Wigner Time-Delay Distribution in Chaotic Cavities and Freezing Transition

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Using the joint distribution for proper time delays of a chaotic cavity derived by Brouwer, Frahm, and Beenakker [Phys. Rev. Lett. **78**, 4737 (1997)], we obtain, in the limit of the large number of channels N, the large deviation function for the distribution of the Wigner time delay (the sum of proper times) by a Coulomb gas method. We show that the existence of a power law tail originates from narrow resonance contributions, related to a (second order) freezing transition in the Coulomb gas.

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The study of scattering theory in chaotic or disordered quantum systems within the random matrix theory (RMT) has been a subject of intense research for many years. Though it originated in nuclear physics (see the review [1]), it has major implications in condensed matter theory where it can be used to model electronic transport in mesoscopic (coherent) conductors [2,3]. The dynamics of an electron of energy E is described through the $N \times N$ on-shell scattering matrix $\mathcal{S}(E)$, where N is the number of scattering channels. A useful concept that characterizes the temporal aspects of the scattering process is time delay [4,5] undergone by an incident wave packet. This is captured by the Wigner-Smith time-delay matrix [6], $Q(E) = -i\mathcal{S}(E)^\dagger \partial \mathcal{S}(E)/\partial E$ (with $\hbar = 1$), whose eigenvalues are the *proper* time delays $\tau_1, \ldots, \tau_N$.

If the system is characterized by some complex dynamics, due to the presence of disorder or chaos, the statistical properties of the time delays exhibit interesting universal characteristics: the universality of the time-delay distribution for 1D-disordered quantum mechanics was demonstrated in [7] (see also [8–12] and [13] for 2D and 3D cases). The situation where the dynamics is chaotic has been extensively studied within RMT: the marginal law of *partial* time delays [14], $\tilde{p}_N(\tau) = (1/N)\sum_a \langle \delta(\tau - \tilde{\tau}_a)\rangle$, was obtained for a Gaussian unitary ensemble (GUE) of RMT indexed by $\beta = 2$ [15,16]. In [17], the time-delay distribution was derived in the $N = 1$ case with $\beta \in \{1, 2, 4\}$, corresponding to orthogonal, unitary, and symplectic symmetry classes. Using the "alternative RMT" introduced in [18], Brouwer and co-workers succeeded in finding the joint distribution of the inverse proper time delays $\gamma_k \equiv 1/\tau_k$ (in the absence of tunable barriers at the contacts) [19,20]:

$$P(\gamma_1, \ldots, \gamma_N) \propto \prod_{i<j} |\gamma_i - \gamma_j|^\beta \prod_k \gamma_k^{\beta N/2} e^{-\beta\gamma k/2} \quad (1)$$

(the times are measured in units of the Heisenberg time $\tau_H = 2\pi\hbar/\Delta$, where Δ is the mean level spacing). This measure, known as the Laguerre ensemble of random matrices, also corresponds to the distribution of the (positive) eigenvalues of Wishart matrices $X^\dagger X$, where the matrix X has size $N \times (2N - 1 + 2/\beta)$ with independent and identically distributed Gaussian matrix elements.

In this article we are interested in the Wigner time delay, defined as the sum of proper (or partial) [14] time delays $\tau_W = \mathrm{Tr}\{Q\}/N = (1/N)\sum_{a=1}^N \tau_a$. This quantity is of great interest due to its close relation to the density of states (DOS) of the *open system*, through the Krein-Friedel relation [21]: $\nu(E) = \mathrm{Tr}\{Q(E)\}/(2\pi) = N\tau_W/(2\pi)$. The Wigner time delay (or related quantities such as injectance or emittance) is a central concept for studying charging effects, e.g., for mesoscopic capacitances [17,22].

We denote by $P_N(\tau) = \langle \delta(\tau - (1/N)\sum_a \tau_a)\rangle$ the Wigner time-delay distribution. Despite the fact that the joint distribution of proper times has been known already for 15 years, little is known about the distribution of τ_W for general N: it has been computed explicitly only for $N = 1$, $P_1(\tau) = (\beta/2)^{\beta/2}/\Gamma(\beta/2)\tau^{-2-\beta/2}e^{-\beta/(2\tau)}$ [17] and $N = 2$, $P_2(\tau) = \beta^{3\beta+2}\Gamma(3(\beta+1)/2)/[\Gamma(\beta+1)\Gamma(3\beta+2)]\tau^{-3(\beta+1)}U((\beta+1)/2, 2(\beta+1); \beta/\tau)e^{-\beta/\tau}$ [24], where $U(a, b; z)$ is the confluent hypergeometric function. The distribution was conjectured to have a power law tail for large τ, $P_N(\tau) \sim \tau^{-2-\beta N/2}$ in [16] (for $\beta = 2$)

