Transition from Amplitude to Oscillation Death via Turing Bifurcation

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Coupled oscillators are shown to experience two structurally different oscillation quenching types: amplitude death (AD) and oscillation death (OD). We demonstrate that both AD and OD can occur in one system and find that the transition between them underlies a classical, Turing-type bifurcation, providing a clear classification of these significantly different dynamical regimes. The implications of obtaining a homogeneous (AD) or inhomogeneous (OD) steady state, as well as their significance for physical and biological applications and control studies, are also pointed out.

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Since the work of Van der Pol and Van der Mark [1], the studies of coupled nonchaotic oscillators have provided a rich source of ideas and insights regarding the role of different coupling types, as well as the dependence on the oscillator structure in the generation of new dynamical regimes [2–4]. It has been shown that even ensembles consisting of identical oscillators may generate a variety of rhythms that differ in their period and phase relations based on the coupling organization [5–8]. Apart from such rhythmogenic activity, coupling can even suppress oscillations in a network by different mechanisms. Here, we distinguish between two main manifestations of oscillation quenching, amplitude and oscillation death phenomena, which are structurally different.

Generally, the amplitude death (AD) refers to a situation where oscillations are suppressed when individual oscillators are coupled and return to the steady state of the system instead. Thus, the amplitude death results in a homogeneous steady state (HSS), since all of the oscillators populate the same state. Three main mechanisms can lead to this phenomenon: (a) a sufficiently large variance of the frequency distribution [2,9], (b) existence of time delay in the coupling [10–13], and (c) coupling of identical oscillators through dissimilar (or conjugate) variables [14,15] (for a recent review on AD, see Ref. [16]). On the other hand, the second manifestation of oscillation quenching—the oscillation death (OD) phenomenon—has a significantly different background of occurrence compared to AD. Namely, OD is a result of breaking the system's symmetry through a pitchfork bifurcation of the unstable steady state, whereby the homogeneous steady state splits, giving rise to two additional branches. In the limiting case of two coupled oscillators, one follows the upper, whereas the second oscillator follows the lower branch. Thus, OD is manifested as a stabilized inhomogeneous steady state (IHSS), displaying further the possibility for the occurrence of additional limit cycle(s) in the same phase space area.

The idea of the broken symmetry steady state pioneered by Turing [17] for stationary media received its mathematical formulation by Prigogine and Lefever [18] for two identical oscillating elements—Brusselators, coupled in a diffusionlike manner. Furthermore, it has been shown theoretically that OD is model independent, persisting for large parametric regions in several models of diffusively coupled chemical [19] or biological oscillators [7,20–24]. Experimentally, the extinction of oscillations in chemical reactors coupled by mutual mass exchange was initially reported by Dolnik and Marek [25]. Later on, Crowley and Epstein demonstrated for two coupled, slightly nonidentical chemical oscillators that the basis for the OD is a specific, vector-type coupling, namely, coupling via a slow recovery variable [26]. Recently, OD has been experimentally observed in chemical nanooscillators (microfluidic Belousov-Zhabotinsky-octane droplets), diffusively coupled via signaling species (Br_2, in this case) [27]. However, in certain systems, i.e., in neurobiology, a manifestation of both oscillation quenching types is present: OD constitutes a well-known phenomenon in neurons, the winner-take-all situation [28], whereas AD mainly serves to suppress neuronal oscillations [29].

Because of their significantly different representations, as inhomogeneous (OD) and homogeneous (AD) steady states, both oscillation quenching types allow the generation of two structurally different dynamical regimes with different meaning. This is important not only from a viewpoint of dynamical control but also from an application aspect: it has been shown that OD can be interpreted as a background mechanism of cellular differentiation [30,31], whereas AD is mainly used as a stabilization control in physical or chemical systems [32,33]. Thus, OD is especially significant for biology, since in contrast to AD, it can provide presence of heterogeneity in a stable homogeneous medium. This possibility is further widened by the fact that OD is a source of a stable inhomogeneous limit cycle

(IHLC) in many systems [22,34], and an IHLC represents in turn an additional mechanism for generating heterogeneity.

Although mathematically the background mechanisms, as well as the manifestations of OD and AD phenomena, can be clearly distinguished, some authors consider these different quenching types without strong discrimination [14]. A recent burst of publications [35–37] devoted to oscillation quenching thus requires further clarifications in this field.

