Quantum Frameness for *CPT* Symmetry

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(Received 31 December 2012; revised manuscript received 2 June 2013; published 11 July 2013)

We develop a theory of charge-parity-time (*CPT*) frameness resources to circumvent *CPT* superselection. We construct and quantify such resources for spin-0, 1/2, 1, and Majorana particles and show that quantum information processing is possible even with *CPT* superselection. Our method employs a unitary representation of *CPT* inversion by considering the aggregate action of *CPT* rather than the composition of separate *C*, *P*, and *T* operations, as some of these operations involve problematic antiunitary representations.

DOI: 10.1103/PhysRevLett.111.020504 PACS numbers: 03.67.Hk, 11.30.Er, 11.30.Fs

Superselection rules such as charge [1,2], orientation [3], chirality [1,4,5], and phase [2,6,7] prohibit certain coherent quantum superpositions and are formally equivalent to the lack of a requisite classical frame of reference [8]. Superselection rules can be circumvented by consuming appropriate *frameness resources*, namely, quantum systems whose states are asymmetric with respect to a group G of transformations associated with the requisite frame of reference [9]. Here, we develop the superselection rule for charge-parity-time (*CPT*) invariance [10–12] and construct the corresponding frameness resources for spins $s=0$, $s=1/2$, and $s=1$ as well as for Majorana particles. We also suggest a procedure whereby such resources can be constructed for higher-order spins.

Constructing the *CPT* operator in the seemingly natural way by composing the separate *C*, *P*, and *T* operations involves undesirable antiunitary projective representations. If *CPT* were an antiunitary projective representation, two phase terms ± 1 arise and cannot be simply eliminated, thereby resulting in a doubling of the representation [13]. Perfunctory use of the antiunitary projective representation unacceptably allows frameness resources to be converted to nonresources under symmetry-respecting evolution, viz., the Hamiltonian commutes with every element of the representation of the group [14].

Therefore, we construct *CPT* as an indecomposable unitary projective representation such that $CPT^2 = \mathbb{1}$, with $\mathbb{1}$ the identity transformation, and the global phase can be removed by defining the operator appropriately. We apply this approach to the distinct cases of integer and half-odd integer s and construct the relevant projective unitary representation for *CPT* as well as the resource states required to lift *CPT* superselection. In addition, our strategy allows for the identification of *CPT*-invariant subspaces capable of storing and transmitting information even in the presence of *CPT* superselection.

In the Feynman-Stueckelberg interpretation [15], the image of a particle with mass m, spin s, linear three-momentum $\boldsymbol{p}$, and energy $E=\sqrt{|\boldsymbol{p}|^2c^2+(mc^2)^2}$ under the action of *CPT* is an antiparticle of the same mass and energy with its spin and three-momentum reversed, and its internal degrees of freedom, such as electric charge, baryon number, and lepton number, inverted. Employing only the *universally conserved* internal symmetries, we define the total internal quantum number

$$u := Q + (B - L) \tag{1}$$

for Q the total electric charge and $B-L$ the difference between total baryon number B and total lepton number L. (In some theories, B and L are not individually conserved, but $B-L$ is; this is known as the chiral anomaly [16].)

As m and E are *CPT* invariant, we denote the state corresponding to a particle with total internal quantum number u, spin s, and linear three-momentum $\boldsymbol{p}$ as $|u,s,\boldsymbol{p}\rangle$. The state of the corresponding antiparticle with the same mass and energy is

$$CPT|u,s,\boldsymbol{p}\rangle = e^{i\theta_{CPT}}|-u,-s,-\boldsymbol{p}\rangle, \tag{2}$$

for $\theta_{CPT}\in[0,2\pi)$ an unimportant global phase. The state $|u,s,\boldsymbol{p}\rangle$ is technically not normalizable for an infinite region with continuous $\boldsymbol{p}$ but is well defined as a distribution in the dual Φ^* to the nuclear space of test functions $\Phi\subseteq\mathcal{H}$, with $\mathcal{H}$ a Hilbert space and $(\Phi,\mathcal{H},\Phi^*)$ the Gel'fand triple [17], also known as a rigged Hilbert space. Observables are complex functionals of test functions and distributions like $|u,s,\pm\boldsymbol{p}\rangle$. In our notation, Dirac "bras" refer to test functions and Dirac "kets" refer to distributions. Here, we employ the Dirac adjoint representation to ensure covariance and unitarity throughout [18].

For reference-frame-establishment protocols, we consider two parties Alice and Bob who can occupy different space-time regions and, moreover, can be moving relative to each other. Thus, Alice's state $|u,s,\boldsymbol{p}\rangle$ is equivalent to

Bob's state only up to a Poincaré transformation Λ that is known by Alice and Bob plus either a CPT transformation or else a $\mathbb{1}$ transformation.

