Intensity Fluctuations of Waves in Random Media: What Is the Semiclassical Limit?

Jakob J. Metzger,[1,2] Ragnar Fleischmann,[1] and Theo Geisel[1,2]

[1]*Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Goettingen, Germany*

[2]*Institute for Nonlinear Dynamics, Department of Physics, University of Goettingen, 37077 Goettingen, Germany*

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Waves traveling through weakly random media are known to be strongly affected by their corresponding ray dynamics, in particular in forming linear freak waves. The ray intensity distribution, which, e.g., quantifies the probability of freak waves is unknown, however, and a theory of how it is approached in an appropriate semiclassical limit of wave mechanics is lacking. We show that this limit is not the usual limit of small wavelengths, but that of decoherence. Our theory, which can describe the intensity distribution for an arbitrary degree of coherence is relevant to a wide range of physical systems, as decoherence is omnipresent in real systems.

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Extreme waves are known to occur in many physical systems where the waves are scattered by a weak random potential. Examples have been reported on a wide range of length scales and for waves which are described by different types of equations, including the branching of electron flows in semiconductor devices [1–4], the formation of freak or rogue waves in optical [5,6], ocean [7,8], sound, [9,10] and microwaves [11]. It is generally accepted that heavy tails in the probability density function (PDF) of the wave amplitude or the corresponding wave intensity can result from nonlinearities in the equations describing the wave propagation, in particular in optical and oceanic rogue waves. However, heavy tails in the PDF have also been found in systems that are well described by linear wave equations, including all of the examples mentioned above. Thus, some authors have argued that a focusing of the waves, i.e., the formation of caustics or branches in the flow [1,12–14], can lead to very high wave amplitudes even in the absence of nonlinearities, and can also act as a trigger for rogue nonlinear waves [7,11,14–16].

Random caustics first appear at a characteristic propagation distance from the source well below the mean free path [12,13]. It is well understood how their appearance leads to branching and a power-law tail in the ray (or classical) intensity PDF. This is also believed to carry over to the corresponding wave (or quantum mechanical) system for small wavelengths [12]. As the rays travel further, however, exponentially many caustics occur and the influence of individual caustics gradually fades. In this regime, the ray intensity PDF has been conjectured to approach a log-normal distribution [9], which is in stark contrast to the intensities of coherent waves which approach the limiting exponential distribution expected for diffusion [17–20]. Thus, the classical intensity distribution remains unknown and a theory of how it is approached in an appropriate semiclassical limit is lacking. The solution of this long-standing fundamental problem and the determination of the correct semiclassical limit lie at the heart of wave propagation in random media before the onset of diffusion, and can also be expected to explain discrepancies between random wave models and experimentally observed wave intensity distributions, e.g., for experimentally observed log-normal distributions in acoustic waves [9,10] and deviations from the exponential distribution in microwaves [11].

In this Letter, we demonstrate that the ray and wave distributions strongly differ and that this discrepancy holds down to the smallest wavelengths. We show, nevertheless, how in the appropriate semiclassical limit, which turns out to be that of the loss of coherence, the classical distribution is recovered. We derive analytical results (confirmed by numerical simulations) for the intensity PDFs of waves with arbitrary degree of coherence, which exhibit a transition from an exponential distribution to a log-normal distribution when coherence is lost, even for only moderately small wavelengths compared to the correlation length. Throughout this Letter, we study elastic propagation of an initially plane wave in time-independent random potentials considering the single particle two-dimensional Schrödinger equation as the wave equation and ensembles of noninteracting Newtonian particles as the classical counterpart. However, we expect our results to hold also for different wave equations and their corresponding ray equations. The random potentials used are Gaussian random fields with a Gaussian correlation function, but we note that our results can be expected to be generalized to other correlation functions [13]. The medium is assumed to be weakly scattering such that propagation below the mean free path is essentially paraxial. We also assume that the wavelength is smaller than the correlation length, which is necessary for branching to occur [21].