One of the most challenging open problems here is to identify the transition scenario(s) between these two very distinct dynamical regimes, allowing us in turn to stress the sharp contrast between them. Thus, in this Letter, we aim to identify the role of coupling in the evolution from AD to OD for the first time in a single paradigmatic system, characterizing additionally the accompanying limit cycles. We confine ourselves primarily to the transition effect and establish a classification of the conditions under which one or both of the oscillation quenching manifestations can be observed.

We analyze a model of two coupled Landau-Stuart oscillators [38] given by the equation of motion

$$\dot{z} = (1 + i\omega - |z|^2)z, \tag{1}$$

with ω being the frequency, and $z(t) = x(t) + iy(t)$. Coupling two such oscillators diffusively, the dynamical equations expressed in Cartesian coordinates give

$$\dot{x}_i = P_i x_i - \omega_i y_i + \varepsilon(x_j - x_i), \qquad \dot{y}_i = P_i y_i + \omega_i x_i. \tag{2}$$

Here, $P_i = 1 - |z_i|^2$, $i, j = 1, 2$, and $i \neq j$. The parameter ε governs the coupling strength. For $\varepsilon = 0$, each oscillator has a stable limit cycle at $|z_i| = 1$ on which it moves at its natural frequency ω_i, rendering the equilibrium solution of system (2) $z_i = 0$ linearly unstable.

We are interested here in stabilization scenarios, i.e., routes that lead to one of the two oscillation quenching types, AD or OD phenomena, as well as the possibilities for transition between them. Previous studies have underlined the fact that different frequencies can stabilize coupled oscillators even in the absence of delays, whereas identical frequencies cannot [2,13]. Since $\omega_i = \omega_j$ is a more stringent case for stability, we consider first the case of oscillators with identical frequencies. The analysis is performed by tracking the detailed bifurcation structure of the system (using the XPPAUT package [39]) as the coupling coefficient is varied.

For identical elements (without any loss of generality, we consider here $\omega_1 = \omega_2 = 2$), we find a manifestation of a stable oscillation death regime. The route to OD in the system of coupled identical Landau-Stuart oscillators follows the classical bifurcation scenario as characterized in previous studies [7,19,21]. In particular, the system (2) has two sets of fixed points: the origin $(0, 0, 0, 0)$ which exists for all ε and is unstable, and the pair $(x_{1^*}, y_{1^*}, -x_{1^*}, -y_{1^*})$, where $x_{1^*} = -(\omega y_{1^*}/(\omega^2 + 2\varepsilon y_{1^*}^2))$ and $y_{1^*} = \sqrt{(\varepsilon - \omega^2 \pm \sqrt{\varepsilon^2 - \omega^2})/2\varepsilon}$. The characteristic eigenvalue equation at the origin determines the conditions for which an IHSS occurs: for $\varepsilon = ((\omega^2 + 1)/2) = 2.5$, a pitchfork bifurcation [PB_1 in Fig. 1(a)] gives rise to the two separate branches of the IHSS [time traces shown in Fig. 3(a)], on which OD is stabilized via Hopf bifurcations [HB_{s1} in Fig. 1(a)].

Although the Hopf bifurcations determined with the bifurcation analysis [HB_1 and HB_{s1} in Fig. 1(a)] give rise to unstable limit cycles, direct numerical simulations identified the existence of stable oscillatory solutions. We detect two main oscillatory solutions: in-phase oscillations, whose region of stability ends with the occurrence of a stable OD regime [time traces given schematically in Fig. 3(a)], and antiphase oscillations with a significantly smaller stability region (results not shown).

In general, oscillation quenching in systems of nonidentical oscillators, i.e., oscillators with a defined frequency [$\Delta = (\omega_i/\omega_j)$] or parameter mismatch, has been vaguely characterized from the aspect of OD [23,36,40], in contrast to its AD counterpart [2,9,19,38]. Thus, we ask here the following question: What is the role of coupling in the evolution from the AD to the OD phenomenon, and under which conditions is this process determined? In order to identify the possibility for OD manifestation if $\Delta > 1$, as well as to identify the transition scenario between both

