

# Polaritonic Solitons in a Bose-Einstein Condensate Trapped in a Soft Optical Lattice

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We investigate the ground state (GS) of a collisionless Bose-Einstein condensate (BEC) trapped in a soft one-dimensional optical lattice (OL), which is formed by two counterpropagating optical beams perturbed by the BEC density profile through the local-field effect (LFE). We show that LFE gives rise to an envelope-deformation potential, a nonlocal potential resulting from the phase deformation, and an effective self-interaction of the condensate. As a result, stable photon-atomic (polaritonic) lattice solitons, including an optical component, in the form of the deformation of the soft OL, in a combination with a localized matter-wave component, are generated in the blue-detuned setting, without any direct interaction between atoms. These self-trapped modes, which realize the system's GS, are essentially different from the gap solitons supported by the interplay of the OL potential and collisional interactions between atoms. A transition to tightly bound modes from loosely bound ones occurs with the increase of the number of atoms in the BEC.

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The profound importance of the use of optical lattices (OLs) in atomic [1,2] and molecular [3,4] physics is well known. These studies assume that the interaction of an atom or molecule with counterpropagating laser beams illuminating the gas generates a periodic lattice potential [5] (a similar method is used for inducing virtual lattices in photorefractive crystals, which support a great variety of patterns [6]). In fact, there are two aspects of the interaction of the atomic gas with optical fields [7]. First, the field produces a density perturbation in the gas through the effective dipole potential. Second, the OL in an inhomogeneous Bose-Einstein condensate (BEC) may be affected by local variations of the density-dependent refraction index, which is called the local-field effect (LFE). This effect is often ignored for far-off-resonance counterpropagating optical beams in dilute atomic and molecular gases; i.e., the respective OL is assumed to be rigid [1–3,5]. However, the deformation of the OL by the LFE correctly explains [8] the asymmetric diffraction (an asymmetric momentum distribution of scattered atoms) of a BEC on counterpropagating beams with unequal intensities [9]. We call such a deformable OL a *soft lattice*.

The present Letter aims to demonstrate that the distortion of the blue-detuned soft OL by the LFE induces effective interactions in the collisionless BEC, which may be accounted for by three terms: an envelope-deformation potential, a nonlocal potential resulting from the phase deformation, and self interaction (SI). Through these terms, the LFE gives rise to (bright) photon-atomic, alias polaritonic, lattice solitons without the  $s$ -wave

interaction between atoms, which represent the system's ground state (GS). The analysis presented below reveals that a transition from the extended GS to a strongly localized one occurs with the increase of the number of atoms in the BEC,  $N$ . The analysis also shows a drastic asymmetry between blue- and red-detuned OLs—namely, stable solitons do not emerge in the latter case.

In this connection, it is relevant to mention that the trapping of BEC in OL potentials provides a versatile platform for the creation of lattice solitons [2,10] and the investigation of underlying mathematical structures [11]. The use of matter-wave lattice solitons as a high-density source for atomic interferometry has been studied in mean-field and quantum regimes [12]. Using the solitons for the design of quantum memory and switching was proposed, too [13]. Lattice solitons (alias gap solitons) have been created in a nearly one-dimensional (1D) condensate of  $^{87}\text{Rb}$  with repulsive interatomic interactions [14]. Note that, although the gap solitons are, generally, dynamically stable modes, they do not represent the GS of the interacting BEC trapped in the OL.

Nonlinear OLs, created by a spatially periodic modulation of the Feshbach resonance, have been proposed as well [15]. Similar nonlinear lattices can be created in optical media by dint of the electromagnetically induced transparency [16]. Studies of solitons in nonlinear lattices is an active topic in photonics and matter-wave optics [17]. However, the LFE has not yet been addressed in the studies of matter-wave solitons in OLs. The photon-atomic solitons generated by the LFE, which are introduced below,

i.e., localized matter-wave modes coupled to local deformations of the underlying OL, are somewhat similar to polariton-exciton solitons in plasmonics [18,19]. They represent the GS of the system and thus are basically different from the OL-supported gap solitons in the self-interacting BEC.