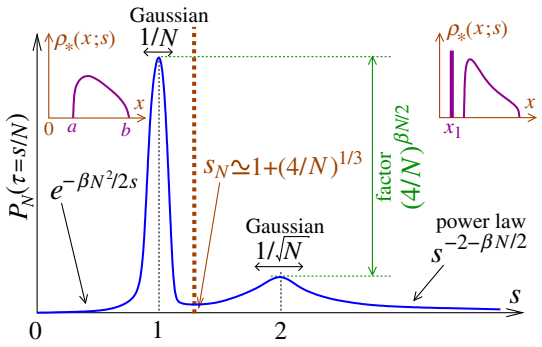

FIG. 1 (color online). Sketch of the distribution of $s = N\tau_W$. The dashed line at $s = s_N \simeq 1 + (4/N)^{1/3}$ separates the two phases of the Coulomb gas with densities represented in the small figures on the left and the right, respectively.

by using the resonance picture allowing one to identify the tails of $P_N(\tau)$ and $\tilde{p}_N(\tau)$ (for a heuristic argument using the relation to resonance width, cf. the review [25]). More recently, the first three cumulants of τ_W were derived by a generating function method [26]. However, a full understanding of its distribution for general N is still missing so far.

In this Letter, by analyzing an underlying Coulomb gas we provide a complete description of $P_N(\tau)$ for large N and show that it has a rather rich behavior including an interesting nonanalytic point which is a consequence of a freezing transition in the Coulomb gas. Limiting behaviors of $P_N(\tau)$ may be summarized as follows (τ_W is measured in unit of τ_H):

$$P_N(\tau) \sim \tau^{-3N^2\beta/4} e^{-N\beta/(2\tau)} \quad \text{for } \tau \ll \frac{1}{N}, \tag{2}$$

$$\sim \exp -\frac{N^4\beta}{8}\left(\tau - \frac{1}{N}\right)^2 \quad \text{for } \tau \sim \frac{1}{N}, \tag{3}$$

$$\sim N^{-\beta N/2} \exp -\frac{N^3\beta}{4}\left(\tau - \frac{2}{N}\right)^2 \quad \text{for } \tau \sim \frac{2}{N}, \tag{4}$$

$$\sim \tau^{-2-\beta N/2} \quad \text{for } \tau \gg \frac{1}{N}. \tag{5}$$

A sketch of the distribution is given in Fig. 1. The Gaussian form around $\tau \sim 1/N$ in (3) allows one to extract the mean time delay and its variance. Reinstating τ_H, we obtain $\langle \tau_W \rangle = \tau_H/N$. Consequently, the mean DOS reads $\langle \nu(E) \rangle = N\langle \tau_W \rangle/(2\pi) = 1/\Delta$, as expected. Similarly, the variance can be read off (3)

$$\mathrm{Var}\,(\tau_W) \simeq \frac{4\tau_H^2}{\beta N^4}, \quad \text{i.e., } \mathrm{Var}(\nu(E)) \simeq \frac{4}{\beta N^2 \Delta^2}. \tag{6}$$

Equation (6) was first obtained in [27] for $\beta = 1$. It agrees with the leading order of the result obtained in Ref. [26], $\mathrm{Var}(\tau_W) = 4\tau_H^2/[(N+1)(N\beta-2)N^2]$. Note also that (5) coincides with the power law tail conjectured by Fyodorov and Sommers [16], $P_N(\tau) \sim \tau^{-2-\beta N/2}$.