Whether Alice and Bob are related by CPT or by $\mathbb{1}$ is unknown to Alice and Bob, hence the reference-frame problem. As Alice and Bob know the transformation Λ, Bob can compensate for its effect on quantum information sent to him from Alice by employing a suitable device that simulates Λ^{-1} after receiving the particle. Specifically, Bob would apply known rotations and boosts as necessary to recover Alice's basis modulo whether CPT or $\mathbb{1}$ should also be applied.

For proper operation of Bob's Λ compensator, we must consider that although rotation generators commute with CPT, boost generators do not. Therefore, the order in which Bob compensates for Λ and CPT matters. We assume that Bob compensates for the effects of Λ first. Thus, upon receiving Alice's particle, which is prepared in some (test-function) state

$$|\psi\rangle = \sum_{u,s}\int dp\,\psi(u,s,\boldsymbol{p})|u,s,\boldsymbol{p}\rangle \tag{3}$$

and hence arrives at Bob's location in the state $\Lambda|\psi\rangle$, Bob's device effects Λ^{-1} in his frame and recovers the original state up to a CPT transformation. Henceforth, we focus only on the superselection rule pertaining to $\{\mathbb{1}, CPT\}$.

To construct unitary projective representations of $\{\mathbb{1}, CPT\}$, we first complete the intermediate step of constructing the set of operators $\{\mathbb{1}, C, PT, CPT\}$, which, under composition, forms the (Abelian) Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then, we reduce this group to the subgroup $\{\mathbb{1}, CPT\}$, which is equivalent to $\mathbb{Z}_2$. The representations are constructed with respect to the states $|u,s,\boldsymbol{p}\rangle$ that span the space of distributions Φ^*. The resultant orthonormal basis for given labels u, s, and $\boldsymbol{p}$ is

$$\begin{aligned}\{|u,s,\boldsymbol{p}\rangle, e^{i\theta_{PT}}|u,-s,-\boldsymbol{p}\rangle &=: PT|u,s,\boldsymbol{p}\rangle,\\ e^{i\theta_C}|-u,s,\boldsymbol{p}\rangle &=: C|u,s,\boldsymbol{p}\rangle,\\ e^{i\theta_{CPT}}|-u,-s,-\boldsymbol{p}\rangle &=: CPT|u,s,\boldsymbol{p}\rangle\},\end{aligned} \tag{4}$$

with $\theta_{CPT} = \theta_{PT} + \theta_C$ and the ":=" and "=:" notation indicating that the side with the colon is defined by the other side.

By restricting to positive u and positive s and "forward" $\boldsymbol{p}$ (i.e., $\boldsymbol{p}$ restricted to a half space of $\mathbb{R}^3$ with respect to some defined forward direction vector), the corresponding bases (4) are mutually orthogonal with the proviso that the continuous nature of $\boldsymbol{p}$ means that this orthogonality is of the Dirac δ form rather than of the Kronecker δ form. The special case $\boldsymbol{p} = \boldsymbol{0}$ case is of zero measure in the set of all such subbases and hence does not require special treatment.

We now apply our strategy to special cases of particles, namely, relativistic particles with $s=0$, $s=1/2$, and $s=1$. These cases cover almost all particles of interest in physics. Subsequent to these cases, we explain how to extend the results to $s>1$ by using Bargmann-Wigner equations [19].

Consider a single, massive spin-0 particle with total internal quantum number u satisfying the Klein-Gordon equation

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi = 0, \tag{5}$$

with $\Box$ the D'Alembertian differential operator [18]. The Klein-Gordon solutions are plane waves of three-momentum $\boldsymbol{p}$ with both positive or negative energies. We interpret an eigenstate with a negative energy as an antiparticle state with positive energy.

The orthonormal subbasis for the massive spin-0 particle is

$$\{|u,0,\boldsymbol{p}\rangle, e^{i\theta_{PT}}|u,0,-\boldsymbol{p}\rangle, e^{i\theta_C}|-u,0,\boldsymbol{p}\rangle, e^{i\theta_{CPT}}|-u,0,-\boldsymbol{p}\rangle\}. \tag{6}$$

Restricting to the subgroup $\{\mathbb{1}, CPT\}$, the representation of the CPT operator in this basis is

$$CPT = e^{i\theta_{CPT}}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}. \tag{7}$$