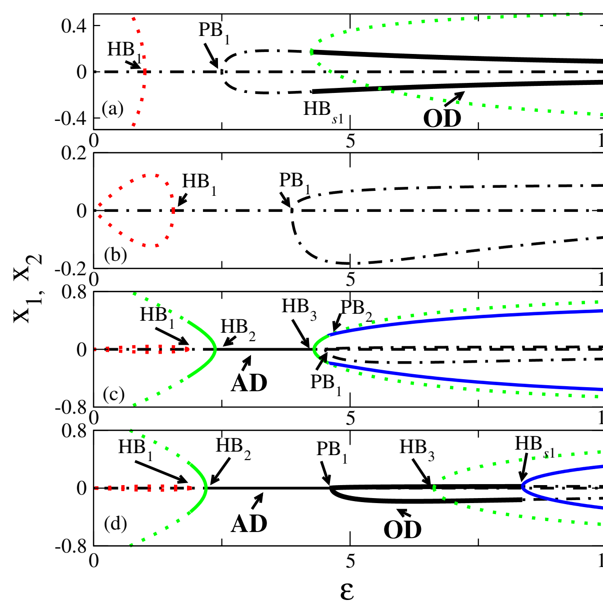

FIG. 1 (color online). Role of coupling in the evolution from AD to OD. Parameters: $\omega_2 = 2$ and (a) $\Delta = 1$, (b) $\Delta = 2$, (c) $\Delta = 3.385$, and (d) $\Delta = 4$. Additionally, the accompanying limit cycles are also shown. Thin solid lines denote a stable HSS, thick solid lines a stable IHSS, dash-dotted lines an unstable steady state; dotted lines denote unstable limit cycles, whereas colored (online) solid lines denote stable limit cycle solutions.

oscillation quenching types, we study next the dynamical structure of the system for increasing frequency mismatch.

If $1.95 < \Delta < 3.1$, the oscillators no longer suffer death: the OD regime (stable for $1 \leq \Delta \leq 1.95$) is now destabilized, and additionally, no presence of stable AD has been identified [Figs. 1(b) and 2]. Thus, for $\varepsilon > 0$, the dynamics is periodic [Fig. 3(b)]. However, at $\Delta \approx 3.1$, a critical transition occurs: for a given coupling strength (here, $\varepsilon = 2.91$), the stability of the origin changes, and the limit cycles collapse into the origin via an inverse Hopf bifurcation [HB_2 occurs for $\varepsilon = 2.37$ in Fig. 1(c), demonstrating this principle for $\Delta = 3.385$]. This means that the oscillations are quenched under the coupling, and the system evolves toward the homogeneous equilibrium which is stable between HB_2 and HB_3 [in Fig. 1(c), HB_3 occurs for $\varepsilon = 4.299$]. This allows for a stable AD effect to be observed (as previous classical analyses have shown [2,38]). However, what previous studies failed to observe is that in this case, the pitchfork bifurcation responsible for the symmetry breaking of the HSS is also still present in a near proximity of HB_3, right after AD is destabilized [PB_1 at $\varepsilon = 4.518$ in Fig. 1(c)]. The frequency mismatch Δ, on the other hand, is not sufficient to stabilize the two branches of the IHSS. What is important to note here, however, is the presence of an additional stable limit cycle from the supercritical Hopf bifurcation HB_3. In particular, stable oscillations [small amplitude, phase-shifted oscillations; see the conjecture lines in Fig. 3(c)] are present in a small parameter region between the supercritical Hopf bifurcation HB_3 and the pitchfork bifurcation PB_2. From this broken symmetry bifurcation point, a secondary stable bifurcation branch emerges, corresponding to an inhomogeneous limit cycle [characterized with a situation when both oscillators perform almost in-phase oscillations—but with shifted amplitudes; time traces are displayed in Fig. 3(c)]. The systematic analysis of the system reveals that the interplay between heterogeneity and coupling strength is sufficient to introduce symmetry breaking, here manifested via a secondary bifurcation structure. However, this level of heterogeneity allows only for one type of oscillation quenching to be stabilized—the AD phenomenon.

Increasing Δ even further leads to a critical value for which a qualitative transition occurs: the AD phenomenon represented via the stable homogeneous steady state transits to a stable inhomogeneous steady state, or an OD regime [Fig. 1(d) and conjecture lines in Fig. 3(d)]. The evolution between these two very distinct dynamical regimes is characterized with a classical Turing-type bifurcation. We demonstrate next the main structure of the corresponding bifurcation scenario.

