Complex Velocity Dependence of the Coefficient of Restitution of a Bouncing Ball

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We investigate the coefficient of normal restitution as a function of the impact velocity, $\varepsilon(v)$, for inelastic spheres. We observe oscillating behavior of $\varepsilon(v)$ which is superimposed to the known decay of the coefficient of restitution as a function of impact velocity. This remarkable effect was so far unnoticed because under normal circumstances it is screened by statistical scatter. We detected its clear signature by recording large amounts of data using an automated experiment. The new effect may be understood as an interplay between translational and vibrational degrees of freedom of the colliders. Both characteristics of the oscillation, the wavelength and the amplitude, agree quantitatively with a theoretical description of the experiment.

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The coefficient of normal restitution (COR) is the key element to describe dissipative collisions of hard spheres. It is of great importance to granular matter research as it is the foundation of both kinetic theory, based on the Boltzmann equation [1], and consequently granular fluid dynamics [2], as well as highly efficient event-driven molecular dynamics of granular matter [3–5]. In all references on event-driven simulations and kinetic theory of nonadhesive granular matter, it is assumed that the COR is either a constant or a monotonically decaying function of the impact velocity [6–8]. In the present Letter we will show that this assumption is not always sufficient to describe the dynamics of collisions accurately, since, at least for a certain time, part of the kinetic energy of the relative motion can be stored in vibrational degrees of freedom. The aim of this work is to report remarkable results derived from an exceptionally large amount of data gathered using an automated experiment. Such experiments, provide clear evidence that the COR is not a monotonically decaying function but oscillates as a function of the impact velocity.

The COR, ε, characterizes the collision of spheres in hard-sphere approximation. When colliding, the particles change the normal component of their relative velocity according to

$$(\vec{v}_j' - \vec{v}_i')\hat{e} = -\varepsilon(\vec{v}_j - \vec{v}_i)\hat{e}, \tag{1}$$

where $\vec{v}_i$ and $\vec{v}_j$ are the precollisional velocities, $\vec{v}_i'$ and $\vec{v}_j'$ are the postcollisional velocities, and $\hat{e} \equiv (\vec{r}_j - \vec{r}_i)/|\vec{r}_j - \vec{r}_i|$ is the intercenter unit vector at the instant of the collision. Thus, ε is defined as the ratio of the pre- and postcollisional values of the normal component of the particles' relative velocity.

The definition, Eq. (1), implies two consequences: (a) the velocities change instantaneously while in physical collisions the interaction force is finite such that collisions last a finite duration, τ, and (b) the unit vector, $\hat{e}$, does not change during collisions which is an approximation for finite τ. Both effects may cause complications for oblique collisions [9,10] but not for central collisions considered in this Letter.

For central collisions of viscoelastic spheres, the COR may be analytically computed from the interaction force [11–13] to obtain ε as a decaying function of the impact velocity [14] in good agreement with experiments [15].

Experimentally, the COR may be measured by means of a ball bouncing on a solid plane, e.g., [16–21]: When the ball is released from a certain height, the times of impact may be determined to high precision from the sound emission using contact microphones:

$$\varepsilon(v) = \frac{\Delta t_{n,n+1}}{\Delta t_{n-1,n}}, \qquad v = \frac{g}{2}\Delta t_{n-1,n}, \tag{2}$$

where $g = 9.81\ \mathrm{m/s^2}$ and $\Delta t_{i,i+1} \equiv t_{i+1} - t_i$, with the impact times t_k. Alternatively, $\varepsilon(v)$ may be obtained from the peak heights, $h_{i,i+1}$, between impacts i and $i+1$:

$$\varepsilon(v) = \sqrt{\frac{h_{n,n+1}}{h_{n-1,n}}}, \qquad v = \sqrt{2gh_{n-1,n}}. \tag{3}$$