The resultant CPT operator over this subbasis is thus unitary and antidiagonal. For the massless spin-0 particle, the Klein-Gordon equation is simply $\Box\psi = 0$ and the space is still spanned by $|\pm u, 0, \pm\boldsymbol{p}\rangle$, so CPT is the same unitary operator as for the massive-particle case. Neglecting the unobservable global phase θ_{CPT}, the eigenstates of CPT (7) are

$$\begin{aligned}|\pm,0,\boldsymbol{p}\rangle &\equiv \frac{1}{\sqrt{2}}(|u,0,\boldsymbol{p}\rangle \pm |-u,0,-\boldsymbol{p}\rangle),\\ |\pm,1,\boldsymbol{p}\rangle &\equiv \frac{1}{\sqrt{2}}(|u,0,-\boldsymbol{p}\rangle \pm |-u,0,\boldsymbol{p}\rangle)\end{aligned} \tag{8}$$

with corresponding eigenvalues ± 1.

We now consider the representation of CPT for a massive Dirac spinor whose state ψ satisfies the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu + mc)\psi = 0. \tag{9}$$

The Dirac matrices are

$$\gamma^0 = \begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}, \qquad \gamma^j = \begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix}, \tag{10}$$

and $\sigma^j|_{j\in(1,2,3)}$ are the Pauli matrices. Analogous to the previous case of massive $s=0$ particles, we construct the eight-dimensional state space spanned by $\{|\pm u, \pm 1/2, \pm\boldsymbol{p}\rangle\}$. As the action of CPT inverts all degrees of freedom (2), the corresponding unitary CPT matrix is

$$CPT = \begin{pmatrix} 0 & 0 & \sigma^1 \\ 0 & \sigma^1 & 0 \\ \sigma^1 & 0 & 0 \end{pmatrix} \qquad (11)$$

with eigenstates

$$\begin{aligned} |\pm, 0, \boldsymbol{p}\rangle &\equiv \frac{1}{\sqrt{2}}(|u, 1/2, \boldsymbol{p}\rangle \pm |-u, -1/2, -\boldsymbol{p}\rangle), \\ |\pm, 1, \boldsymbol{p}\rangle &\equiv \frac{1}{\sqrt{2}}(|u, -1/2, \boldsymbol{p}\rangle \pm |-u, 1/2, -\boldsymbol{p}\rangle), \\ |\pm, 2, \boldsymbol{p}\rangle &\equiv \frac{1}{\sqrt{2}}(|u, 1/2, -\boldsymbol{p}\rangle \pm |-u, -1/2, \boldsymbol{p}\rangle), \\ |\pm, 3, \boldsymbol{p}\rangle &\equiv \frac{1}{\sqrt{2}}(|u, -1/2, -\boldsymbol{p}\rangle \pm |-u, 1/2, \boldsymbol{p}\rangle). \end{aligned} \qquad (12)$$

The eigenvalue for each state $\{|\pm, \iota, \boldsymbol{p}\rangle|_{\iota\in(0,\ldots,3)}\}$ is ± 1.

Dirac spinor states that are invariant under the action of CPT are defined as Majorana spinors [18,20]. Majorana spinors are also invariant under CPT when obtained as solutions to the Majorana equation

$$i\hbar\gamma^\mu\partial_\mu\psi_c + mc\psi = 0, \qquad (13)$$

where, in the Majorana basis, $\psi_c := i\psi^*$. Hence, $\{\mathbb{1}, CPT\}$ is a projective unitary representation of $\mathbb{Z}_2$ for both Dirac and Majorana spinor states.

Massless $s = 1/2$ particles are described by the Weyl equation

$$i\hbar\sigma^\mu\partial_\mu\psi = 0, \qquad (14)$$

where the σ^μ are the usual Pauli matrices for $\mu = j \in \{1, 2, 3\}$ and $\sigma^0 := \mathbb{1}$. The solutions to the Weyl equation (14), known as Weyl spinors, can be represented as four-component spinors. For $m \equiv 0$, Eq. (14) is identical to the Dirac equation [18], in which case the solutions are identical to those of the Dirac equation with states of $\pm u$ representing right-handed and left-handed spinors, respectively. Therefore, the CPT operator is the same for both massive and massless $s = 1/2$ particles.

