Reviving Oscillations in Coupled Nonlinear Oscillators

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By introducing a processing delay in the coupling, we find that it can effectively annihilate the quenching of oscillation, amplitude death (AD), in a network of coupled oscillators by switching the stability of AD. It revives the oscillation in the AD regime to retain sustained rhythmic functioning of the networks, which is in sharp contrast to the propagation delay with the tendency to induce AD. This processing delay-induced phenomenon occurs both with and without the propagation delay. Further this effect is rather general from two coupled to networks of oscillators in all known scenarios that can exhibit AD, and it has a wide range of applications where sustained oscillations should be retained for proper functioning of the systems.

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Coupled nonlinear oscillators constitute an excellent framework for understanding the various complex collective dynamics that spontaneously emerge in real-life systems [1–3]. Amplitude death (AD) [3–21], the quenching of oscillations under coupling despite the fact that each isolated unit remains oscillating is one such basic phenomena, which has been explored in diverse areas of science and technology [4]. AD is desirable in some real-world applications such as in synthetic genetic networks [7,8] and in laser systems [9,10]. In contrast, in many other real situations AD is detrimental and should be avoided, as oscillatory behavior needs to be retained for proper functioning. Typical examples include cardiac arrest due to cessation of normal sinus rhythm of pacemaker cells [22], states that mimic brain death involving temporary loss of parts of brain function and their related motor and sensory organs resulting in paralysis [23–26], and maintaining the output intensity of arrays of power generators [11], mass synchronization (to a fixed point) leading to pathological disorders [27]. Hence, the emergence of AD in several such circumstances should be circumvented.

A few recent investigations proposed methods to avoid AD in their corresponding parameter space [12,13]. Nevertheless, there lacks a general technique to efficiently overcome AD arising from any known scenarios, which remains an open challenge as emphasized in [4]. In addition, to the typically studied propagation delay, a processing delay [28–30] emerges in dynamical systems due to a finite response time required for internal processing of the input information. For instance, it is a key component in network delay (specifically at nodes with high degrees), in control systems, in navigation, in machine learning, etc. Relaxation time of dynamical systems, latency time in lasers, epidemics, economics, etc. are also a class of a system's internal processing delay.

In this letter, we present exclusive evidences that a processing delay in the coupling competes with the quenching effects of frequency mismatch [14,15] and propagation delays [16,17] in circumventing the onset of AD by any known scenarios including dynamic and conjugate couplings [18,19]. It also effectively destabilizes AD due to both stable homogeneous steady states (HSS) and inhomogeneous steady states (IHSS). This peculiar effect of a processing delay, in reviving oscillations in the parameter regime of AD, may provide a valuable clue to understanding sustained oscillatory behaviors of many natural systems. Examples include, repair and regeneration mechanisms of animal and plant physiology, cortical networks, circadian rhythm, epidemics, population dynamics, etc.

Let us start with a paradigmatic model of two coupled Stuart-Landau limit-cycle oscillators [14–17],

$$\dot{Z}_j(t) = (1 + iw_j - |Z_j(t)|^2)Z_j(t) + K[Z_k(t-\delta-\tau) - Z_j(t-\delta)], \quad (1)$$

where $j, k = 1, 2$ and $Z_{1,2} = x_{1,2} + iy_{1,2}$ are complex state variables, $w_{1,2}$ are the rotational frequencies of the two uncoupled limit cycle oscillators, and K quantifies the strength of coupling, and the delays δ and τ physically account for the processing time and the transmission time, respectively. The processing delay is the time taken by the node Z_j to process the information $Z_k(t-\tau)$ reached Z_j prior by δ. When $K = 0$, both oscillators exhibit stable limit cycle oscillations $|Z_1| = |Z_2| = 1$ with frequencies w_1 and w_2, respectively. For $\delta = 0$, the unstable HSS $Z_1 = Z_2 = 0$ of the uncoupled systems is stabilized for appropriate K and for large $\Delta = |w_1 - w_2|$ with $\tau = 0$ [14,15]

or in the presence of an appropriate $\tau > 0$ for $w_1 = w_2$ [16,17,31] leading to AD with the collapse of the stable limit cycles.