Experiment.—We performed bouncing ball experiments using a stainless steel sphere (diameter 6 mm) and a massive glass plate ($40 \times 20 \times 2\ \mathrm{cm^3}$), supported by a leveled steel baseplate (~ 8 kg). The plates are mechanically decoupled via a 1 mm layer of sorbothane. When released from the initial height, $8\ \mathrm{cm} \le h_0 \le 12\ \mathrm{cm}$, the ball comes to rest after typically 80–100 bounces, yielding the same amount of data points, $\varepsilon(v)$, via Eq. (2). The times t_k are obtained by recording the sound using a piezoelectric sensor mounted to the glass plate [22] and edge detection (sample rate 5×10^5/s, resolution 16 Bit). The resulting error $\delta\varepsilon$ leads to an uncertainty $\delta\varepsilon/\varepsilon \lesssim 2.5 \times 10^{-4}$ in the range of v considered here.

To obtain good statistics, the experiment was fully automated using a robot to position a vertical vacuum nozzle to a desired starting point: Initially, the steel sphere is held by the nozzle at a position (x_0, y_0) in the horizontal plane, at a distance h_0 above the plate. By breaking the vacuum, it is released to bounce repeatedly off the plane. When the sphere comes to rest, it is blown by a fan into a catch tank. From there, the robot picks it up again using the vacuum nozzle and moves it to the next initial position. Each single experiment lasts for about one minute.

We performed about 5, 000 experiments, resulting in about 400, 000 impacts. From these impacts we selected those taking place at impact velocities ranging from 0.3 to 0.7 m/s corresponding to dropping heights from the interval (5, 25) mm. The resulting CORs are shown in Fig. 1. Clearly, the expected decay of $\varepsilon(v)$ is superimposed by an undulation. This can be seen from the data directly but more obviously from their median. The extraordinary degree of fluctuations in the data which was earlier noticed [23] and explained [24], requires a very large set of data to observe the undulation which is otherwise hidden by noise. This might be the reason why despite the very large body of literature available on COR measurements, this effect seems to be overlooked so far. We wish to note that King *et al.* [23] reported a nonmonotonic function $\varepsilon(v)$ (a single hump). However the height of the jumps of particles considered there is in the region of 50 nm, that is about 10^5 times smaller than in our case. Therefore, we believe that the effects reported in [23] are of different origin [25]. Additionally it is known [26–28] that the COR drops to zero for low impact velocities due to adhesion, which is not relevant in our setup.

The data shown in Fig. 1 take several days of continuous measurements. To assure constant conditions, we controlled air temperature and humidity. To avoid damages of the glass by repeatedly dropping the sphere from the same position, the initial position, (x_0, y_0), was chosen randomly from a dropping zone of size 5×5 cm^2 close to the center of the plate. The surface of the sphere was inspected before and after the experiment and no

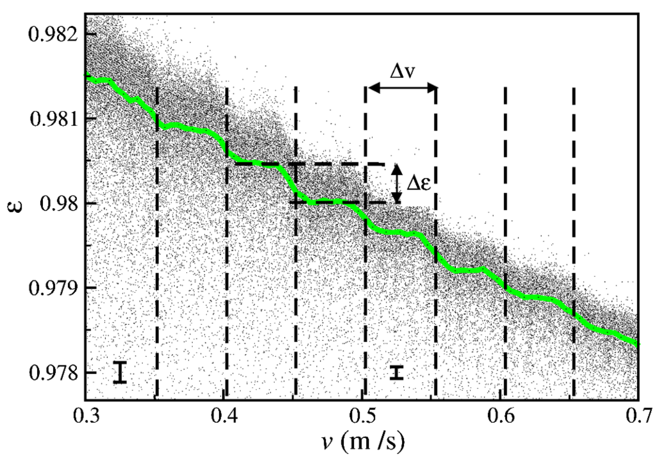

FIG. 1 (color online). COR as a function of the impact velocity. The green line displays the median. Equidistant vertical lines guide the eyes to the steplike shape of $\varepsilon(v)$. Typical error bars are given at $v = 0.325$ m/s and $v = 0.525$ m/s.

significant differences were found. Moreover, to check stationarity, we analyzed the first 10% and the last 10% of the data separately. Both sets of data lead to identical statistics up to fluctuations. Thus, no significant influence of wear was apparent. During the entire measurement we did not observe any effect of electrostatics.