We consider the atomic BEC irradiated by counterpropagating optical fields  $E_1$  and  $E_2$  with common frequency  $\omega$ . If  $\omega$  is far detuned from the electronic transition frequency of atoms, the condensate wave function  $\Phi(\mathbf{R}, T)$  obeys the Gross-Pitaevskii equation [7],

$$i\hbar \frac{\partial \Phi}{\partial T} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + \frac{|\mathbf{d} \cdot \mathbb{E}|^2}{\hbar \Delta} \right] \Phi, \quad (1)$$

where  $\mathbf{d}$  is the transition matrix element,  $\Delta$  is the detuning, and normalization  $\int |\Phi|^2 d\mathbf{r} = 1$  is imposed. Direct interactions between atoms are disregarded here, to focus on interaction mechanisms induced by the LFE. Experimentally, the  $s$ -wave scattering length can be tuned to nearly zero by means of the Feshbach resonance technique [20,21]. The vectorial optical field  $\mathbb{E}$  obeys the wave equation,

$$\frac{n^2}{c^2} \frac{\partial^2 \mathbb{E}}{\partial T^2} - \nabla^2 \mathbb{E} = 0, \quad (2)$$

with  $c$  the light speed in vacuum, and the refraction index of the condensate,  $n(\mathbf{R}, T) = \sqrt{1 - Nd^2/(\varepsilon_0 \hbar \Delta)} |\Phi(\mathbf{R}, T)|^2$ , which provides for the feedback of the condensate onto the propagation of the electromagnetic field.

We focus on the 1D situation, assuming that the matter wave is confined, with radius  $w_\perp$ , in the transverse plane with radial variable  $R_\perp$ , i.e.,  $\Phi = (\pi w_\perp^2)^{-1/2} \tilde{\Psi}(X, T) \times \exp[-R_\perp^2/(2w_\perp^2)]$ . The confinement may be provided by the magnetic trap, or an optical one making use of a wavelength different from the OL-building one. Because of negligible light scattering in the transverse direction for optical fields with a transverse width far larger than the condensate width [22], we omit the transverse variation of the field and assume it polarized in the  $z$  direction,  $\mathbb{E}(\mathbf{R}, T) = \tilde{E}(X) \mathbf{e}_z e^{-i\omega T}$ , where  $\tilde{E}$  is a slowly varying amplitude. Then, multiplying Eq. (2) by  $\Phi^2$  and integrating the result in the transverse plane, we derive the coupled 1D equations,

$$i \frac{\partial \Psi}{\partial t} = -\frac{\partial^2}{\partial x^2} \Psi(x, t) + \frac{V_0}{4} |E|^2 \Psi, \quad (3)$$

$$2i \frac{n^2 \omega_r}{\omega} \frac{\partial E}{\partial t} + \frac{\partial^2}{\partial x^2} E + n^2 E = 0, \quad (4)$$

with rescaling  $E(x, t) = \tilde{E}(X, T)/E_1$ ,  $\Psi(x, t) = \tilde{\Psi}(X, T)/\sqrt{k_L}$ ,  $x = k_L X$ ,  $t = \omega_r T$ ,  $V_0 = 2d^2 I_1 / (\hbar^2 \Delta \varepsilon_0 c \omega_r)$ ,  $\omega_r = \hbar k_L^2 / (2m)$ . Here  $I_1$  and  $k_L$  are the intensity and wave number of the incident field,  $E_1$ , and  $n \approx \sqrt{1 - N\alpha |\tilde{\Psi}(X, T)|^2}$  with  $\alpha \equiv d^2 / (\pi \varepsilon_0 \hbar \Delta w_\perp^2)$ .