For massive spin-1 particles, we use the Weinberg-Shay-Good equation [18]

$$[i\hbar\partial_\mu(\gamma^{\mu\nu} - g^{\mu\nu})i\hbar\partial_\nu + 2m_0^2c^2]\psi = 0 \qquad (15)$$

with $\gamma^{\mu\nu}$ the 6×6 matrices

$$\gamma^{ij=ji} = \begin{pmatrix} 0 & \delta_{ij}\mathbb{1} + M^{(ij)} + M^{(ji)} \\ \delta_{ij}\mathbb{1} + M^{(ij)} + M^{(ji)} & 0 \end{pmatrix},$$
$$\gamma^{0i} = \gamma^{i0} = \begin{pmatrix} 0 & S^i \\ -S^i & 0 \end{pmatrix}, \qquad \gamma^{00} = -\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad (16)$$

and

$$S^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S^2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$S^3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M^{(ij)} = iS^j iS^i. \qquad (17)$$

We construct a twelve-dimensional state space spanned by the orthonormal basis $\{|\pm u, \pm s, \pm \boldsymbol{p}\rangle, s \in (-1, 0, 1)\}$. The CPT operator is then given by a 12×12 antidiagonal matrix with unit entries, and the eigenvectors are

$$\begin{aligned} |\pm, 0, \boldsymbol{p}\rangle &= \frac{1}{\sqrt{2}}(|u, 1, \boldsymbol{p}\rangle \pm |-u, -1, -\boldsymbol{p}\rangle), \\ |\pm, 1, \boldsymbol{p}\rangle &= \frac{1}{\sqrt{2}}(|u, 0, \boldsymbol{p}\rangle \pm |-u, 0, -\boldsymbol{p}\rangle), \\ |\pm, 2, \boldsymbol{p}\rangle &= \frac{1}{\sqrt{2}}(|u, -1, \boldsymbol{p}\rangle \pm |-u, 1, -\boldsymbol{p}\rangle), \\ |\pm, 3, \boldsymbol{p}\rangle &= \frac{1}{\sqrt{2}}(|u, 1, -\boldsymbol{p}\rangle \pm |-u, -1, \boldsymbol{p}\rangle), \\ |\pm, 4, \boldsymbol{p}\rangle &= \frac{1}{\sqrt{2}}(|u, 0, -\boldsymbol{p}\rangle \pm |-u, 0, \boldsymbol{p}\rangle), \\ |\pm, 5, \boldsymbol{p}\rangle &= \frac{1}{\sqrt{2}}(|u, -1, -\boldsymbol{p}\rangle \pm |-u, 1, \boldsymbol{p}\rangle), \end{aligned} \qquad (18)$$

with eigenvalues ± 1. Hence, $\{\mathbb{1}, CPT\}$ forms a projective unitary representation of $\mathbb{Z}_2$ for massive spin-1 particles.

Now, we consider massless $s = 1$ particles. Photons are the only known particles of this type. The corresponding expression for wave function dynamics is the Białynicki-Birula–Sipe equation [21–27]

$$i\hbar(\partial_0 + c\beta^3 S^j\partial_j)\psi = 0, \qquad \beta^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad (19)$$

with $\{S^j\}$ given by Eq. (17). The solutions are six-component spinors with Weyl represention $\psi = \binom{\Psi_+}{\Psi_-}$ where $\Psi_\pm$ represent opposite helicities [26].

For the solutions of Eq. (19) to describe a photon correctly, we require the auxiliary condition [21]

$$\psi = \beta^1\psi^*, \qquad \beta^1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \qquad (20)$$

The state space of solutions of Eq. (19) is the same as for Eq. (14). Consequently, the CPT operator has the same form as that for the massive $s = 1$ particles and thus the same eigenvectors. Physically, these eigenvectors correspond to linear superpositions of states with opposite helicities. Such states are usually assumed not to mix and are often treated separately [21,26]. We also note that the photon does not possess a state of zero total spin (corresponding to a lack of longitudinal polarization in classical optics), and thus the states $|\pm, 1, \boldsymbol{p}\rangle$ and $|\pm, 4, \boldsymbol{p}\rangle$ in Eq. (18) are unphysical.

For particles of arbitrarily higher spin, solutions may be constructed using the Bargmann-Wigner equations [19], which are individually indexed Dirac equations. For instance, $s = 3/2$ particles obey a version of the Bargmann-Wigner equations known as the Rarita-Schwinger equation [28] whose solutions are sixteen-component spinors or equivalently four four-component spinors, each of which is individually a solution of the Dirac equation.

We are now ready to formulate a *CPT* superselection rule. Because of Schur's lemmas [29], unitary representations of finite groups can be fully reduced into their irreducible components (irreps). In particular, $\mathbb{Z}_2$ has two one-dimensional irreps given by $\pm$. As *CPT* superselection implies that coherent superpositions between eigenstates of the *CPT* operator cannot be observed [9], the state space $\mathcal{H}$ of any system subject to *CPT* superselection may conveniently be written as

$$\mathcal{H} \cong \bigoplus_{\epsilon\in\{\pm\}} \mathcal{H}^{(\epsilon)}, \tag{21}$$

with irrep label ϵ denoting the two inequivalent irreps of $\mathbb{Z}_2$ and $\mathcal{H}^{(\epsilon)}$ the corresponding eigenspaces.