In particular, for increasing frequency mismatch Δ, the broken symmetry bifurcation points of the HSS [PB_1 in Fig. 1(c)] and the limit cycle [PB_2 in Fig. 1(c)] come closer together, and at $\Delta_{\text{critical}} \approx 3.45$ they merge. This gives birth to a supercritical pitchfork bifurcation [PB_1 in Fig. 1(d)], which allows for symmetry breaking of the stable HSS (AD regime) and a transition to an inhomogeneous steady state and a stable OD regime. On the other hand, the supercritical Hopf bifurcation HB_3 which marked the AD stability region [Fig. 1(c)] now moves to the right, resulting in the birth of an unstable limit cycle solution [as shown in the bifurcation diagram and Fig. 3(d)]. The bifurcation scenario which we here identify as the main evolution transition between the AD and OD regimes resembles the key idea underlying the Turing mechanism: a homogeneous equilibrium is stable to homogeneous perturbations but unstable to certain spatially varying perturbations, leading to a spatially inhomogeneous steady state, that is, a spatial pattern. Thus, the transition from a homogeneous to an inhomogeneous steady state (HSS → IHSS), which we observe here, resembles a Turing scenario, only without the space variable. In more abstract terms, when considering a system of coupled oscillators, the oscillator number plays the role of a space coordinate. It is also important to stress here that in the classical Turing definition, the homogeneity of the medium does not arise due to the coupling present in the system, as in the current case.

By tracing the interdependence of the frequency mismatch Δ and the coupling strength ε, a more systematic characterization of the AD → OD transition can be established, as determined by the critical parameter values (Fig. 2). Clearly, the parametric stability region of newly stabilized OD regime is further increased with growing Δ values, and the Δ is also responsible for the occurrence of the AD phenomenon in this example. Moreover, it is critical to note that the inhomogeneous limit cycle solution which resulted from the broken symmetry bifurcation point of the limit cycle in Fig. 1(c) retains its structure after the formation of the stable inhomogeneous steady state [Fig. 3(d)]. In particular, the stable IHLC generated by the Hopf bifurcation which marks the stability region of

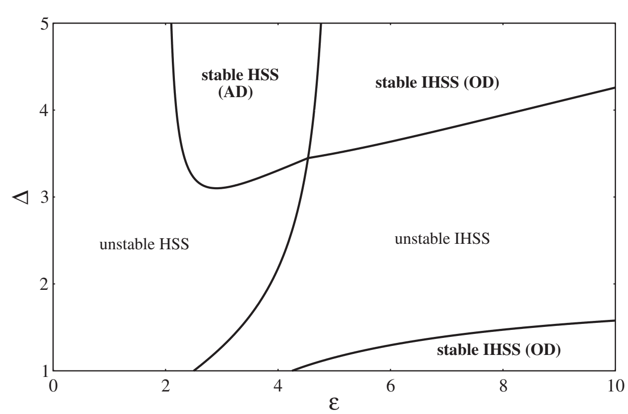

FIG. 2. Two parameter (ε vs Δ) bifurcation diagram denoting the transition between (un)stable homogeneous and inhomogeneous steady states for the system of two coupled Stuart-Landau oscillators [Eqs. (2)].

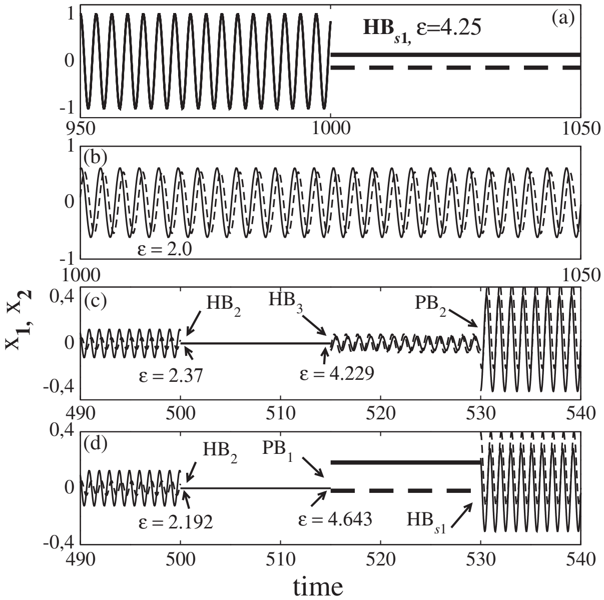

FIG. 3. Conjecture lines denoting stable limit cycles. The parameters are the same as in Fig. 1. Solid lines correspond to time traces of the first oscillator (x_1), whereas dashed lines correspond to the second oscillator (x_2).

the OD regime [HB_{s1} in Fig. 1(d)] has the same characteristics: both oscillators display almost in-phase oscillations, differing from each other by an amplitude shift [Fig. 3(d)].

