Thermodynamics of "Exotic" Bañados-Teitelboim-Zanelli Black Holes

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A number of three-dimensional (3D) gravity models, such as 3D conformal gravity, admit "exotic" black hole solutions: the metric is the same as the Bañados-Teitelboim-Zanelli metric of 3D Einstein gravity but with reversed roles for mass and angular momentum, and an entropy proportional to the length of the *inner* horizon instead of the event horizon. Here we show that the Bañados-Teitelboim-Zanelli solutions of the exotic 3D Einstein gravity (with parity-odd action but Einstein field equations) are exotic black holes, and we investigate their thermodynamics. The first and second laws of black hole thermodynamics still apply, and the entropy still has a statistical interpretation.

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In three spacetime dimensions (3D) the Einstein gravitational field equations imply that the metric has a constant curvature and, hence, in the case of a negative cosmological constant, that it is locally isometric to anti–de Sitter (AdS) space. One solution is therefore the AdS vacuum, but there is also a two-parameter family of stationary black hole solutions, found by Bañados, Teitelboim, and Zanelli [1]; the Bañados-Teitelboim-Zanelli (BTZ) metric takes the form

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\phi + N^\phi dt)^2. \quad (1)$$

The functions N^2 and N^ϕ are

$$N^2 = -8Gm + \frac{r^2}{\ell^2} + \frac{16G^2j^2}{r^2}, \qquad N^\phi = -\frac{4Gj}{r^2}, \quad (2)$$

where ℓ is the AdS radius and G the 3D Newton constant. The parameters (m, j) can be interpreted as the mass M and angular momentum J of the black hole in units for which $\hbar = 1$; i.e., $M = m$ and $J = j$. The laws of black hole thermodynamics [2] apply, with the entropy given by the Bekenstein-Hawking formula [3,4], which states (in the 3D context) that the entropy is $1/4$ of the length of the event horizon in Planck units. There is even a statistical mechanics interpretation of this entropy [5,6]; the microstates are those of a holographically dual conformal field theory (CFT) implicit in an earlier work of Brown and Henneaux [7], who showed that the central charges are

$$c_L = c_R = \frac{3\ell}{2G}. \quad (3)$$

The gravity description of the CFT is valid at large ℓ/G.

Because the BTZ metric is locally isometric to AdS_3, it solves the field equations of *any* 3D gravity model admitting an AdS_3 vacuum, and the mass and angular momentum may then be equal to some other linear combination of the parameters. Rather surprisingly, a number of 3D gravity models have been found [8–11] for which the roles of mass and angular momentum are *reversed*, in the sense that

$$M = j/\ell, \qquad J = \ell m. \quad (4)$$

Such BTZ black holes have been called "exotic." The latest addition [12] to the class of 3D gravity models for which BTZ black holes are exotic is 3D conformal gravity [13,14], for which the simplest action is the Lorentz-Chern-Simons term, introduced in [15] in the context of "topologically massive gravity." The central charges of the CFT holographically dual to generic 3D gravity models may be computed by a variety of methods [12,16], and for models admitting exotic BTZ black holes it is found, e.g. for conformal 3D gravity, that

$$c_R = -c_L. \quad (5)$$

This is what one might expect of a gravity model with a parity-odd action.

Like the Kerr solution of the 4D Einstein equations, the BTZ metric has two Killing horizons, located at $r = r_\pm$, where $r_\pm$ are the zeros of N^2:

$$r_\pm = \sqrt{2G\ell(\ell m + j)} \pm \sqrt{2G\ell(\ell m - j)}. \quad (6)$$

We may assume without loss of generality that $j \geq 0$. We shall also assume that $\ell m \geq j$, to ensure the existence of an event horizon at $r = r_+$. Note that the parameters (m, j) can be simply expressed in terms of the radii of the inner and outer horizons:

$$\ell m = \frac{r_+^2 + r_-^2}{8G\ell}, \qquad j = \frac{r_+ r_-}{4G\ell}. \quad (7)$$

The entropy of BTZ black holes has also been computed by various means, which include holographic methods [17] and Wald's Noether charge formula [18], or its extension to allow for parity violation [19]. Applied to the original BTZ black hole, these methods give a result in agreement with the Bekenstein-Hawking formula, but applied to exotic BTZ black holes they give a different result: the entropy is proportional to the length $2\pi r_-$ of the *inner* horizon.

This result calls into question the validity of the laws of black hole thermodynamics. For example, a perturbation of a "normal" BTZ black hole will *decrease* the length of its inner horizon [20,21]. This is not problematic in itself but it illustrates the point that Hawking's area theorem (length in 3D) need not apply to inner horizons, and as the entropy of an exotic BTZ black hole is proportional to the length of the inner horizon, it becomes unclear whether the second law of thermodynamics still applies. In an attempt to resolve some of these difficulties, Park proposed a new formula for the BTZ black hole entropy in the context of higher-derivative 3D gravity theories [22]. His proposal was supported by a computation of the statistical entropy but it conflicts with results obtained by other methods and implies various thermodynamic abnormalities, such as negative temperature.