Coulomb gas.—To derive our main results (2)–(5), we use the Coulomb gas method, originally introduced by Dyson [28]. Recently, this method has been suitably adopted and successfully used in a number of different contexts: e.g., the distribution of the conductance of chaotic cavities [29–31], or the quantum entanglement in a random bipartite state [32–34]. Our starting point is to rewrite the joint distribution (1) of the rescaled rates $x_i = \gamma_i/N$ as a Gibbs measure, $P(\gamma_1, \ldots, \gamma_N) \propto \exp\{-\beta N^2 \mathcal{E}[\rho]/2\}$, with the "energy" $\mathcal{E}[\rho]$ expressed as a functional of the density of the rescaled rates $\rho(x) = (1/N)\sum_{i=1}^N \delta(x - x_i)$. The energy reads

$$\mathcal{E}[\rho] = \int_0^\infty dx (x - \ln x)\rho(x) - \int_0^\infty dx dx' \rho(x)\rho(x') \ln|x - x'|. \tag{7}$$

The rescaled time delay is $s = N\tau_W = \sum_i \gamma_i^{-1}$ (i.e., the DOS of the cavity in appropriate units $s = \nu(E)\Delta$). In the limit $N \to \infty$, the density $\rho(x)$ may be treated as continuous and the distribution $P_N(\tau = s/N)$ can be derived via a saddle point method. The optimal (saddle point) distribution minimizes (7) with two constraints: normalization $\int dx \rho(x) = 1$ and $\int dx\, \rho(x)/x = s$. This requires the minimization of the "free energy" $\mathcal{F}[\rho] = \mathcal{E}[\rho] + \mu_0(\int dx \rho(x) - 1) + \mu_1(\int dx \rho(x)/x - s)$, where μ_0 and μ_1 are two Lagrange multipliers that enforce the two constraints (we neglect the subdominant contribution of entropy [35]). Setting the functional derivative $\delta\mathcal{F}/\delta\rho(x_0) = 0$ gives

$$\mu_0 + x_0 - \ln x_0 + \frac{\mu_1}{x_0} - 2\int_a^b dx \rho(x) \ln|x - x_0| = 0, \tag{8}$$

where we assume that the optimal density has support over the interval $x_0 \in [a, b]$. Deriving once more with respect to x_0 gives

$$\frac{1}{2}\left(1 - \frac{1}{x_0} - \frac{\mu_1}{x_0^2}\right) = \fint_a^b dx \frac{\rho(x)}{x_0 - x}, \tag{9}$$

where $\fint$ represents the principal part. This equation expresses the force balance at equilibrium, for any charge at $x_0 \in [a, b]$, between the confining force $-V'_{\mathrm{eff}}(x)$ coming from the effective potential $V_{\mathrm{eff}}(x) = x - \ln x + \mu_1/x$ and the Coulomb repulsion force from other charges. We denote by $\rho_*(x; s)$ the solution of (9). The time-delay distribution then takes the scaling form

$$P_N(\tau) \underset{N\to\infty}{\sim} \exp\left\{-\frac{1}{2}\beta N^2 \Phi_-(N\tau)\right\}, \tag{10}$$

where the large deviation function is $\Phi_-(s) = \mathcal{E}[\rho_*(x; s)] - \mathcal{E}[\rho_*(x; 1)]$ (note that when the two constraints are fulfilled, $\mathcal{F}[\rho_*] = \mathcal{E}[\rho_*]$). The term $\mathcal{E}[\rho_*(x; 1)]$ emerges from the normalization of (1), obtained by solving the same equation in the absence of the second constraint, i.e., for $\mu_1 = 0$, which we will show to coincide with $s = 1$. Using (8), we may rewrite the energy of the optimal distribution as

$$\mathcal{E}[\rho_*(x;s)] = \frac{\mu_1}{2}\left(\frac{1}{x_0} - s\right) + \int_a^b dx \rho_*(x;s) \times \left[\frac{x - \ln x + x_0 - \ln x_0}{2} - \ln|x - x_0|\right]. \tag{11}$$

Optimal distribution.—The integral equation (9) may be solved thanks to a theorem due to Tricomi [36]. We find the optimal distribution

$$\rho_*(x;s) = \frac{1}{2\pi}\frac{x+c}{x^2}\sqrt{(x-a)(b-x)}, \tag{12}$$

where the three parameters a, b, and $c = \mu_1/\sqrt{ab}$ can be found by solving the three algebraic equations obtained by imposing the vanishing of the density at the two boundaries and the condition $\int_a^b dx\rho_*(x;s)/x = s$. These equations are conveniently written in terms of the variables $v = \sqrt{ab}$ and $u = \sqrt{a/b}$. A few steps of algebra shows that u solves

$$s = \sigma(u) \overset{\text{def}}{=} (1-u)^2\frac{(-u^4+16u^3+2u^2+16u-1)}{16u^2(3u^2-2u+3)}. \tag{13}$$

Then v, μ_1, and c are given by $v = 2u(3u^2-2u+3)/(1-u^2)^2$, $\mu_1 = -4u^2(u^2-6u+1)(3u^2-2u+3)/(1-u^2)^4$, and $c = \mu_1/v = -2u(u^2-6u+1)/(1-u^2)^2$.