We have also studied the general case of N oscillators connected with global diffusive coupling and have found that the transition scenario between the homogeneous and inhomogeneous steady states is preserved, since both AD and OD manifestations persist in the same form. By definition, in the case of AD, the oscillators return to the same steady state of the system, whereas the inhomogeneous steady state (OD) is manifested via a two cluster decomposition independently of N [7]. Thus, following a similar bifurcation scenario as for $N = 2$, the stable OD that exists for N coupled identical oscillators is destabilized when the eigenfrequencies of the oscillators start to spread apart, and the oscillators are then located on the limit cycles. Then, in a given (Δ, ε) interval, an inverse Hopf bifurcation gives rise to AD, until at the critical parameter values, a transition to stable OD occurs. Again, a Turing-type bifurcation is characteristic for this scenario. Local coupling, on the other hand, has been shown to induce novel manifestations of OD characterized with multiple stable clusters in the vicinity of the two classical OD branches [41]. Because of the complexity of the dynamical structures and the number of stable attractors that can appear under these conditions, the possibility for a slightly modified transition scenario is not excluded. Thus, this problem awaits further investigations.

Generally, the transition between AD and OD is not restricted to the methodological example studied here but also persists for different coupling types. We have studied additionally the Wilson-Cowan system describing the interaction between populations of inhibitory and activatory neurons (for simplification, each population was represented with a single oscillators, as demonstrated in Ref. [29]). Given that the system admits an orbitally stable periodic solution under a wide range of parameters, increasing the strength of connections between the excitatory neurons can cause the oscillations to disappear, and the system transits to an AD regime [29]. The background mechanism of AD is, however, different than the previously studied case of coupled Landau-Stuart oscillators: the AD here appears due to lack of uniformity of the local frequency along the limit cycle of the coupled system. A necessary feature for this mechanism is that the coupling does not vanish identically when the oscillators are in the same phase. Introducing now inhibitory connections between the neurons allows the system to break symmetry. Thus, for a critical strength of the inhibitory coupling, a supercritical pitchfork bifurcation determines the transition from stable HSS (AD regime) to an IHSS and a stable OD regime, as previously characterized.

The importance of oscillation quenching, both AD and OD, has been noted by many authors in relation to various physical and biological phenomena [2,7,19,21,28]. A suppression of oscillations is, in particular, significant in biology, and especially in neuronal systems, where oscillation quenching has been related to short term memory [42] and to selection and switching in the basal ganglia, both under normal and pathological conditions [43,44]. Our findings suggest, however, that special attention must be paid to the particular oscillation quenching type. In contrast to previous studies, where the existence of a frequency mismatch has only been considered as a route to AD phenomenon, we show here that for one-dimensional diffusive coupling, AD evolves towards OD, manifested as an inhomogeneous steady state with significantly different dynamical features. Our analysis has uncovered that a classical Turing-type bifurcation characterizes the transition from AD to OD. This in turn implies that both quenching types represent rather robust and common dynamical behavior for interacting oscillatory processes, since many real-life networks inevitably involve oscillatory processes with varying frequencies. Moreover, considering that one-dimensional diffusion is a frequent mechanism of coupling in chemical and biological systems, we expect that the transition scenario which we investigated here for the paradigmatic model of coupled Landau-Stuart oscillators is a characteristic feature of more realistic coupled systems. Additionally, the observation of the same bifurcation scenario under very different conditions (e.g., nondiffusive coupling and a different generation mechanism of AD in the Wilson-Cowan example), is in favor of the generality of the considered principle. Thus, another possible application of the discussed scenario could be maintaining different stable steady states (HSS or IHSS) of a laser output.

In this manner, it will also be interesting to investigate in future whether the proposed transition mechanism can be used as a stabilization control technique of coupled chaotic systems.