Theory.—In dissipative collisions, part of the energy of the macroscopic translational degree of freedom (DOF) of the particle is transferred into other forms of energy such as heat, sound emission, and vibration. The influence of vibrational DOFs of colliding bodies on the dynamics was intensively investigated experimentally (e.g., [17,19,23,29]) and theoretically (e.g., [19,30]). If no vibrational DOFs are excited before the impact, energy will be transferred from linear motion to vibrational DOFs, that is, the coupling between vibrational and translational DOFs leads to dissipation of translational energy [31–34], which implies $\varepsilon < 1$. If, however, particles with excited vibrational DOFs collide, the energy stored in the vibration may be transformed back into translational energy, in dependence on the phase of the vibration at the instant of the impact. In extreme cases, even superelastic collisions, $\varepsilon > 1$, may be observed [35,36]. Thus, coupling of translational and vibrational DOFs may result in both, reduction or enhancement of the COR. For uncorrelated collisions as in a granular gas, this behavior may be captured by a statistical description of the COR [37,38].

To explain the undulation of $\varepsilon(v)$, we consider the vibrational DOFs of the colliders, their interaction with the linear motion, and their effect on the COR in detail. We show that the time scale due to the oscillation period of the excited vibrational DOFs relates to other characteristic times of the system which gives rise to the experimentally found undulations, Fig. 1.

The characteristic times of the system are (a) the period of the sphere's fundamental oscillation, τ_s, (b) the period of the baseplate's fundamental mode, τ_b, (c) the typical contact duration, τ_c, and (d) the time of free flight, τ_f. Note that effects arising from higher modes of the sphere and the plate are not observed in our experiment. For the coupling between translation and vibration, there are four relevant combinations of time scales: (τ_s, τ_c), (τ_s, τ_f), (τ_b, τ_c), and (τ_b, τ_f). While all of them lead to interesting phenomena [25], only the last combination may be relevant here. The other combinations are either not in agreement with a constant period, Δv, of the undulation which is clearly seen in Fig. 1 or the value of Δv would be by orders of magnitude off for our material parameters [25].

Model.—To introduce the model, for clarity we first provide a simplified version using phenomenological parameters. Then we refer to the concrete details of the experiment to obtain quantitative agreement between model and experiment.

The model is sketched in Fig. 2: We describe the baseplate as a point mass, m_b, connected via a linear-dashpot

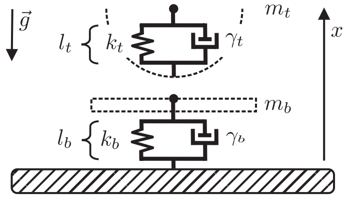

FIG. 2. Model of the bouncing ball experiment.

force to a fixed point (spring constant k_b, length l_b, damping γ_b). The impacting sphere of mass m_t interacts with m_b also via a linear-dashpot force (k_t, l_t, γ_t).

Newton's equations read

$$m_t\ddot{x}_t = -m_t g + F_{bt}, \qquad m_b\ddot{x}_b = -m_b g + F_{gb} - F_{bt}, \tag{4}$$

with $g = 9.81\ \mathrm{m/s^2}$ and the interaction forces

$$F_{gb} = -k_b\xi_b - \gamma_b\dot{\xi}_b \tag{5}$$

$$F_{bt} = \begin{cases} 0, & \xi_t \geq 0 \\ \max(-k_t\xi_t - \gamma_t\dot{\xi}_t, 0), & \xi_t < 0, \end{cases} \tag{6}$$

where $\xi_b \equiv x_b - l_b$ and $\xi_t \equiv x_t - x_b - l_t$. The max function in Eq. (6) assures the force between the baseplate and the sphere to be strictly repulsive. We solve Eq. (4) with the initial conditions

$$x_b = l_b - m_b g/k_b; \qquad x_t = x_b + l_t + h_{0,1}; \qquad \dot{x}_b = \dot{x}_t = 0, \tag{7}$$

where $h_{0,1}$ is the dropping height.