Here we investigate these issues in the context of what must surely be the simplest 3D gravity model for which BTZ black holes are exotic. This is the *parity-odd* action for 3D Einstein gravity with negative cosmological constant that Witten called (coincidentally) exotic [23]. To contrast the normal and exotic versions of 3D Einstein gravity, we give here their respective Lagrangian 3-forms in terms of independent dreibein 1-forms e^a and Lorentz connection 1-forms ω^a ($a = 0, 1, 2$), and their torsion and curvature 2-form field strengths,

$$T^a = de^a + \epsilon^{abc}\omega_b e_c, \qquad R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c, \tag{8}$$

where the exterior product of forms is implicit. In the normal case the Lagrangian 3-form is

$$L = \frac{1}{8\pi G}\left[e_a R^a - \frac{1}{6\ell^2}\epsilon^{abc}e_a e_b e_c\right]. \tag{9}$$

This is the well-known Einstein-Cartan formulation, so we do not elaborate further. The Lagrangian 3-form for the exotic theory is

$$L_E = \frac{\ell}{8\pi G}\left[\omega_a\left(d\omega^a + \frac{2}{3}\epsilon^{abc}\omega_b\omega_c\right) - \frac{1}{\ell^2}e_a T^a\right]. \tag{10}$$

The first term is the Lorentz-Chern-Simons term (except that the connection is an independent one). Although the action obtained by integration of L_E does not preserve parity, its parity transform is minus itself (i.e., it is parity-odd) so the field equations do preserve parity. Varying with respect to e^a, we get the torsion constraint $T^a = 0$, which allows us to solve for ω^a; using this in the equation obtained by varying ω^a, we recover the same Einstein equations that follow from the standard parity-even Lagrangian 3-form (9).

An explanation of how the exotic action was found will also explain why its BTZ black holes are exotic. The 3D AdS isometry group is $SO(2,2) \cong Sl(2;\mathbb{R}) \times Sl(2;\mathbb{R})$. It was shown in [24] that the field equations of the $SO(2,2)$ Chern-Simons (CS) action are equivalent to the Einstein equations with negative cosmological constant. Actually, since one may consider any linear combination of the two CS terms for the $Sl(2;\mathbb{R})$ factors, for which the 1-form potentials are

$$A^a_\pm = e^a \pm \ell\omega^a, \tag{11}$$

the choice of $SO(2,2)$ Chern-Simons term is not unique, although there is a unique choice that yields a parity-preserving action. Because CS terms are intrinsically parity odd, and because the 1-forms A^a_+ and A^a_- are interchanged by parity, only the *difference* of the two $Sl(2;\mathbb{R})$ CS terms preserves parity, and for a suitable overall constant this combination yields the standard action of 3D general relativity. This observation was first made by Achúcarro [25] and later by Witten [23], who also wrote down an explicit form for the parity-odd action obtained by taking the sum of the two $Sl(2;\mathbb{R})$ CS terms. Written in terms of the 1-forms (e^a, ω^a) this exotic version has the Lagrangian 3-form of Eq. (10).

Let us now examine the implications for BTZ black holes. For each of the two $Sl(2;\mathbb{R})$ subgroups of $SO(2,2)$, we can define a conserved charge $Q_\pm$ as the holonomy of an asymptotic $U(1)$ connection [26]. Using Eq. (11) one then finds, in the normal case, that

$$\ell M = Q_+ + Q_-, \qquad J = Q_+ - Q_-, \qquad \text{(normal)} \tag{12}$$

so parity, which exchanges $Q_\pm$, leaves M unchanged but flips the sign of J, as expected. In contrast, the exotic case gives

$$\ell M = Q_+ - Q_-, \qquad J = Q_+ + Q_-, \qquad \text{(exotic)} \tag{13}$$

so the roles of ℓM and J are reversed. For the BTZ solution of normal 3D gravity we have $M = m$ and $J = j$, so we will find that $\ell M = j$ and $J = \ell m$ when we view it as a solution of the exotic theory. In other words, BTZ black holes are normal as solutions of normal 3D Einstein gravity, and exotic as solutions of exotic 3D Einstein gravity.

We now turn to black hole thermodynamics. The Hawking temperature T of the BTZ black hole and the angular velocity Ω of its event horizon are

$$T = \frac{r_+^2 - r_-^2}{2\pi r_+\ell^2}, \qquad \Omega = \frac{r_-}{\ell r_+}. \tag{14}$$

These expressions for the intensive thermodynamic variables are geometrical in the sense that they depend only on the location of the Killing horizons and are model-independent, i.e., independent of the particular field equations that are solved by the BTZ metric.