In order to appreciate the effect of the processing time δ, we first treat the coupled system (1) with $\tau = 0$. The stability conditions for AD for this case with $\delta = 0$ are $1 < K < (1 + \Delta^2/4)/2$ and $\Delta > 2$ [15], from which it is clear that AD occurs for a certain interval of K if the frequencies of both oscillators are sufficiently different. To our surprise, with the introduction of the processing time $\delta > 0$ in the coupling, we find that there is a critical threshold $\delta_c(K)$ above which the stable HSS (AD) becomes destabilized; this is in strong contrast to the propagation delay τ which has the tendency to induce a stable HSS [16,17]. The stability of the HSS is changed if the complex eigenvalue λ of the characteristic equation [31] crosses the imaginary axis $\lambda = i\beta$, which implies that the regaining of oscillations by the processing time δ is via a Hopf bifurcation [32]. The critical curve corresponding to $\delta_c(K)$ is analytically deduced as [33]

$$\delta_c(K) = \min\left\{\delta_{c\pm}(K) = \frac{\cos^{-1}\left[\frac{(\beta-\bar{w})^2_\pm + 1 + \Delta^2/4}{2K + 2K(\beta-\bar{w})^2_\pm}\right]}{\bar{w} \pm \sqrt{(\beta-\bar{w})^2_\pm}}\right\}, \quad (2)$$

where $\bar{w} = (w_1 + w_2)/2$ and $(\beta - \bar{w})^2_\pm = 1 + \Delta^2/4 + 2(K^2 - 1) \pm 2\sqrt{(1 + \Delta^2/4)(K^2 - 1) + (K^4 - K^2 + 1)}$. For an illustrative example, Fig. 1(a) depicts the numerically obtained stable HSS [31], AD regime (shaded region), in the $K - \delta$ space for $w_1 = 5$ and $w_2 = 15$, where the boundary is exactly enclosed by the minimum of the critical curves $\delta_{c\pm}(K)$. The spread of AD region becomes smaller and smaller for increasing δ and completely disappears if $\delta > \delta_c = 0.073$.

For a global perspective, Fig. 1(b) depicts the spread of AD regions in the (K, Δ) space for $\delta = 0$, 0.03, 0.04, 0.05, and 0.06, where $w_1 = 10 - (\Delta/2)$ and $w_2 = 10 + (\Delta/2)$. The AD region shrinks monotonically as δ is increased and vanishes if δ surpasses the critical value $\delta_c = \max\{\delta_c(K){:}\, 1 < K < (1 + \Delta^2/4)/2 \text{ and } \Delta > 2\} = 0.08$. Thus, the processing delay δ competes with the quenching effect of the frequency mismatch in switching the stability of the stable HSS to revive stable oscillations above δ_c in the parameter regimes of stable HSS (AD).

Reddy *et al.* [16] have established that the system (1) experiences AD even for $\Delta = 0$ for appropriate τ. Now, we will show the influence of the processing delay δ in this scenario. Figure 2(a) depicts the AD islands in the τ-K space for $\delta = 0$, 0.005, 0.01, and 0.02, respectively, where $w_1 = w_2 = 10$. It is evident that increasing δ monotonically reduces the AD island and it becomes unstable for $\delta > \delta_c$ reviving stable oscillations in the whole parameter space. To quantify the degree of the spread of AD island with respect to δ, we introduce a normalized ratio $R = S(\delta)/S(\delta = 0)$, where $S(\delta)$ denotes the area of AD island for δ. The dependence of R on δ is displayed in Fig. 2(b). We find that R monotonically decreases as δ increases acquiring $R = 0$ for $\delta > \delta_c = 0.065$. This indicates that stabilization of the unstable HSS leading to AD does not occur for any K and τ above δ_c. Hence, it is also clear that the processing time-delay δ can destabilize stable HSS (AD) even in delay-coupled identical oscillators, thereby reviving oscillations.