Most probable values.—We first analyze the distribution $P_N(\tau)$ in the vicinity of its maximum. $\mathcal{E}[\rho_*(x;s)]$ is minimized, i.e., $P_N(\tau)$ is maximized, by removing the constraint $\int_a^b dx\rho(x)/x = s$, i.e., by setting $\mu_1 = 0$. For convenience we introduce the roots $x_\pm = 3 \pm 2\sqrt{2}$ of the polynomial $u^2 - 6u + 1$. For $\mu_1 = 0$, Eq. (13) has the solution $u = \sqrt{x_-/x_+} = x_-$ with $v = 1$ and $s = 1$ and consequently $a = x_- = 0.171\ldots$ and $b = x_+ = 5.828\ldots$ In this case we recover the Marčenko-Pastur (MP) law [37]

$$\rho_*(x;1) = \frac{1}{2\pi x}\sqrt{(x-x_-)(x_+-x)}. \tag{14}$$

Expansion of Eq. (13) around the MP point leads to $s-1 \simeq -x_+(u-x_-)/\sqrt{2}$, hence $v \simeq 1 + 3x_+(u-x_-)/(2\sqrt{2}) \simeq 1 - 3(s-1)/2$ and $c \simeq \mu_1 \simeq -(s-1)/2$. The corresponding energy (11) may be conveniently obtained by choosing $x_0 = 1$: we see that the first term is quadratic $(s-1)^2/4$; we check numerically that the remaining integral term is constant, equal to $\mathcal{E}[\rho_*(x;1)] = 3 - 2\ln 2$, up to higher order corrections [numerical fit gives a correction $(s-1)^3/4$]. Therefore we conclude that $\Phi_-(s) \underset{s\sim 1}{\simeq} (1/4)(s-1)^2$, i.e., Eq. (3) [the parabolic behavior is compared to the numerical calculation of the integral (11) in Fig. 3].

Large deviations for $s \to 0$.—Expansion of (13) for $s \to 0$ gives $u = 1 - \sqrt{2s} + s + \mathcal{O}(s^{3/2})$, hence $v = 1/s + \mathcal{O}(s^0)$. The support of the distribution is given by $a = (1/s)[1 - \sqrt{2s} + \mathcal{O}(s)]$ and $b = (1/s)[1 + \sqrt{2s} + \mathcal{O}(s)]$ (the Lagrange multiplier is $\mu_1 = s^{-2} + \mathcal{O}(s^{-1})$). The optimal distribution resembles the semicircle law centered around $1/s$:

$$\rho_*(x;s) \underset{s\to 0}{\simeq} \frac{1}{\pi}\sqrt{2s-(sx-1)^2}. \tag{15}$$

This was expected: when $s \to 0$, the eigenvalues $\{x_i\}$ of the Wishart matrix are constrained to be very large and they do not feel the spectrum boundary at $x = 0$. Hence, their distribution coincides with the Wigner semicircle law for the usual Gaussian ensembles of random matrices. The energy may be conveniently calculated by choosing $x_0 = 1/s$; this makes the first term of (11) vanish. The leading order of the integral term is straightforwardly calculated from (15): we deduce $\Phi_-(s) \underset{s\to 0}{\simeq} 1/s + (3/2)\ln s - 5(1-\ln 2)/2$, thus proving (2). The factor $\exp -(N\beta/2\tau)$ is in perfect agreement with the exact results for $N = 1\&2$ mentioned earlier.

Large deviations for $s \geq 1$: Freezing transition.—As s increases, it eventually reaches a finite value corresponding to the maximum of the function $\sigma(u)$ (inset of Fig. 2), at $u_c = [1 + 2(2^{1/3} - 2^{2/3})]/3 = 0.115\ldots$ giving $s_c = \sigma(u_c) = [10 + 6\times 2^{1/3} - 11\times 2^{2/3}]/[3(6 - 6\times 2^{1/3} + 2^{2/3})] = 1.1738\ldots$. Then $a = -c$, which leads to a somewhat unusual form

