime a turbulent regime takes over. Crucially, this turbulent regime is characterized by a Kolmogorov-Zakharov scaling. We have obtained similar results for the power spectrum in AdS_5 and AdS_4 spacetimes; we have also considered two initial profiles for the scalar field (Gaussian and sum of eigenfunctions). Thus, by considering various dimensions, two types of initial conditions, and various parameters of the initial conditions, we have found that the range of the power

P H Y S I C A L   R E V I E W   L E T T E R S

spectrum is $s = 1.7 \pm 0.1$; we will report on a more detailed numerical error analysis elsewhere.

We would like to highlight some of the key open problems that our results motivate.

We have discussed that the steady-state assumption in turbulent flows is achieved in our problem through a separation of scales. Given the connections between gravity and the Navier-Stokes equation [16], it will be interesting to pursue a description of collapse more along the lines of turbulent fluid dynamics. Namely, we have indications that the phenomena we are witnessing in gravitational collapse is along the lines of wave turbulence [10,11]. The wave turbulence approach provides an explanation for various aspects that are not typical of full-blown Kolmogorov turbulence such as the similarity of our results in AdS$_5$ and AdS$_4$. More importantly, wave turbulence provides a semianalytical approach to our Kolmogorov-Zakharov spectrum. Aspects of wave turbulence in gravitational collapse will be reported elsewhere [17]. Similar applications of wave turbulence have been developed for a relativistic system [18,19] and provide a tentative explanation of why we find a Kolmogorov-Zakharov (rather than pure Kolmogorov) spectrum in a relativistic system.

It has recently been noted that boson stars might be stable in asymptotically AdS spacetimes because the eigenmodes fail to meet the resonance condition [20]. It would be interesting to scrutinize this claim under the nonlinear perturbation microscope since the very concept of resonance has to be subjected to numerical constraints.

Finally, given that the process of gravitational collapse is dual to thermalization in field theory, it makes sense to speculate that the universality in the turbulent regime means that for strongly coupled field theories the process of thermalization always goes through a similarly turbulent channel.

[1] P. Bizon and A. Rostworowski, Phys. Rev. Lett. **107**, 031102 (2011).

[2] J. Jalmuzna, A. Rostworowski, and P. Bizon, Phys. Rev. D **84**, 085021 (2011).

[3] O. J. C. Dias, G. T. Horowitz, and J. E. Santos, arXiv:1109.1825.

[4] D. Garfinkle, L. A. Pando Zayas, and D. Reichmann, J. High Energy Phys. 02 (2012) 119.

[5] D. Garfinkle and L. A. Pando Zayas, Phys. Rev. D **84**, 066006 (2011).

[6] H. P. de Oliveira, L. A. Pando Zayas, and C. A. Terrero-Escalante, Int. J. Mod. Phys. D **21**, 1242013 (2012).

[7] A. N. Kolmogorov, Doklady Acad. Sci. USSR **31**, 301 (1941).

[8] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, England, 2000).

[9] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).

[10] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I—Wave Turbulence* (Springer, New York, 1992).

[11] S. Nazarenko, *Wave Turbulence* (Springer, New York, 2011).

[12] J. Boyd, *Chebyshev and Fourier Spectral Methods* (Dover, New York, 2001), 2nd ed.

[13] A. Ishibashi and R. M. wald, Classical Quantum Gravity **21**, 2981 (2004).

[14] S. Newhouse, D. Ruelle, and F. Takens, Commun. Math. Phys. **64**, 35 (1978).

[15] A. E. Motter, Phys. Rev. Lett. **91**, 231101 (2003).

[16] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, J. High Energy Phys. 07 (2012) 146.

[17] G. Düring, C. Falcón, and L. A. Pando Zayas, "The Wave Turbulence Path to Gravitational Collapse" (to be published).

[18] R. Micha and I. I. Tkachev, Phys. Rev. Lett. **90**, 121301 (2003).

[19] R. Micha and I. I. Tkachev, Phys. Rev. D **70**, 043538 (2004).

[20] O. J. C. Dias, G. T. Horowitz, D. Marolf, and J. E. Santos, arXiv:1208.5772.