Recent investigations have revealed that AD can also occur in coupled identical oscillators even without propagation delays using dynamic [18] and conjugate couplings [19], where the underlying mechanism for the onset of AD is significantly different from each other. Surprisingly, we find that the processing delay δ is also capable of inhibiting the onset of stable steady states in both these scenarios. Consider a system of two coupled Stuart-Landau oscillators with dynamic coupling [18]

$$\begin{aligned} \dot{x}_j &= p_j x_j - w y_j + K(u_j(t-\delta) - x_j(t-\delta)), \\ \dot{y}_j &= w x_j + p_j y_j, \qquad \dot{u}_j = -u_j + x_k, \end{aligned} \quad (3)$$

and conjugate coupling [19]

$$\begin{aligned} \dot{x}_j &= p_j x_j - w y_j + K(y_k(t-\delta) - x_j(t-\delta)), \\ \dot{y}_j &= w x_j + p_j y_j + K(x_k(t-\delta) - y_j(t-\delta)), \end{aligned} \quad (4)$$

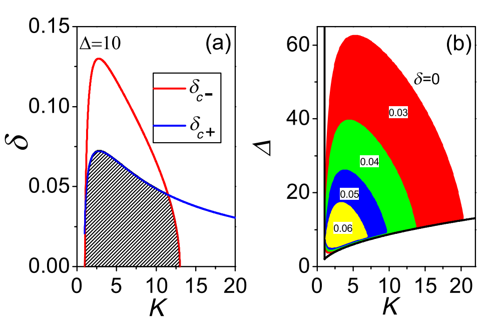

FIG. 1 (color online). Destabilizing the stable HSS in coupled system (1), $\tau = 0$. (a) AD region enclosed by the critical stability curves $\delta_{c\pm}(K)$ and (b) The effect of processing delay δ on the AD region.

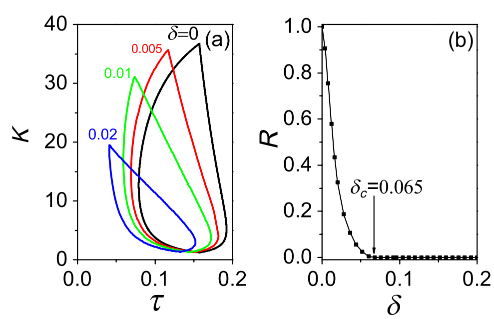

FIG. 2 (color online). Destabilizing the stable HSS in coupled system (1), $\tau > 0$. (a) The effect of processing delay δ on the AD island and (b) The ratio of the AD island area $R = S(\delta)/S(\delta = 0)$ vs δ.

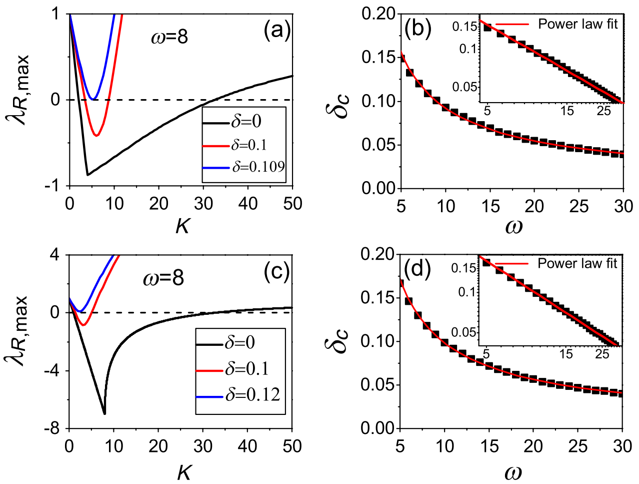

FIG. 3 (color online). Destabilizing the stable HSS in coupled system (3) with dynamic coupling [(a) and (b)] and (4) with conjugate coupling [(c) and (d)]. (a) and (c): The largest real part $\lambda_{R,\max}$ of λ vs K [34]. (b) and (d): the critical value δ_c vs. w.