Starting from the dropping height, $h_{0,1}$, the first impact excites the vibrational DOF of the lower spring and the ball bounces back to a certain height $h_{1,2}$. Using Eq. (3) [39] we then determine ε_2 and the corresponding v_2 from $h_{1,2}$ and the maximum height after the second impact, $h_{2,3}$.

Let us first look to the undamped case, $\gamma_b = \gamma_t = 0$. Naïvely (disregarding energy transfer between vibrational and translational DOFs) here one expects $\varepsilon = 1$ since no dissipative forces are involved. Because of the coupling of the DOFs, however, we obtain a different result, see Fig. 3(a). During the first impact, kinetic energy of the ball is transferred to the lower spring leading to oscillations of m_b. During the second impact again energy is transferred between m_t and m_b; however, since m_b oscillates with its eigenfrequency this energy may be positive or negative, depending on the phase of the oscillation. Consequently, for elastic interaction, ε may be larger or smaller than one. The vibrational DOF of the baseplate acts as a phase dependent source or sink for the translational energy, thus, leading to oscillations of $\varepsilon(v)$.

Consider now the corresponding dissipative system (γ_b, $\gamma_t \neq 0$). Here the amplitude of the plate's vibration decreases during the free flight, τ_f, reducing the amount of transferable energy. As τ_f increases with impact velocity,

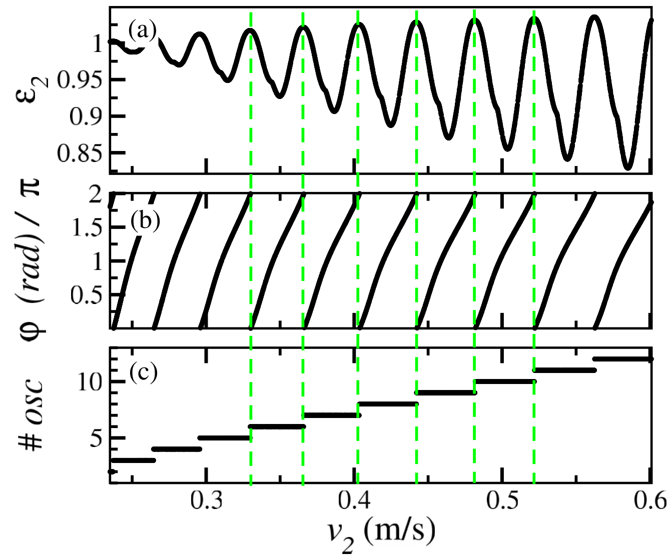

FIG. 3 (color online). Elastic ball impacting an elastic plate ($\gamma_b = \gamma_t = 0$). All quantities are shown as functions of the impact velocity, v. (a) COR, $\varepsilon(v_2)$. (b) Baseplate's oscillation phase at impact. (c) Number of complete baseplate oscillations during the period of free flight. Equidistant dashed lines lead the eye to the relation between the maxima of $\varepsilon(v)$ and the vibration of the baseplate. Parameters: $m_b = m_t = 1$ g, $k_b = k_t = 500$ N/m, $l_b = l_t = 1$ mm, $\gamma_b = \gamma_t = 0$.

$\varepsilon(v)$ is a decreasing function, superimposed with the oscillation due to the phase of the baseplate's oscillation. From the numerical solution of Eq. (4) we obtain the COR shown in Fig. 4.

The functional form of $\varepsilon(v)$ obtained from the experiment, shown in Fig. 1, resembles Fig. 4. In particular, both figures show a decaying function $\varepsilon(v)$ superimposed by an oscillation of constant width. Therefore, we believe that the match or mismatch of the baseplate's vibration and period of free flight is responsible for the steplike structure of $\varepsilon(v)$ found in experiments.