Different parts of the wave front begin to interfere when the first caustics occur, resulting in typical diffraction patterns at the branches [22]. Although still well below the mean free path, the waves interfere multiply when they have traveled further than the first caustics, allowing a random wave model to be applied. Here, the wave amplitude Ψ is assumed to be composed of a superposition of a number of plane waves with random amplitudes and wave vectors. The real and imaginary parts of the amplitude are then distributed normally due to the central limit theorem, and the distribution of the intensity is thus exponential, i.e., $P(I) = 1/ae^{-I/a}$ where a is the mean intensity and $I = |\Psi|^2$, which is known as Rayleigh's law [11,20,23].

On the other hand, it has been conjectured that the classical ray density should be distributed log-normally. Here, the argument is that the time evolution of the stability matrix, which describes the evolution of the infinitesimal surrounding of a trajectory in phase space, is determined by a product of random matrices, and thus, that the elements of the stability matrix can be written as products of independent random numbers. Therefore, their logarithms should be distributed normally [9]. This argument has been confirmed for the trace of the stability matrix, but not for individual elements of the matrix, which are directly related to the classical ray density. The first result of our paper will therefore be the numerical confirmation of the conjecture that the classical ray density is also distributed log-normally. To obtain the classical density, we follow the propagation of small phase space elements along rays and project them onto position space. This leads to more accurate results than simple ray counting. In order to compare this to quantum mechanical calculations, we need to smooth the singularities (caustics) present in the classical flow. The scale of the smoothing should be comparable to that of the quantum flow, which is naturally smooth on a scale of half the wavelength λ. The simplest way to obtain a similar smoothing is to choose the size of the spatial bins for the classical intensity count to be the same as $\lambda/2$. The quantum intensity is obtained by a simple binning of the modulus squared of the wave function at a given distance from the source.

The classical and the quantum cases are illustrated in Fig. 1. Our results show that the classical PDF $P(I)$ follows a log-normal distribution very well. Slight deviations to higher I in the tail of the distribution become visible for very small smoothing lengths (or corresponding wavelengths) and are expected to scale algebraically [12]. The quantum intensity distribution is well described by an exponential distribution. We note that the argument for the exponential PDF does not depend on the relative size of the wavelength compared to the scale of the medium (here the correlation length), and thus the effective $\hbar$ (cf. Fig. 1). In other words, in the semiclassical limit $\hbar \to 0$ we do not recover the classical distribution. We will show that, in order to recover the classical statistics, we need to break the phase coherence of the waves, which in the random wave picture leads to the exponential distribution. Decoherence can either occur through inelastic scattering of the waves as they propagate through the medium or by phase fluctuations of the source. Our

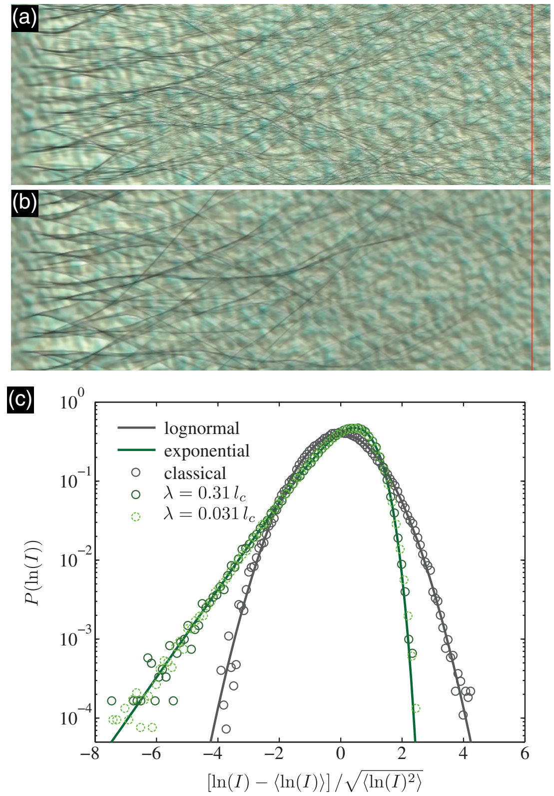

FIG. 1 (color online). Comparison of quantum mechanical and classical density calculations. (a),(b) Numerical simulations of a quantum mechanical (a) and of a classical flow (b) propagating through the same weak random potential (shaded background) starting from a plane wave initial condition. The gray scale superimposed on the potential indicates the flow intensity. The statistics of the intensities in (c) is measured at the location of the vertical lines to the right of the top panels. (c) Classical and two quantum intensity distributions (obtained from 100 realizations of the random potential) in a rescaled log-log plot in which the log-normal distribution appears as an inverse parabola. The classical flow can be approximated by a log-normal distribution, whereas the quantum mechanical statistics follows Rayleigh's exponential law.