Since the atomic motion is nonrelativistic ( $n^2 \omega_r / \omega \sim 10^{-13}$ ), one may drop the first term in Eq. (4), assuming that the incident light has the form of a long pulse in the temporal direction. Then, a solution to Eq. (4) is a superposition of counterpropagating waves with slowly varying amplitudes and phase shifts [23],

$$E = n^{-1/2} \{ A^+ \exp[i(x + \delta\phi)] + A^- \exp[-i(x + \delta\phi)] \}, \quad (5)$$

where  $\delta\phi = \int_0^x [\sqrt{1 - k_L N \alpha |\Psi|^2} - 1] dx$ , and amplitudes  $A^+$  and  $A^-$  of the right- and left-propagating waves obey equations  $\partial A^\pm / \partial x = S^\mp A^\mp$  with  $S^\pm \equiv (2n)^{-1} (dn/dx) \times \exp[\pm 2i(x + \delta\phi)]$ . As shown in Ref. [8], the approximation based on Eq. (5) agrees well with underlying Eq. (4).

We now focus on the basic case with  $I_1 = I_2$ , i.e.,  $A^+ = \sqrt{I^+} e^{i\theta^+}$  and  $A^- = \sqrt{I^+} e^{i\theta^+ - i\delta\theta}$ , with phase difference  $\delta\theta$ . Seeking for stationary solutions to Eq. (3) as  $\Psi(x, t) = \Psi(x) e^{-i\mu t}$ , and making use of Eq. (5), one obtains

$$\mu \Psi = -\Psi'' + [V_0 \cos^2(x) + \delta V(x)] \Psi, \quad (6)$$

in which the LFE-induced potential, added to the conventional lattice potential, is  $\delta V(x) = V_0 I^+ n^{-1} \cos^2(x + \delta\phi + \delta\theta/2) - V_0 \cos^2 x$ . For typical physical parameters of the system (which are given below), an estimate yields  $k_L \alpha \sim 10^{-6}$ ; hence,  $\delta\phi$  and  $\delta\theta$  are small, and the potential may be approximated by

$$\delta V(x) \approx \delta V^{\text{appr}} = \delta V^{\text{def}}(x) + \delta V^{\text{nonloc}}(x) + \delta V^{\text{SI}}(x). \quad (7)$$

Here, the term induced by the deformation of the field envelope is

$$\delta V^{\text{def}} = V_0 \delta I^+ \cos^2 x, \quad (8)$$

with  $\delta I^+ \equiv I^+ - 1$ ; the nonlocal interaction potential is

$$\delta V^{\text{nonloc}} = -(1/2) V_0 I^+ \sin(2x) \sin(\delta\Theta), \quad (9)$$

with  $\delta\Theta \equiv 2\delta\phi + \delta\theta$  being the phase difference between the right- and left-traveling electromagnetic waves induced by the LFE, and

$$\delta V^{\text{SI}} = \gamma I^+ (\cos^2 x) |\Psi|^2 \quad (10)$$

represents the effective spatially modulated self-repulsion of the condensate, whose strength is periodically modulated in space, with amplitude  $\gamma = Nd^4 k_L I_1 / (\pi c \varepsilon_0^2 \hbar^3 w_\perp^2 \omega_r \Delta^2)$ .

Without terms  $\delta V^{\text{nonloc}}$  and  $\delta V^{\text{SI}}$ , Eq. (6) features a combination of the linear and nonlinear OLs; hence, lattice solitons [2,10,17] may be expected in this setting. However, the deformation of the OL is a part of the present setting; i.e., the nonlocal and deformation potentials make the situation different from the straightforward combination of the linear and nonlinear OLs.

To produce accurate results, we solved Eqs. (3) and (4) numerically, with parameters taken for transition

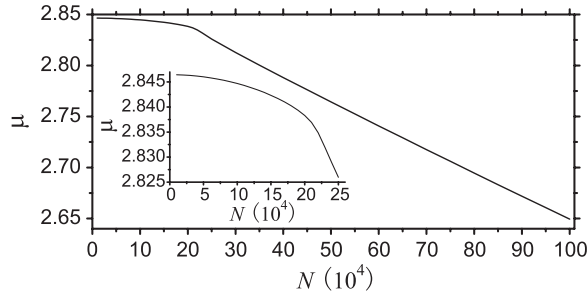


FIG. 1. The chemical potential of the numerically found ground state vs the number of atoms,  $N$ . The inset shows details for relatively small values of  $N$ .