$$\rho_*(x;s_c) = \frac{1}{2\pi x^2}(x-a)^{3/2}(b-x)^{1/2}. \tag{16}$$

For $s > s_c$, (13) no longer has physical (real) solutions. In this case, the saddle point turns out to have a different solution where a single isolated charge, say at x_1, splits off the main body of the density and carries a macroscopic weight (see Fig. 1). A similar scenario occurs in the study of quantum entanglement in the random bipartite state [32–34]. We decompose the density as $\rho(x) = (1/N)\delta(x-x_1) + \tilde{\rho}(x)$ where $\tilde{\rho}(x) = (1/N)\sum_{i>1}\delta(x-x_i)$ is still treated as a continuous density. The energy

$$\mathcal{E}[\rho] = \mathcal{E}[\tilde{\rho}] + \frac{x_1 - \ln x_1}{N} - \frac{2}{N}\int dx\tilde{\rho}(x)\ln(x-x_1) \tag{17}$$

must be minimized under the two constraints $\int dx\tilde{\rho}(x) = 1 - 1/N$ and $\int dx\tilde{\rho}(x)/x = s - 1/(Nx_1)$. This leads to the two equilibrium conditions

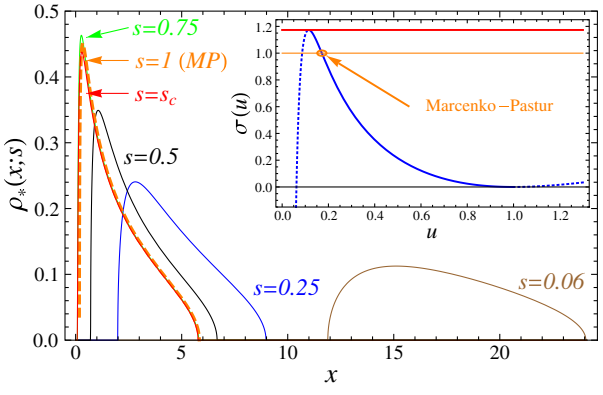

FIG. 2 (color online). The optimal density of eigenvalues for different values of s; when s increases, the density eventually freezes to the MP law (dashed line).

$$\frac{1}{2}\left(1-\frac{1}{x_0}-\frac{\mu_1}{x_0^2}\right)-\frac{1}{N}\frac{1}{x_0-x_1}=\fint_a^b dx'\,\frac{\tilde{\rho}(x')}{x_0-x'} \quad (18)$$

$$\frac{1}{2}\left(1-\frac{1}{x_1}-\frac{\mu_1}{x_1^2}\right)=\int_a^b dx'\,\frac{\tilde{\rho}(x')}{x_1-x'}, \quad (19)$$

$\forall\, x_0 \in [a,b]$ and $x_1 < a$. We now demonstrate that a consistent scenario is the following: freezing of the density $\tilde{\rho}(x)$ while the isolated charge goes to zero $x_1 \to 0$. When $N \to \infty$, the right-hand side of (19) reaches a constant value as $x_1 \to 0$; so does the left-hand side if and only if $\mu_1 \simeq -x_1 \to 0^-$. Hence the solution of (18) is the MP law: $\tilde{\rho}_*(x;s) = \rho_*(x;1) + \mathcal{O}(N^{-1})$. The rescaled time delay splits into the contribution of the isolated charge and of $\tilde{\rho}$ as $s = 1/(Nx_1) + 1$, i.e., $x_1 = 1/[N(s-1)]$. In fact this analysis holds for any $s > 1$ (and not only $s \geq s_c$): the energy (17) of this new phase coincides with the energy of the MP solution, up to $1/N$ corrections. Therefore, for $1 < s \leq s_c$ we have found another phase with a lower energy, which shows that the branch obtained previously (with compact solution (12) over $[a, b]$ for $s < s_c$ as well as (16) for $s = s_c$) is actually *metastable* (Fig. 3). In the (thermodynamic) limit $N \to \infty$, the energy of the gas vanishes for all $s > 1$, while for $s < 1$, it behaves as $(1-s)^2/4$ as mentioned earlier (Fig. 3). This then results in a *second order* phase transition at $s = 1$. We call this a *freezing* transition, because for $s > 1$, energy freezes to the value 0 in the thermodynamic limit and also the bulk density freezes to the MP distribution.