In contrast, the extensive thermodynamic variables are model dependent. For generality, we shall consider the case in which the mass and angular momentum are given by

$$M = \alpha m + \gamma j/\ell, \qquad J = \alpha j + \gamma \ell m, \tag{15}$$

for constants (α, γ). The cases $(\alpha, \gamma) = (1, 0)$ and $(\alpha, \gamma) = (0, 1)$ correspond, respectively, to the normal and exotic BTZ black holes. In the normal case the second of the relations (7) tells us that the product of the lengths of the inner and outer horizons is independent of M, as expected on general grounds [27–30], but in the exotic case it is independent of J.

Let us now write the first law of black hole thermodynamics as

$$dM - \Omega dJ = TdS. \tag{16}$$

We claim that this law is satisfied for M and J given by Eq. (15) if, and only if, the entropy S is given by

$$S = \frac{\pi}{2G}(\alpha r_+ + \gamma r_-). \tag{17}$$

In fact, given only M as a linear combination of m and j, the expressions for both J and S can be fixed by requiring the validity of the first law. This can be verified by computing dS in terms of dM and dJ using Eqs. (6) and (14). When $(\alpha, \gamma) = (1, 0)$ the entropy reduces to the usual Bekenstein-Hawking formula, but when $(\alpha, \gamma) = (0, 1)$ it is given by the exotic formula $S = S_E \equiv \pi r_-/(2G)$; i.e., it is now $1/4$ the length, in Planck units, of the *inner* horizon.

Next we show that the entropy, as given by Eq. (17), obeys the second law, at least for the quasistationary process of infalling matter. Let p be the 3-momentum of a particle falling through the event horizon. Since the event horizon is a Killing horizon for the Killing vector

$$\xi = \partial_t + \Omega \partial_\phi, \tag{18}$$

we have $-p \cdot \xi \geq 0$ at the horizon. This implies that the changes dM and dJ in the mass and angular momentum of the black hole satisfy the inequality $dM \geq \Omega dJ$, and, hence, that

$$\ell r_+ dM \geq r_- dJ. \tag{19}$$

Using the expressions (15) for the mass and angular momentum, and the formulas (6) for $r_\pm$, we find that

$$dS \geq 0. \tag{20}$$

This is true for any constants (α, γ); in particular, it is true for the exotic case $(\alpha, \gamma) = (0, 1)$, even though the entropy is then proportional to the length of the inner horizon.

It should be noted that, in arriving at these results, we have used the angular momentum and temperature associated to the *event* horizon, not to the inner horizon. We are *not* discussing here the "thermodynamics of inner horizons."

Finally, we consider the statistical entropy of exotic BTZ black holes. In the high-temperature $(\beta \to 0)$ limit, the partition function $Z(\beta) = \mathrm{tr} e^{-\beta H}$ of a CFT with Hamiltonian H can be approximated by the integral

$$Z(\beta) = \int_0^\infty d\Delta \rho(\Delta) e^{-\beta\Delta}, \tag{21}$$

where Δ is the eigenvalue of the Virasoro generator L_0, and $\rho(\Delta) = e^{S(\Delta)}$ is the (smoothed) density of states, equal to the exponential of the entropy function $S(\Delta)$. Using the fact that $S \propto \sqrt{\Delta}$, the partition function can be evaluated in a saddle-point approximation. Then, using modular invariance, the result can be related to the low temperature $(\beta \to \infty)$ limit, which is dominated by the ground state, with energy determined by the central charge. In this way Cardy found a large-Δ approximation to $\rho(\Delta)$, such that [31]

$$S(\Delta) \approx 2\pi\sqrt{c\Delta/6}, \tag{22}$$

where c is the central charge. Applying this formula to the left and right sectors of the Brown-Henneaux CFT with central charges $c_L = c_R = 3\ell/(2G)$, and using the relations

$$2\Delta_L = \ell M + J, \qquad 2\Delta_R = \ell M - J, \tag{23}$$

one finds that [5,6]

$$S_L = \pi\sqrt{\frac{\ell(\ell M + J)}{2G}}, \qquad S_R = \pi\sqrt{\frac{\ell(\ell M - J)}{2G}}. \tag{24}$$

The sum $S = S_L + S_R$ equals the Bekenstein-Hawking entropy $\pi r_+/(2G)$. We may also conclude from this result that the partition function of the Brown-Henneaux CFT takes the holomorphically factorized form $Z = Z_L Z_R$ within the limits of the leading-order saddle-point approximation. Whether this form holds exactly is not known; we defer to [32] for a discussion of this point.