motivation is to understand elastic scattering in weak random potentials. Thus, we concentrate on the latter; we expect our results, however, to qualitatively hold for other mechanisms of decoherence. For the spatially extended initially plane wave it is natural to study the influence of a loss of spatial coherence. Temporal decoherence is neglected for the sake of simplicity. We construct a model in which we split up the initial plane wave $\Psi(\mathbf{x})$ into N small wave packets $\psi_j(\mathbf{x})$, which we can assign random additional phases φ_j, over which we average in the final expressions. The total wave function is then given by $\Psi(\mathbf{x}) = \sum_j \psi_j(\mathbf{x}) e^{i\varphi_j}$. If all φ_j are the same, this implies a coherent initial wave function, while a completely random phase for each wave packet corresponds to an incoherent initial condition. We model the initial wave packets as minimum uncertainty (Gaussian) wave packets $\psi_j^0(\mathbf{x}) = (2\pi\kappa^2)^{-1/4}\exp\{-(\mathbf{x}-\mathbf{x}_0)^2/4\kappa^2 + i\mathbf{p}_0\mathbf{x}/\hbar\}$ with $\kappa = \sqrt{\hbar/2}$ to obtain equal spread in position and momentum space, and with $\mathbf{p}_0$ the initial momentum. To vary the degree of coherence we draw the phases φ_j from a distribution with zero mean and standard deviation σ_φ. This allows us to go from the totally coherent case ($\sigma_\varphi = 0$, all phases equal) to the incoherent case ($\sigma_\varphi \gg 1$, all phases random). An example of a single initial wave packet propagating through a random potential is shown in red in Fig. 2.

The intensity $I = \langle \Psi^* \Psi \rangle_{\boldsymbol{\varphi}}$ averaged over many realizations of the N phases $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_N)$ can be calculated for an arbitrary distribution function of the phases $F(\boldsymbol{\varphi})$ as

$$I = \iint d^N\varphi d^N\varphi' \sum_{j,k} \psi_j^* \psi_k e^{i(\varphi_j - \varphi_k')} F(\boldsymbol{\varphi}) F(\boldsymbol{\varphi}').$$

Assuming that the phases are independent of each other, their distribution factorizes to $F(\boldsymbol{\varphi}) = F(\varphi_1)F(\varphi_2)\ldots F(\varphi_N)$ and the expression simplifies to

$$\begin{aligned} I &= \sum_{j,k} \iint d\varphi_j d\varphi_k \psi_j^* \psi_k e^{i(\varphi_j - \varphi_k)} F(\varphi_j) F(\varphi_k) \\ &= \sum_j \psi_j^* \psi_j + \gamma \sum_{j,k,j\neq k} \psi_j^* \psi_k, \end{aligned}$$

where we have introduced the coherence parameter $\gamma = \int d\varphi \int d\varphi' e^{i(\varphi - \varphi')} F(\varphi) F(\varphi')$. Introducing the fully coherent and incoherent intensities $I_{\mathrm{coh}} = \sum_{j,k} \psi_j^* \psi_k$ and $I_{\mathrm{inc}} = \sum_j \psi_j^* \psi_j$, we can further simplify the expression by reinserting the $j = k$ terms in the second summand and subtracting them from the first to obtain

$$I = (1 - \gamma) I_{\mathrm{inc}} + \gamma I_{\mathrm{coh}}. \tag{1}$$

Examples of I_{coh} and I_{inc}, as well as a mixture with $\gamma = 1/2$ are given in Fig. 2.