$5^2S_{1/2} \rightarrow 5^2P_{3/2}$  in  $^{87}\text{Rb}$ , blue detuning  $\Delta = +2$  GHz, transverse radius  $10 \mu\text{m}$ , and  $V_0 = 10$  (the OL strength measured in units of the recoil energy), which corresponds to optical intensity  $I_1 = 10.26 \text{ mW/cm}^2$ . Chemical potential  $\mu$  of the so found GS is plotted as a function of  $N$  in Fig. 1. When  $N$  is large ( $N \gtrsim 2 \times 10^5$ ),  $\mu$  decreases almost linearly with  $N$ , the situation being different at smaller  $N$ . Note that a similar linear dependence between  $\mu$  and  $N$  occurs for solitons of the 1D nonlinear Schrödinger equation with the nonlinearity of power  $7/3$ , which occurs in some settings for interacting fermionic [24] and bosonic [25] gases. In contrast, for usual gap solitons,  $\mu$  is a growing, rather than decreasing, function of  $N$  [10], although a combination of linear and nonlinear OLs may change this [26].

When  $N$  is relatively small, the numerically found GS is loosely bound, featuring slowly decaying spatial undulations, as shown in Fig. 2 for  $N = 10^5$  and  $2 \times 10^5$ . In contrast, for larger  $N$ , such as  $3 \times 10^5$  and  $6 \times 10^5$ , which correspond to the nearly linear  $\mu(N)$  curve in Fig. 1, the GS is found in the form of stable tightly bound solitons, see Fig. 3. The range of values of  $N$  considered here is realistic, as BEC can be readily created with  $N$  up to  $\sim 10^8$  [27]. A closer inspection of the profiles of the loosely and tightly bound modes demonstrates that, although undulating, they never cross zero (unlike gap solitons in the model with the fixed OL), which corroborates that they are GSs. The stability of the modes has been verified by simulations of

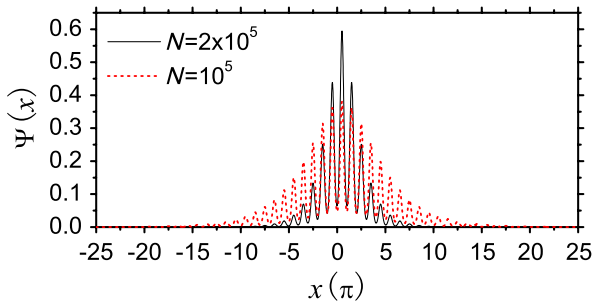


FIG. 2 (color online). Wave functions of loosely bound ground states for different numbers of atoms.

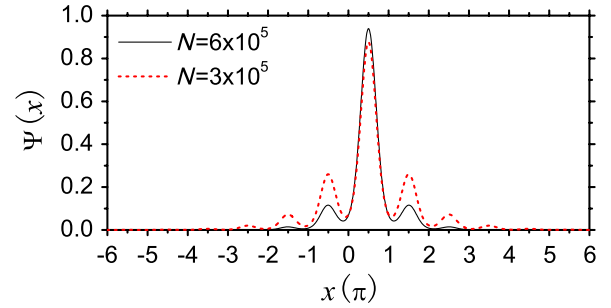


FIG. 3 (color online). The ground-state wave functions for larger  $N$  than in Fig. 2, in the form of tightly bound solitons.

their evolution within the framework of Eqs. (3) and (4). It is relevant to note that the gradual transition from the loosely to tightly bound GS is similar to the transformation of solitons reported in Ref. [28], where the effective interaction between bosons was modified by admixing fermions to the system.