One can analyze more precisely this new *frozen* phase by computing the $1/N$ corrections to the energy. For large enough s, Eq. (17) is dominated by the logarithmic term $-(1/N)\ln x_1$, i.e., $\mathcal{E}[\rho_*(x;s)] \simeq (\cdots) + (1/N)\ln[N(s-1)]$. We get the power law tail $P_N(\tau) \sim (s-1)^{-\tilde{\theta}-\beta N/2}$, where $\tilde{\theta}$ is some exponent of order N^0 introduced in order to account for N^{-2} corrections to $\mathcal{E}[\rho]$. This exponent may be determined as follows: when $\tau_W > 1/N$, most of the proper times are described by the frozen density (the MP law), i.e., $\tau_i \in [x_-/N, x_+/N]$ for $i > 1$ with $\sum_{i>1} 1/\tau_i = 1$, while one proper time becomes much larger and carries a "macroscopic" contribution, $\tau_1 = s - 1 = N\tau_W - 1$. In the scattering problem, this is interpreted as the large contribution of a narrow *resonance*. Writing $P_N(\tau) = \int d\gamma_1, \ldots, d\gamma_N \delta(N\tau - 1/\gamma_1 - 1) P(\gamma_1, \ldots, \gamma_N)$ and using (1) leads to $\tilde{\theta} = 2$, hence Eq. (5).

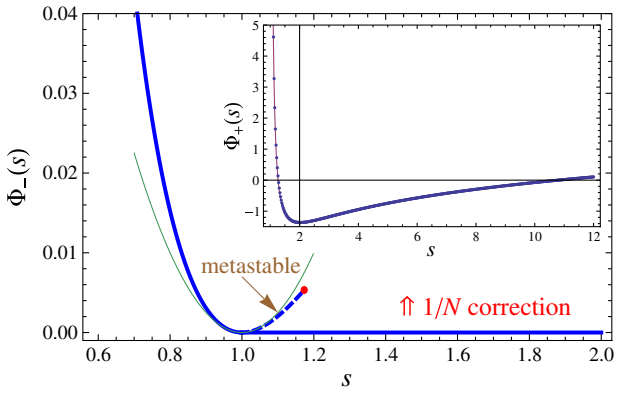

FIG. 3 (color online). Large deviation function $\Phi_-(s)$ (i.e., rescaled energy of the gas). The freezing transition takes place at $s_\infty = 1$. The metastable branch terminates at $s_c = 1.1738\ldots$ Inset: Large deviation function $\Phi_+(s)$ (i.e., $1/N$ correction to the rescaled energy).

A more precise analysis of Eqs. (18) and (19) leads us to introduce the large deviation function $\Phi_+(s) = N(\mathcal{E}[\rho_*(x;s)] - \mathcal{E}[\rho_*(x;1)]) - \ln N$ giving the scaling form

$$P_N(\tau) \sim N^{-\beta N/2} \exp\left\{-\frac{\beta N}{2}\Phi_+(N\tau)\right\} \quad \text{for } \tau > \frac{s_N}{N}. \quad (20)$$

One obtains that $\Phi_+(s) = 1/(s-1) + \ln(s-1) - 1 - 2\ln 2$ (cf. inset of Fig. 3). The local minimum at $s = 2$ is related to (4) while the logarithmic behavior to the power law tail (5). For finite N, the energy functions characterizing the two phases cross for $s = s_N$ such that $\Phi_-(s_N) = [\Phi_+(s_N) + \ln N]/N$. Using the limiting behaviors for $s \to 1$, we obtain the finite N correction to the position of the phase transition: $s_N \simeq 1 + (4/N)^{1/3}$.

Conclusion.—In summary, by using a Coulomb gas approach, we have analyzed the large deviation functions controlling the Wigner time-delay distribution in the limit of a large number of conducting channels. We have shown that the distribution exhibits a rich structure. In particular, its power law tail is related to a freezing transition in the Coulomb gas, corresponding to large contributions to τ_W of resonant states in the original scattering problem. We have also performed a Monte Carlo simulation of the Coulomb gas up to 1600 charges and found good agreement with our analytical results (details will be published elsewhere).

Several questions remain open. (i) A more precise treatment of $1/N$ corrections would be desirable. (ii) The role of tunneling couplings at the contacts and the crossover between GOE and GUE symmetries were studied in [38] for the marginal law $\tilde{p}_N(\tau)$. Similar questions naturally arise for the Wigner time delay distribution and might be relevant for experimental purposes. (iii) The starting point of our calculation, Eq. (1), describes the usual random matrix ensembles; the distribution of τ_W was also obtained in [39] for a chiral-GUE ensemble when $N = 1$. Extension of our analysis to such cases would be certainly interesting, in particular with the growing interest in the study of new symmetry classes of disordered systems.

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