The Hilbert space $\mathcal{H}$ in expression (21) is applicable only for subbases corresponding to a fixed label $\boldsymbol{p}$. In other words, expression (21) is replaced by $\Phi^*_{\boldsymbol{p}} \cong \bigoplus_{\epsilon\in\{\pm\}} \Phi^{(\epsilon)}_{\boldsymbol{p}}$ with Φ^* replacing $\mathcal{H}$ because including $\boldsymbol{p}$ means the states are now distributions, i.e., $\{|u, s, \boldsymbol{p}\rangle\}$. The full space of states corresponds to the space of distributions Φ^*, which is a (continuous) sum of all $\Phi^*_{\boldsymbol{p}}$. Partitioning by irrep label $\epsilon \in \{\pm\}$ holds in the full Φ^* as well, thereby yielding the space of distributions being partitioned into $\Phi^{*(+)}$ and $\Phi^{*(-)}$. These $\pm$ eigenspaces are spanned by the *CPT* eigenvectors with positive and negative eigenvalues, respectively.

The states that can be prepared in the absence of a *CPT* frame of reference (*CPT*-invariant states) are test functions that belong in either $\Phi^{(+)}$ or $\Phi^{(-)}$, which are dual to the spaces $\Phi^{*(+)}$ or $\Phi^{*(-)}$. Hence, a linear superposition of eigenstates of *CPT* is a resource and can be brought, via *CPT*-invariant operations, to the standard form

$$|\psi\rangle = \sqrt{q_0}|+\rangle + \sqrt{q_1}|-\rangle, \qquad q_0 \in [0, 1], \qquad q_1 = 1 - q_0, \tag{22}$$

with $|\pm\rangle$ arbitrary states from $\Phi^{*(\pm)}$. The important point is that state (22) is a superposition of two states chosen from two $\mathbb{Z}_2$ irrep labels $\pm$. For simplicity, we can consider the state as being in a fixed momentum state, i.e., a plane wave. As a perfect plane wave is unphysical, a more realistic treatment would have the state prepared in a wave packet with support over a continuum of momentum values $\boldsymbol{p}$.

The frameness inherent in Alice's *CPT* reference-frame token $|\psi\rangle$ is quantified by the alignment rate $R(\psi)$. This alignment rate quantifies the amount of information Bob obtains on average from each reference-frame token $|\psi\rangle$ in the limit of asymptotically many copies [30]. For the unitary representations of $\mathbb{Z}_2$ [30],

$$R(\psi) = -2\log|q_0 - q_1|. \tag{23}$$

If $q_0 = q_1 = 1/2$ in (22), the alignment rate is effectively infinite.

Our strategy for constructing a projective unitary representation of the *CPT* operator allows for Alice and Bob to communicate information even in the presence of *CPT* superselection. Consider the case that Alice and Bob possess spinless particles. Alice prepares the plane-wave state

$$|\phi\rangle = \alpha|+, 0, \boldsymbol{p}\rangle + \beta|+, 1, \boldsymbol{p}\rangle. \tag{24}$$

As the state (24) is an eigenstate of the *CPT* operator for all $\alpha, \beta \in \mathbb{C}$, Bob's state is represented exactly the same as Alice's after correcting for the known Poincaré transformation Λ between their reference frames. By choosing the coefficients α and β appropriately, Alice can encode a logical qubit, which Bob can retrieve by performing the appropriate decoding.

Here, we have shown that the superselection rule arising from *CPT* symmetry can be circumvented using *CPT* frameness resources. We have identified the ultimate frameness resources for the cases of both massive and massless spin-0, 1/2, and 1 particles including Majorana spinors and have suggested a strategy for finding solutions for states of arbitrary spin. We have also shown that communication is possible, even in the presence of *CPT* superselection, except for the case of spinless particles with three-momentum equal to zero.

M. S. acknowledges financial support from NSERC, USARO, and Austrian Science Fund (FWF) Grants No. Y535-N16, No. P20748-N16, No. P24273-N16, and No. F40-FoQus F4011/12-N16. B. T. acknowledges financial support from NSERC, AITF, MITACS, NSF CCI center, "Quantum Information for Quantum Chemistry (QIQC)" Grant No. CHE-1037992 and NSF Grant No. PHY-0955518, and B. T. thanks I. Marvian for valuable discussions. I. T. D. acknowledges financial support from FQXi. B. C. S. acknowledges support from NSERC, AITF, and CIFAR.

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