Further simulations of Eqs. (3) and (4) demonstrate that the photon-atomic solitons are immobile (they absorb a suddenly applied kick), which is explained by the fact that they are pinned to the underlying OL. In that sense, they are similar to discrete solitons pinned to the underlying lattice [29].

The deformation of the envelope of the right-traveling electromagnetic wave,  $\delta I^+$ , which is a photonic component of the GS, is plotted in Fig. 4. Local peaks of the intensity appear at  $x = 2n\pi + \pi/2$  (with integer  $n$ ), coinciding with local maxima of the density in the matter-wave component, cf. Figs. 2 and 3. Thus, the GS is indeed a coupled self-trapped photon-atomic mode (a “symbiotic” one, cf. this concept developed for solitons in other settings [30]).

To further illustrate the intrinsic structure of the solitons, the deviation of the effective LFE-affected potential from the periodic one induced by the unperturbed OL,  $\delta V(x)$ , is plotted in Figs. 5(a) and 5(b). The figures feature a self-sustained “valley” in the central region, superimposed on the nearly periodic potential. Note that modulated potentials, unlike strictly periodic ones, can maintain localized states, a well-known example being the Anderson

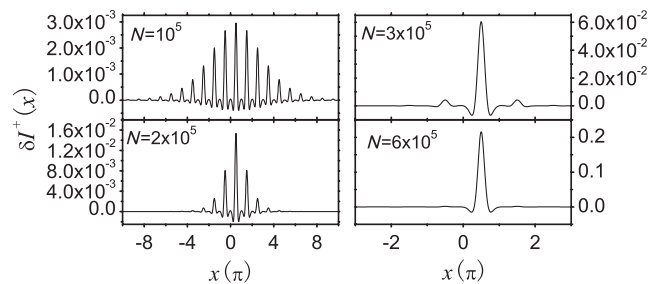


FIG. 4. The deformation of the envelope of the right-traveling electromagnetic wave for different values of  $N$ .

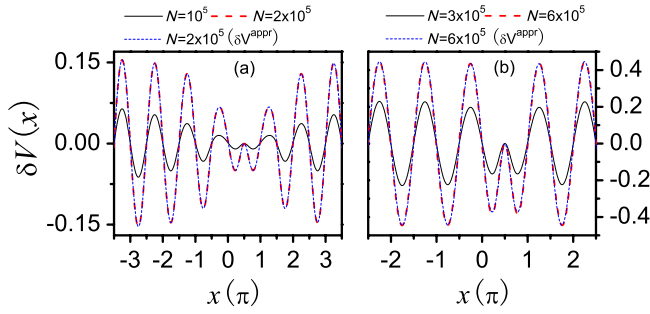


FIG. 5 (color online). The numerically found LFE-induced potential,  $\delta V$  for  $N = 10^5$  and  $2 \times 10^5$  (a), and  $N = 3 \times 10^5$  and  $6 \times 10^5$  (b). The approximate potential  $\delta V^{\text{appr}}$ , given by Eq. (7), is shown, for  $N = 2 \times 10^5$  and  $6 \times 10^5$ , by the dotted lines.

localization in quasiperiodic potentials [31]. This analogy helps to understand the trapping of the atomic wave function in the soliton. The valley becomes narrower with the increase of  $N$ , leading to the tighter bound modes.

To check how well the full numerically found distortion of the potential,  $\delta V(x)$ , is fitted by approximation  $\delta V^{\text{appr}}$  given by Eq. (7), we plot  $\delta V^{\text{appr}}(x)$  for  $N = 2 \times 10^5$  and  $6 \times 10^5$  in Fig. 5, which shows a negligible difference between  $\delta V$  and  $\delta V^{\text{appr}}$ . Further, the relation of the envelope-deformation, nonlocal-interaction, and self-interaction potential terms,  $\delta V^{\text{def}}$ ,  $\delta V^{\text{nonloc}}$ , and  $\delta V^{\text{SI}}$ , in the approximate potential to the atom number is presented in Fig. 6. When  $N$  is small,  $\delta V^{\text{nonloc}}$  is the dominant term. It features the above-mentioned central valley which supports the loosely bound GS wave function. With the increase of  $N$ , the shape of the potentials simplifies, the valley shrinks, and terms  $\delta V^{\text{def}}$  and  $\delta V^{\text{SI}}$ , featuring a trapping region, quickly grow. Being strongly localized, they maintain tightly bound GS wave functions, as seen in Fig. 3.