where $j, k = 1, 2, j \neq k$, and $p_j = 1 - |Z_j|^2 = 1 - x_j^2 - y_j^2$. The characteristic equations determining the stability of the origin (AD state) in the coupled systems (3) and (4) can be directly obtained [34]. For $\delta = 0$ and a sufficiently large w, the coupled systems (3) and (4) can experience AD for a certain range of K depending on w. However, when turning on the processing delay $\delta > 0$, the range of AD decreases and even vanishes if $\delta > \delta_c$; see Figs. 3(a) and 3(b) for the dynamic coupling as well as Figs. 3(c) and 3(d) for the conjugate coupling. For both coupling types, δ_c is an exponentially decreasing function of w, which well obeys a power law relation as shown in the insets with the log-log fittings [Figs. 3(b) and 3(d)]. This reveals that AD is efficiently avoided for all values of K above δ_c for a given w. Hence, the competing effect, destabilizing the stable HSS, of the processing delay δ in regaining the stability of oscillations in the coupled systems are fairly general for different coupling scenarios that stabilize the unstable HSS giving rise to AD.

Now, we illustrate that the processing delay δ is also capable of annihilating the onset of inhomogeneous AD, which is induced by the birth of a new set of stable IHSS due to the coupling. For this purpose, we consider a system of two coupled Brusselators [6]:

$$\dot{x}_j = -(B+1)x_j + x_j^2 y_j + A + K(x_k(t-\delta) - x_j(t-\delta)),$$
$$\dot{y}_j = Bx_j - x_j^2 y_j + K(y_k(t-\delta) - y_j(t-\delta)), \qquad (5)$$

where $j, k = 1, 2, j \neq k$. For $B = 10$ and $A = 2$, each uncoupled Brusselators exhibit a stable limit cycle oscillation and has an unstable HSS ($x_{ss} = A$, $y_{ss} = B/A$). AD in (5) with $\delta = 0$ was investigated quite extensively in [6]. The HSS is unstable for all values of K, but at intermediate levels of K, stable IHSS exists; i.e., inhomogeneous AD appears. We have reproduced the stability diagram of steady-state solutions of [6] for $\delta = 0$ in Fig. 4(a). The dependence of the steady states, $x_{1,2}$, on K is shown there, where red bold lines represent stable steady states and black thin lines correspond to unstable ones. The presence of processing delay δ in the coupling does not change the structure of the steady-state solutions, but just switches their stability. Figures 4(b)–4(d) depict the solution diagrams of the steady states for $\delta = 0.3$, 0.5, and 0.7, respectively. It is evident that the stable IHSS region is gradually reduced for increasing δ. To clearly manifest the effect of δ on the stable IHSS, a normalized factor $R = L(\delta)/L(\delta = 0)$ is introduced, where $L(\delta)$ denotes the length of the stable IHSS (inhomogeneous AD) for a given δ. Figure 4(e) illustrates the dependence of R on δ. Interestingly, we find that R is a nonmonotonic function of δ reaching $R = 0$ for $\delta > \delta_c = 5.13$, which indicates that the AD region ceases to exist. This asserts that inhomogeneous AD induced by the coupling can also be destabilized by the processing delay for $\delta > \delta_c$ leading to sustained oscillations.