Beyond the simplified model.—The parameters of the model are not directly related to the experiment but are chosen rather arbitrarily. To check the validity of the

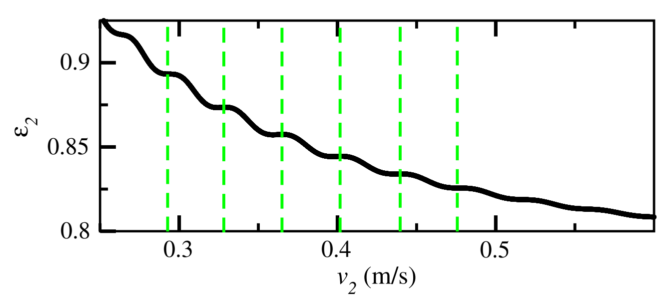

FIG. 4 (color online). COR, $\varepsilon(v)$, due to Eq. (4), including dissipative forces. Equidistant vertical lines correspond to the lines shown in Fig. 3. Parameters: same as in Fig. 3 except $\gamma_b = 0.09$ kg/s and $\gamma_t = 0.025$ kg/s.

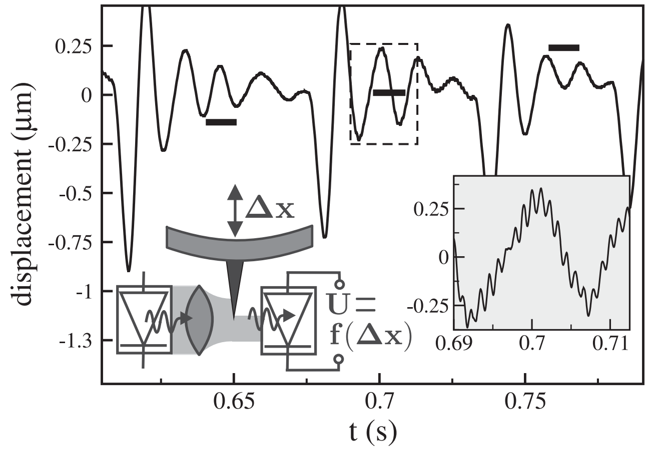

FIG. 5. Smoothed vertical displacement of the plate's surface over time for impact velocity $v \approx 0.33$ m/s. The left inset illustrates the applied light shading technique. The right inset shows a magnification of the region marked by the dashed rectangle. Horizontal bars highlight the period obtained from the step width, $f^{\text{plate}} \approx 98.1$ Hz, being in very good agreement with the period of the damped oscillation measured here.

model, we refer now quantitatively to the details of the experiment.

Consider first the length of the undulation, Δv, of $\varepsilon(v)$, see Fig. 1. The maximum deformation of the plate caused by the impacting sphere is about 0.25 μm as obtained from light shading experiments (see Fig. 5). This amplitude is small compared to the thickness of the soft sorbothane layer (1 mm) such that the plate may be considered as unsupported. The eigenfrequencies of a free rectangular, elastic plate (thickness h, density ρ) read [40]

$$f_{ij}^{\text{plate}} = \frac{\lambda_{ij}}{2\pi a^2}\sqrt{D/\rho} \quad \text{with} \quad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{8}$$

where E is the Young's modulus, and a is the length of the shorter side of the plate. For the plate's aspect ratio 1/2 and Poisson's ratio $\nu = 1/6$, we have $\lambda_{02} \approx 5.593$ for the lowest eigenmode [40]. For the glass plate used in the experiment ($E \approx 70$ GPa, $\rho = 2569$ kg/m^3, $a = 0.2$ m, $h = 0.02$ m) we obtain $f_{02}^{\text{plate}} \approx 96.2$ Hz. The expected period of the oscillation of $\varepsilon(v)$ follows from

$$\Delta v = g/(2f^{\text{plate}}) \approx 0.051 \text{ m/s}. \tag{9}$$

This value is in very good agreement with the length of the undulation of $\varepsilon(v)$ shown in Fig. 1 ($\Delta v \approx 0.05$ m/s corresponding to $f^{\text{plate}} \approx 98.1$ Hz).