Additional analysis demonstrates that, quite naturally, the self-trapped modes do not exist if direct repulsive

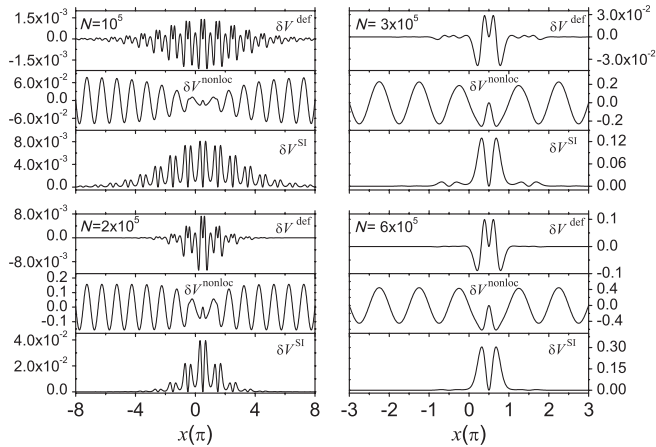


FIG. 6. The potential terms accounting for the envelope-deformation effect,  $\delta V^{\text{def}}$ , the effective nonlocality,  $\delta V^{\text{nonloc}}$ , and the light-induced self-interaction,  $\delta V^{\text{SI}}$ , are shown for different  $N$ .

interactions between atoms dominate over the LFE, and, on the other hand, the usual mechanism of the formation of 1D solitons [32] supplants the LFE if direct attractive interactions are strong enough. However, the Feshbach-resonance technique makes it possible to reduce the direct interactions [20,21], if necessary, and thus, allow the LFE-based mechanism to manifest itself.

In the case of the red detuning of the electromagnetic waves ( $\Delta < 0$ ), the same model does not produce solitons. In this case, the nonlinear potential term  $\delta V^{\text{SI}}$  and the linear OL potential are in competition (recall that the sign of  $\gamma \sim \Delta^{-2}$  is independent of the sign of  $\Delta$ , unlike  $V_0 \sim \Delta^{-1}$ ), which hampers the creation of matter wave solitons. Simultaneously, a “potential barrier,” rather than a trap, is generated for optical fields in the Helmholtz equation (2) by the condensate refraction index  $n \geq 1$ , prohibiting the creation of solitons.

In conclusion, we have studied the interaction of the collisionless BEC with counterpropagating optical waves, including the LFE, which deforms the potential of the “soft” OL. This effect leads to the deformational and nonlocal-interaction potentials, along with the self-repulsive interaction. The so induced nonlinearity, acting along with the linear OL, gives rise to stable photon-atomic solitons, which realize the ground state of the system, provided that the light is blue-detuned from the atomic transition. The increase of the number of atoms leads to the transition from loosely bound to tightly bound solitons. It may be interesting to extend the analysis by constructing bound complexes of such solitons, and generalizing the analysis to the 2D setting.

These findings add to recent results demonstrating the potential of the LFE for generating novel physical phenomena, such as nonclassical matter-wave and photonic states [33], and “photon bubbles” (which are of great interest to astrophysics), predicted through the interaction of diffuse light with BEC via the LFE in a magneto-optic trap [34]. Further, the deformation of the OL by the LFE is akin to “irregular” gratings observed in asymmetric superradiance of matter waves for blue- and red-detuned settings [35], while the conventional superradiance theory [36] assumes a regular OL. Thus, the theory may be extended to explore the super-radiance of matter waves. The LFE was also recently investigated in the framework of thermal molecular dynamics [4], suggesting the possibility of generating thermal photon-atomic solitons in that context.

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