Finally, we demonstrate that the processing delay δ is even capable of reviving oscillations in AD regimes of large (arbitrary) networks. This is illustrated in a network of N ($N \geq 2$) coupled Stuart-Landau oscillators:

$$\dot{Z}_j = (1 + iw_j - |Z_j|^2)Z_j + \frac{K}{d_j}\sum_{s=1}^{N} g_{js}(Z_s(t-\delta-\tau) - Z_j(t-\delta)), \qquad (6)$$

where $j = 1, 2, \ldots, N$. The network topology is determined by g_{js} as follows: if two oscillators j and s are connected, then $g_{js} = g_{sj} = 1$ (undirected), otherwise $g_{js} = g_{sj} = 0$ and $g_{jj} = 0$. d_j denotes the degree of oscillator j. If $N = 2$, (6) is degenerated to (1). For a network of

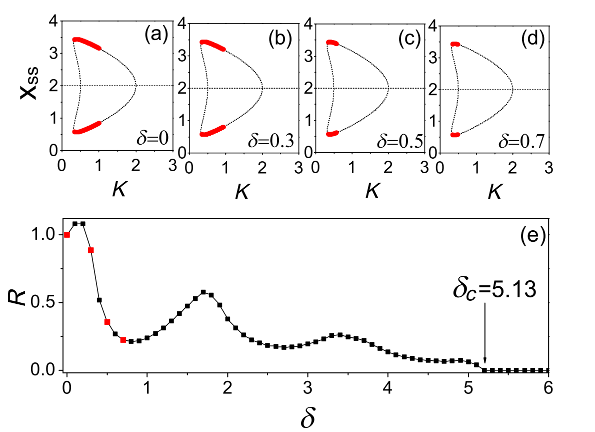

FIG. 4 (color online). Destabilizing the stable IHSS in two coupled Brusselators (5). (a)–(d) Solution diagrams of steady states, where red bold lines correspond to stable steady states and black thin lines to unstable ones. (e) The ratio of the length of stable IHSS interval $R = L(\delta)/L(\delta = 0)$ vs δ. The points with values of δ used in (a)–(d) are highlighted with red color.

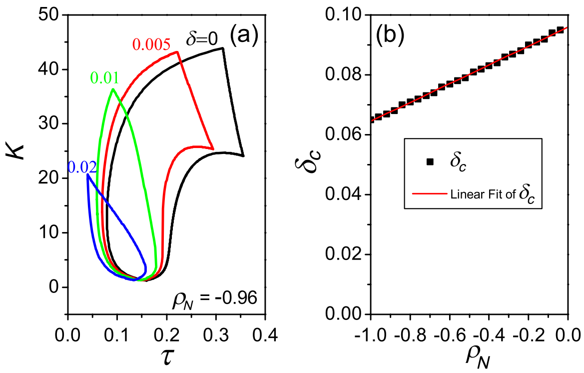

FIG. 5 (color online). Destabilizing the stable HSS in a network of delay-coupled oscillators, Eq. (6). $w_j = 10$. (a) The effect of processing delay δ on AD island with $\rho_N = -0.96$. (b) The critical value δ_c linearly increasing as ρ_N increases with the slope of 0.0313.

N identical oscillators ($w_j = w$), performing a standard linear stability analysis of (6) around the origin yields the following N characteristic equations:

$$\lambda = 1 + iw - Ke^{-\lambda\delta} + K\rho_j e^{-\lambda(\tau+\delta)}. \quad (7)$$

Here, ρ_j's are eigenvalues of the network matrix $G = (g_{js}/d_j)_{N\times N}$, which are ordered as $1.0 = \rho_1 \geq \rho_2 \geq \cdots \geq -1/(N-1) \geq \rho_N \geq -1.0$. The network (6) suffers AD if and only if all the roots of Eq. (7) have negative real parts for every ρ_j ($j = 1, 2, \ldots, N$). Ordering ρ_j's by studying the qualitative dependence of $\mathrm{Re}(\lambda)$ on ρ_j, it is proved that the boundaries of AD island are defined by only two extreme eigenvalues: $\rho_1 (\rho_1 = 1)$ and ρ_N [11,35]. ρ_N completely characterizes the effect of the connection topology (arbitrary network) on the occurrence of AD; the size of the corresponding AD island monotonically increases with the increase of ρ_N. In Fig. 5(a), we plot the AD islands of the coupled system (6) with $\rho_N = -0.96$ for a ring network with $N = 11$ nodes in the (τ, K) space for $\delta = 0$, 0.005, 0.01, and 0.02, and $w = 10$. Increasing δ rapidly decreases the AD island and completely eliminates AD if $\delta > \delta_c$. The value of δ_c depends on ρ_N, e.g., $\delta_c = 0.066$ for $\rho_N = -0.96$ in Fig. 5(a), while $\delta_c = 0.065$ if $\rho_N = -1.0$ for two coupled oscillators in Fig. 2. Figure 5(b) shows the relation between δ_c and ρ_N, where the moderate linear increase of δ_c with ρ_N reveals the high efficiency of the processing time delay δ in avoiding AD in an arbitrary network.