Second, we look to the height of the undulation steps, $\Delta\varepsilon$ (Fig. 1): Assume the baseplate oscillates with $-A\sin(2\pi f^{\text{plate}}t)$. The energy transfer from the plate to the particle is, in very good approximation, maximal when the impact takes place at an instant where the plate has maximum upward velocity

$$v_b = 2A\pi f^{\text{plate}} \quad \text{at } t_n^{\max} = \frac{(2n-1)\pi}{2\pi f^{\text{plate}}}, \qquad n = 1, 2, 3, \ldots \tag{10}$$

These values $t_n^{\max}$ for the time of free flight correspond to the impact velocity [see Eq. (2)]

$$v_n^{\max} = -\frac{g}{2}\frac{2n-1}{2\pi f^{\text{plate}}}\pi. \tag{11}$$

Similarly, the minimal energy transfer is achieved when the plate has maximum downward velocity

$$v_b = -2A\pi f^{\text{plate}} \quad \text{at } t_n^{\min} = \frac{2n\pi}{2\pi f^{\text{plate}}}, \qquad n = 1, 2, 3, \ldots, \tag{12}$$

where $t_n^{\min}$ corresponds to

$$v_n^{\min} = -\frac{g}{2}\frac{2n}{2\pi f^{\text{plate}}}\pi. \tag{13}$$

Consequently, the postcollisional velocities for impacts taking place at $t_n^{\max}$ and $t_n^{\min}$ are

$$\begin{aligned} V_n^{\max} &= A2\pi f^{\text{plate}} - v_n^{\max} \\ V_n^{\min} &= -A2\pi f^{\text{plate}} - v_n^{\min} \end{aligned} \tag{14}$$

and the corresponding CORs are $\varepsilon_n^{\min} = -V_n^{\min}/v_n^{\min}$ and $\varepsilon_n^{\max} = -V_n^{\max}/v_n^{\max}$, respectively, see Eq. (1). The amplitude of the undulation, $\Delta\varepsilon(v)$, shown in Figs. 1 and 4 is, hence, given by

$$\Delta\varepsilon = \varepsilon_n^{\max} - \varepsilon_n^{\min} = \frac{A(2\pi f^{\text{plate}})^2}{g\pi}\frac{4n-1}{2n^2-n}. \tag{15}$$

Consider an impact velocity, $v = 0.5$ m/s, corresponding to the ratio of the free flight time to the period of the baseplate's oscillation $n = (2v/g)f^{\text{plate}} = 10.0062 \approx 10$. With the measured amplitude of the oscillation, $A \approx 0.25$ μm, from Eq. (15) we obtain $\Delta\varepsilon \approx 0.0006$ which is in good agreement with the measured value $\Delta\varepsilon \approx 0.0005$ (see Fig. 1).

The oscillation of the plate can be measured using a light shading technique, Fig. 5. After each impact (large spikes directed downward), the plate performs a damped oscillation with almost exactly the predicted fundamental frequency, $f_{02}^{\text{plate}} = 96.2$ Hz being in good agreement with the frequency derived from the step width, $f^{\text{plate}} \approx 98.1$ Hz. Higher modes of much smaller amplitude are also observed, Fig. 5 (inset). These modes could lead, in principle, also to steps in $\varepsilon(v)$ of about 10 times smaller width; however, in this experiment they are not found.

Conclusion.—We performed automated bouncing ball experiments and observed unexpected equidistant steplike features in the COR as a function of impact velocity. This behavior could be explained by a model describing the energy transfer between the vibration of the baseplate and the translational motion of the bouncing sphere. By detailed analysis of the geometry and material properties of the experiment we could explain the length and height of the steps, Δv and $\Delta \varepsilon$, found in the experiment quantitatively up to very good precision. Our findings give clear evidence that internal vibrational degrees of freedom of colliding bodies may be essential for the dynamics of granular systems.

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