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[1] B. Van der Pol and J. Van der Mark, Philos. Mag. **6**, 763 (1928).
[2] G. B. Ermentrout, Physica (Amsterdam) **41D**, 219 (1990).
[3] M. Golubitsky, M. Nicol, and I. Stewart, J. Nonlinear Sci. **14**, 207 (2004).
[4] V. Casagrande and A. S. Mikhailov, Physica (Amsterdam) **205D**, 154 (2005).
[5] J. J. Collins and I. Stewart, J. Nonlinear Sci. **3**, 349 (1993).
[6] R. C. Elson, A. I. Selverston, R, Huerta, N. F. Rulkov, M. I. Rabinovich, and H. D. I. Abarbanel, Phys. Rev. Lett. **81**, 5692 (1998).
[7] A. Koseska, E. Volkov, A. Zaikin, and J. Kurths, Phys. Rev. E **75**, 031916 (2007).
[8] E. I. Volkov and M. N. Stolyarov, Biol. Cybern. **71**, 451 (1994).
[9] R. E. Mirollo and S. H. Strogatz, J. Stat. Phys. **60**, 245 (1990).
[10] D. V. Ramana Redy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998).
[11] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Physica (Amsterdam) **129D**, 15 (1999).
[12] S. Strogatz, Nature (London) **394**, 316 (1998).
[13] F. M. Atay, Phys. Rev. Lett. **91**, 094101 (2003).
[14] R. Karnatak, R. Ramaswamy, and A. Prasad, Phys. Rev. E **76**, 035201(R) (2007).
[15] Y. Zhu, X. Qian, and J. Yang, Europhys. Lett. **82**, 40 001 (2008).
[16] G. Saxena, A. Prasad, and R. Ramaswamy, Phys. Rep. **521**, 205 (2012).
[17] A. Turing, Phil. Trans. R. Soc. B **237**, 37 (1952).
[18] I. Prigogine and R. Lefever, J. Chem. Phys. **48**, 1695 (1968).
[19] K. Bar-Eli, Physica (Amsterdam) **14D**, 242 (1985).
[20] E. I. Volkov and M. N. Stolyarov, Phys. Lett. A **159**, 61 (1991).
[21] K. Tsaneva-Atanasova, C. L. Zimliki, R. Bertram, and A. Sherman, Biophys. J. **90**, 3434 (2006).
[22] E. Ullner, A. Zaikin, E. I. Volkov, and J. García-Ojalvo, Phys. Rev. Lett. **99**, 148103 (2007).
[23] A. Koseska, E. I. Volkov, and J. Kurths, Europhys. Lett. **85**, 28 002 (2009).
[24] A. Kuznetsov, M. Kaern, and N. Kopell, SIAM J. Appl. Math. **65**, 392 (2004).
[25] M. Dolnik and M. Marek, J. Phys. Chem. **92**, 2452 (1988).
[26] M. F. Crowley and I. R. Epstein, J. Phys. Chem. **93**, 2496 (1989).
[27] M. Toiya, V. K. Vanag, and I. R. Epstein, Angew. Chem., Int. Ed. **47**, 7753 (2008).
[28] R. Curtu, Physica (Amsterdam) **239D**, 504 (2010).
[29] B. Ermentrout and N. Kopell, SIAM J. Appl. Math. **50**, 125 (1990).
[30] A. Koseska, E. Ullner, E. I. Volkov, J. Kurths, and J. García-Ojalvo, J. Theor. Biol. **263**, 189 (2010).
[31] N. Suzuki, C. Furusawa, and K. Kaneko, PLoS One **6**, e27232 (2011).
[32] M. Y. Kim, R. Roy, J. L. Aron, T. W. Carr, and I. B. Schwartz, Phys. Rev. Lett. **94**, 088101 (2005).
[33] Y. Zhai, I. Z. Kiss, and J. L. Hudson, Phys. Rev. E **69**, 026208 (2004).
[34] J. Tyson and S. Kauffman, J. Math. Biol. **1**, 289 (1975).
[35] A. Prasad, M. Dhamala, B. M. Adhikari, and R. Ramaswamy, Phys. Rev. E **81**, 027201 (2010).
[36] W. Zou, X. Wang, Q. Zhao, and M. Zhan, Front. Phys. China **4**, 97 (2009).
[37] W. Liu, E. Volkov, J. Xiao, W. Zou, M. Zhang, and J. Yang, Chaos **22**, 033144 (2012).
[38] D. G. Aronson, G. B. Ermentrout, and N. Koppel, Physica (Amsterdam) **41D**, 403 (1990).
[39] B. Ermentrout, *Simulating, Analyzing, and Animating Dynamical Systems: A Guide to Xppaut for Researchers and Students (Software, Environments, Tools)* (SIAM, Philadelphia, 2002).
[40] A. Koseska, E. I. Volkov, and J. Kurths, Chaos **20**, 023132 (2010).
[41] A. Koseska and J. Kurths, Chaos **20**, 045111 (2010).
[42] S. Grossberg, J. Math. Biol. **4**, 237 (1977).
[43] K. Gurney, T. J. Prescott, and P. Redgrave, Biol. Cybern. **84**, 401 (2001).
[44] Z. H. Mao and S. G. Massaquoi, IEEE Trans. Neural Networks **18**, 55 (2007).