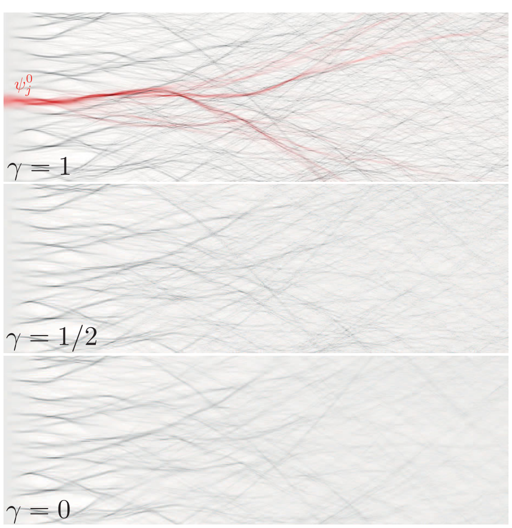

FIG. 2 (color online). Examples of wave flows averaged over 100 realizations of the random phases φ_j for $\gamma = 1$ (coherent), $\gamma = 1/2$ (intermediate), and $\gamma = 0$ (incoherent). The propagation of a single initial wave function ψ_j^0 is superimposed (red online) in the top panel. Darker shades of gray (gray and red online) indicate higher flow intensity.

It is instructive to compare the coherence parameter γ with other coherence measures, e.g., from optics. Since γ measures the initial spatial coherence, we compare it with the degree of coherence of the initial condition as defined in Ref. [24], $g = |\langle \Psi(y_1)\Psi^*(y_2)\rangle_\varphi / \langle I \rangle_\varphi|$, where we take the two positions y_1 and y_2 along the initial extension of the plane wave. We now assume these two positions to be separated by a distance which is large enough to ensure that the phases are not correlated and that the amount of coherence is related only to the magnitude of the fluctuations of the phases, as in our model. The degree of coherence g then becomes $\gamma I_{\mathrm{coh}}/[\gamma I_{\mathrm{coh}} + (1-\gamma) I_{\mathrm{inc}}]$, which can easily be inverted to give $\gamma = \alpha/[\alpha - 1 + 1/g]$ where $\alpha = I_{\mathrm{inc}}/I_{\mathrm{coh}}$. We note that I_{coh} is just the mean intensity of the initial coherent wave, while I_{inc} denotes the average of an ensemble of initial conditions with completely randomized phases. Also, $\gamma = g = 0$ implies the minimally coherent initial condition of the model, but coherence is retained on scales of the Gaussian wave packets and therefore, the coherence length does not vanish.

We now proceed to derive the distribution of the intensities. Assuming that the coherent wave function is exponentially distributed and that the incoherent wave function follows a log-normal distribution, the distribution of the weighted sum in Eq. (1) is given by the convolution of the two distributions. Multiplying the exponential distribution by γ results in rescaling the mean a of the exponential distribution to $a\gamma$. Analogously, the prefactor $(1-\gamma)$ in front of the log-normal distribution leads to a shift in the mean of the log-normal distribution of $\ln(1-\gamma)$. The distribution of the intensity in Eq. (1) is then given by

$$\begin{aligned} P(I, \gamma, \sigma, \mu, a) = &\int_0^I \frac{dx}{\sqrt{2\pi}\sigma x} e^{-([\ln(x) - \mu - \ln(1-\gamma)]^2/2\sigma^2)} \\ &\times \frac{1}{a\gamma} e^{-(I - x/a\gamma)}, \end{aligned} \tag{2}$$

where μ and σ are the usual mean and standard deviation parameters of the log-normal distribution. An analytical form of the integral in Eq. (2) is not known; however, it can be easily evaluated numerically and compared to numerical simulations. In order to compare this prediction with simulations we propagate individual wave packets through 100 realizations of the random potential. We assign phases φ_j drawn from a normal distribution with mean zero and standard deviation σ_φ to the wave packets. The coherence parameter γ can be explicitly calculated for the case of the normal distribution and is given by

$$
\begin{aligned}
\gamma &= \int d\varphi \int d\varphi' e^{i(\varphi-\varphi')} F(\varphi) F(\varphi') \\
&= \frac{1}{2\pi\sigma_\varphi^2} \int_{-\infty}^{\infty} d\varphi \int_{-\infty}^{\infty} d\varphi' e^{i(\varphi-\varphi')} e^{-(\varphi^2+\varphi'^2)/2\sigma_\varphi^2} \\
&= e^{-\sigma_\varphi^2}. \qquad (3)
\end{aligned}
$$