In the exotic case we expect to have $c_R = -c_L$, as in the 3D conformal gravity case, because the action is parity odd. This means that we expect

$$c_L = \frac{3\ell}{2G}, \qquad c_R = -\frac{3\ell}{2G}. \tag{25}$$

We might also expect to have $2\Delta_L = \ell M_E + J_E$ and $2\Delta_R = J_E - \ell M_E$ because this is what one gets from Eq. (23) by setting $J = \ell M_E$ and $\ell M = J_E$. The weak cosmic censorship condition $J_E \geq \ell M_E$, needed for the existence of a horizon, then ensures that $\Delta_R > 0$. However, if this were true then we would have $c_R \Delta_R < 0$ and the Cardy formula would give an imaginary entropy.

Given that $c_R < 0$, the exotic version of the Brown-Henneaux CFT must be nonunitary, but this is only because we consider both the left and right sectors together. We may consider each sector in isolation because there is no interaction between them (at least within the approximation reviewed above). Taken in isolation, there is no significance to the sign of the central charge c_R because we can change its sign from positive to negative by declaring that physical states have negative norm instead of positive norm, so that the energy is now bounded from above rather than from below. This is just a change of conventions; only the sign of c_R relative to c_L is convention independent.

We suggest that $c_R < 0$ for exotic 3D gravity theories precisely because the conventions for the right movers are the reverse of the usual ones, in the sense just explained. This would flip the sign of Δ_R so that

$$2\Delta_L = \ell M_E + J_E, \qquad -2\Delta_R = J_E - \ell M_E. \quad (26)$$

Observe that the weak cosmic censorship condition $J_E - \ell M_E$ now ensures that $\Delta_R < 0$, and hence $c_R \Delta_R > 0$. Applying the Cardy formula (22), we find that

$$S_L = \pi\sqrt{\frac{\ell(J + \ell M)}{2G}}, \qquad S_R = \pi\sqrt{\frac{\ell(J - \ell M)}{2G}}. \quad (27)$$

Now we must ask how the partition function Z_R changes if we change the sign of the norm of all physical states so that $c_R \to -c_R$ and all energies change sign. We have seen that the laws of black hole thermodynamics continue to apply to exotic BTZ black holes, so we expect standard thermodynamic relations such as $E = -\partial \ln Z/\partial\beta$ to remain valid, but this relation is maintained when $E \to -E$ only if we also take $Z \to Z^{-1}$. We conclude from this that whereas $Z = Z_L Z_R$ in the normal case, we must have $Z = Z_L/Z_R$ in the exotic case, and this means that we should now *subtract* S_R from S_L to get the total exotic entropy:

$$S_E = \pi\sqrt{\frac{\ell(J + \ell M)}{2G}} - \pi\sqrt{\frac{\ell(J - \ell M)}{2G}} = \frac{\pi r_-}{2G}. \quad (28)$$

This is precisely the entropy required for the validity of the laws of black hole thermodynamics for exotic BTZ black holes. As far as we are aware, this is the first time that a statistical interpretation has been found for a black hole entropy that is *not* given by the Bekenstein-Hawking formula.

In this Letter, we have addressed puzzles that have arisen in the study of exotic BTZ black holes, for which the three-dimensional spacetime is that of the usual BTZ black hole but with a reversal of the roles of mass and angular momentum. We have shown that such black holes occur even for 3D Einstein gravity in the context of its parity-odd exotic action. The relationship of the normal to the exotic versions of 3D Einstein gravity explains the reversal of the roles of mass and angular momentum. It also goes a long way to explaining why the BTZ black holes of conformal 3D gravity are exotic: it is because the exotic Einstein gravity can be viewed as a truncation of conformal 3D gravity; from the Chern-Simons perspective one is just restricting to an $SO(2,2)$ subgroup of $SO(2,3)$.

Our investigation of the thermodynamics of BTZ black holes allowed for the mass and angular momentum to be given by arbitrary linear combinations of the parameters of the BTZ metric. In fact, given only the mass, the first law determines the entropy, which then satisfies the second law, at least for quasistatic perturbations, and this is a *general result* applicable to all 3D gravity models. For exotic black holes the entropy is equal to $1/4$ of the length of the *inner* horizon, and we have presented a statistical interpretation of this (non-Bekenstein-Hawking) result in terms of a dual CFT, using holomorphic factorization of the leading order partition function for 3D Einstein gravity. It is not known whether the exact partition function factorizes, but if it does not there will be nonholomorphic interactions that will almost certainly imply instabilities if $c_R = -c_L$, so holomorphic factorizability may be an essential requirement for any 3D quantum gravity for which the BTZ black holes are exotic.

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