In summary, we have shown that the processing delay, different from typical propagation delay, offers a general technique to overcome AD in coupled systems to sustain oscillations. The phenomenon of AD can be completely annihilated by the processing delay by switching the stability of stable HSS and IHSS in all known coupling scenarios that can give rise to AD from two coupled systems to an arbitrary network. Reviving of oscillations by the processing time δ in all the cases occurs via a Hopf bifurcation. This is also confirmed in other dynamical systems exhibiting more complex dynamics such as modified Stuart-Landau limit-cycle oscillators by including a delay in its variable, chaotic Rössler oscillators, excitable FitzHugh-Nagumo elements, and Mackey-Glass time-delayed systems even in its hyperchaotic regimes; which underscores that the underlying phenomenon is very generic. The introduction of the processing time in the coupling is capable of retaining rhythmic activity and enhancing the oscillatory intensity in the parameter regimes of AD above δ_c. This ability in coupled oscillators or networks is important for several applications despite a large heterogeneity in their intrinsic frequency [15] and a large spread of propagation (distributed) delays due to the spatial separation [20] of the individual oscillatory units. The presence of even a small fraction of the processing time will have a significant implication on the proper functioning and robustness of large networks such as power grids, neural networks, lasers, epidemics, ecology, etc.

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[31] Using the standard linear stability analysis, the characteristic eigenvalue equation of Eq. (1) around the origin is given by $(1 + iw_1 - Ke^{-\lambda\delta} - \lambda)(1 + iw_2 - Ke^{-\lambda\delta} - \lambda) - K^2 e^{-2\lambda(\delta+\tau)} = 0$. If all roots lie in the open left half of complex plane, i.e., the real part of the rightmost root is negative $\lambda_{R,\max} < 0$, the homogeneous steady state $Z_1 = Z_2 = 0$ is linearly stable, then AD occurs.

[32] The Hopf bifurcation route for the regaining of oscillations by the processing time δ in the coupling is generically valid for all the AD scenarios.

[33] The critical curves $\delta_c(K)$ cannot be analytically tractable for the other AD scenarios discussed in rest of the paper except for the conjugate coupling case [34].

[34] The characteristic eigenvalue equations of Eq. (3) with dyanmic coupling and Eq. (4) with conjugate coupling around the origin are $(1 - \lambda^2)(1 - Ke^{-\lambda\delta} - \lambda) + w^2(1 + \lambda) \pm (1 - \lambda)Ke^{-\lambda\delta} = 0$ and $[(1 - \lambda)^2 + w^2 - 2(1 - \lambda)Ke^{-\lambda\delta}]^2 = 0$, respectively. HSS becomes stable in the range of coupling strength K with $\lambda_{R,\max} < 0$ resulting in AD. For conjugate coupling case, we also analytically derive the critical value of $\delta_c(K) = (1/\beta)\cos^{-1}[(\beta^2 + 1 + w^2)/(2K(\beta^2 + 1))]$ for $1 < K < (1 + w^2)/2$, where $\beta^2 = 2K^2 - 1 + w^2 + \sqrt{(2K^2 - 1 + w^2)^2 + K^2 - (1 + w^2)^2/4}$. The value of $\delta_c = \max\{\delta_c(K)\colon 1 < K < (1 + w^2)/2\}$ gives the threshold above which the origin cannot be stabilized for any coupling strength.

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