By choosing the phases from a normal distribution with appropriate σ_φ, we can specify the amount of coherence of the total wave function. We normalize the mean intensity and extract the parameters of the log-normal distribution from the incoherent numerical simulation. We can then compare the analytical results of our model to the numerical calculations for any degree of coherence. In the simulations, we use a paraxial approximation in which only the force of the random potential transverse to the flow direction is considered [12,13]. We use a Gaussian correlated random potential $V(\mathbf{x})$ with $\langle V(\mathbf{x}')V(\mathbf{x}'+\mathbf{x})\rangle = V_0^2 e^{-\mathbf{x}^2/\ell_c^2}$, and simulate the propagation of 62 wave packets at a standard deviation of $V_0 = 8\%E$, where E is the energy of the wave, and a correlation length of $\ell_c = 0.02$. The results in Fig. 3 are shown for a value of $\hbar$ which corresponds to approximately six wavelengths per correlation length. The figure shows the intensity distribution obtained at a distance of 100 times the correlation length, where the waves have scattered many times. We note that we have also checked the results for different sets of parameters, and that they are also independent of the precise number of wave packets used to reconstruct the initial plane wave.

The data are in excellent agreement with our analytical prediction for $P(I)$, Eqs. (2) and (3). The exponential and log-normal distributions correspond to the limiting cases $\gamma = 1$ and $\gamma = 0$. Interestingly, the incoherent case does not produce higher intensities than the coherent one for the wavelengths considered here. However, the lower panel of Fig. 3 shows that, compared to the standard deviation of the intensities, the tail of the coherent distribution is always the smallest and the broadest tails are those that are almost entirely incoherent.

In conclusion, we have shown that the intensity distribution of waves in random media does not approach the ray intensity distribution in the small wave length limit. Rather, a loss of phase coherence of the waves can mediate the transition from the exponential intensity distribution expected for a random wave model to the log-normal ray distribution. Thus, we have closed a significant gap in the fundamental understanding of wave propagation in random media. Our theory can quantitatively describe the distribution of wave intensities for arbitrary degrees of coherence. As in realistic systems, some amount of decoherence can never be avoided, our theoretical results are important for characterizing the intensity fluctuations of waves in diverse systems ranging from the scattering of electron waves in semiconductor nanostructures to the propagation of sound waves in the oceans on length scales of several thousand kilometers.

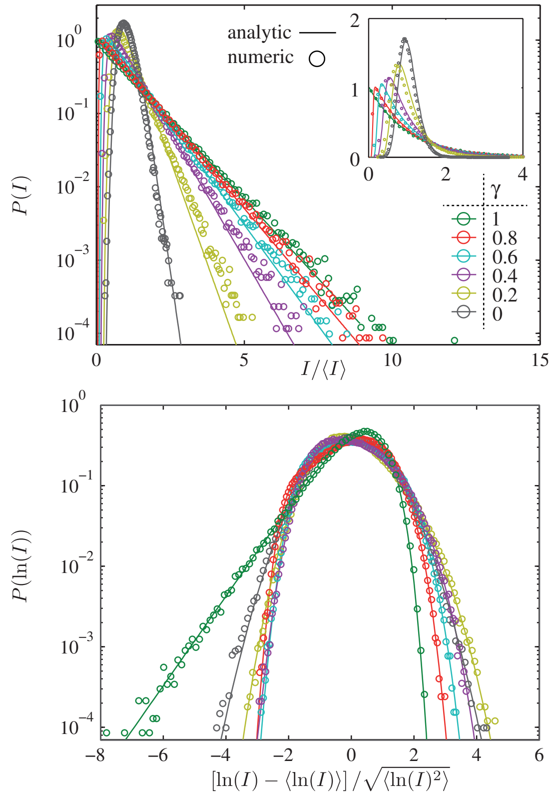

FIG. 3 (color online). Intensity distribution $P(I, \gamma, \mu, \sigma, a)$ for different values of the coherence parameter γ. The lines are the analytical results from Eq. (2), the circles are the numerical data. In the upper panel, a log-linear plot illustrates the correct scaling of the tails as predicted by the analytics. In the inset, a linear plot shows that the rest of the probability distribution is in excellent agreement also. The lower panel shows the same curves, rescaled, as in Fig. 1. It illustrates the growth of the heavy tail of $P(I)$ as it changes from exponential to log-normal for decreasing γ.

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