Gravitoelectromagnetic Perturbations of Kerr-Newman Black Holes: Stability and Isospectrality in the Slow-Rotation Limit

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The most general stationary black-hole solution of Einstein-Maxwell theory in vacuum is the Kerr-Newman metric, specified by three parameters: mass M, spin J, and charge Q. Within classical general relativity, one of the most important and challenging open problems in black-hole perturbation theory is the study of gravitational and electromagnetic fields in the Kerr-Newman geometry, because of the indissoluble coupling of the perturbation functions. Here we circumvent this long-standing problem by working in the slow-rotation limit. We compute the quasinormal modes up to linear order in J for any value of Q and provide the first, fully consistent stability analysis of the Kerr-Newman metric. For scalar perturbations the quasinormal modes can be computed exactly, and we demonstrate that the method is accurate within 3% for spins $J/J_{\max} \lesssim 0.5$, where $J_{\max}$ is the maximum allowed spin for any value of Q. Quite remarkably, we find numerical evidence that the axial and polar sectors of the gravitoelectromagnetic perturbations are isospectral to linear order in the spin. The extension of our results to nonasymptotically flat space-times could be useful in the context of gauge-gravity dualities and string theory.

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Introduction.—In Einstein-Maxwell theory, black holes (BHs) that are stationary, asymptotically flat end states of gravitational collapse must be axisymmetric [1]. Classic uniqueness theorems reviewed in Ref. [2] show that regular, stationary electrovacuum BH space-times in four dimensions are described by the Kerr-Newman (KN) metric [3], characterized by mass M, angular momentum J, and electromagnetic charge Q. When $Q = 0$ the KN solution reduces to the Kerr metric, and for $J = 0$ it reduces to the Reissner-Nordström (RN) metric. When both Q and J are nonvanishing the space-time is endowed with an induced magnetic field, and its magnetic dipole moment corresponds to the same gyromagnetic ratio $g = 2$ as the electron [4]. This observation led to some speculation that the KN metric could be used as a classical model for elementary particles (see, e.g., Ref. [5]).

Charge is unlikely to play a significant role in astrophysics [6,7], but the KN metric is still a precious theoretical laboratory to investigate Einstein-Maxwell theory in curved space-time. For this reason the linearized dynamics of test fields on a KN background have been intensively studied in the past. The scalar [8], neutrino [9], massive spin-1/2 [10,11], and Rarita-Schwinger [12] equations in the KN metric can all be solved by separation of variables. The scattering of charged scalars and fermions in near-extremal KN space-times recently acquired special interest in the context of the KN/conformal field theory (KN/CFT) conjecture [13,14].

The KN space-time is one of the simplest prototypes of the interplay between matter and curvature summarized by Wheeler's famous statement that "matter tells space-time how to curve, and space-time tells matter how to move." Despite their importance, theoretical investigations of the interplay between gravitational and electromagnetic perturbations in the KN metric are still in their infancy. The reason is a major technical stumbling block: most methods to compute quasinormal modes (QNMs, see Refs. [15–18] for reviews), graybody factors, and scattering amplitudes rely on separability, and despite several attempts [19–21], at present no one has been able to separate the angular and radial dependence of the gravitoelectromagnetic eigenfunctions. The last chapter of Chandrasekhar's monumental 1983 monograph [22] is dedicated to an incomplete treatment of this problem. Quoting from Ref. [22]: "It does not appear that the methods developed [...] for the treatment of the gravitational perturbations of the Kerr [black hole] can be extended in any natural way to the treatment of the coupled electromagnetic and gravitational perturbations of the KN [black hole]. The origins of this apparently essential difference in the perturbed Kerr and KN space-time may lie deep in the indissoluble coupling of the spin-1 and spin-2 fields in the perturbed KN space-time—a coupling which it was possible to break only for very special reasons in the perturbed RN space-time".

This work is the first consistent analysis of gravitoelectromagnetic perturbations of KN BHs. We circumvent the decades-old coupling problem using a recent framework to study generic perturbations of spinning BHs in the slow-rotation limit [23,24], which is based on a similar approach used in the past to study slowly rotating compact stars [25–28]. We summarize here some of our most interesting findings. (i) We present the first self-consistent calculation of scalar, electromagnetic. and gravitational QNMs of the KN metric. (ii) Since none of these modes is unstable, our calculation provides solid evidence for the stability of the (nonextremal) KN metric. (iii) In the scalar case we can compare our results to an exact calculation that does not rely on the slow-rotation limit. We find that the perturbative analysis is valid when $J/J_{\max} \ll 1$ (where $J_{\max}$ is the maximum allowed spin for any given Q) and that scalar QNM frequencies are accurate within 3% for spins $J/J_{\max} \lesssim 0.5$, which suggests a similar level of accuracy for the gravitoelectromagnetic modes. (iv) Last but not least, we find the remarkable result that axial and polar QNMs (corresponding to perturbations that have odd or even parity, respectively) are isospectral to linear order in the spin.

Formalism.—The KN metric is the most general stationary electrovacuum solution of Einstein-Maxwell theory. Its full form in Boyer-Lindquist coordinates can be found, e.g., in Ref. [22]. Here and in the following we linearize all quantities in the spin parameter $\tilde{a} \equiv a/M \equiv J/M^2$ (in geometrical units $G = c = 1$), neglecting terms of order $\mathcal{O}(\tilde{a}^2)$. To this order, the KN metric reads

$$ds_0^2 = -Fdt^2 + F^{-1}dr^2 - 2\varpi \sin^2\vartheta d\varphi dt + r^2 d^2\Omega, \quad (1)$$

where $F(r) = (r - r_-)(r - r_+)/r^2$, $r_\pm = M \pm \sqrt{M^2 - Q^2}$ are the horizons of a RN BH, the gyromagnetic term is

$$\varpi(r) = 2\tilde{a}M^2/r - \tilde{a}Q^2M/r^2, \quad (2)$$

and the background electromagnetic potential is given by

$$A_\mu = \left(\frac{Q}{r}, 0, 0, -\frac{\tilde{a}QM}{r}\sin^2\vartheta\right). \quad (3)$$

Note that the presence of both rotation and charge ($\tilde{a}Q \neq 0$) induces a magnetic field in the (ϑ, φ) directions.

We derive the equations describing gravitoelectromagnetic oscillations in the slow-rotation approximation by linearizing the Einstein-Maxwell equations with respect to both the oscillation amplitude and the BH spin parameter $\tilde{a}$, and by expanding the perturbations in a complete basis of tensor spherical harmonics. As a consequence of using this basis in a nonspherical background, the linearized equations display mixing between perturbations with different harmonic indices and opposite parity [23–25,27]. However, the latter do not contribute to the QNM spectrum to first order in $\tilde{a}$ [24,26,29].

Our main analytical result consists of two sets of coupled, second-order equations (one for the axial and one for the polar sector, respectively) which fully describe gravitoelectromagnetic oscillations of a KN BH to first order in the spin. In the frequency domain, and assuming a time dependence $e^{-i\omega t}$, they read (schematically)

$$\begin{aligned}\hat{\mathcal{D}}Z_i^\pm &\equiv V_0^{(i,\pm)}Z_i^\pm + m\tilde{a}[V_1^{(i,\pm)}Z_i^\pm + V_2^{(i,\pm)}Z_i^{\pm\prime}] \\ &\quad + m\tilde{a}Q^2[W_1^{(i,\pm)}Z_j^\pm + W_2^{(i,\pm)}Z_j^{\pm\prime}], \end{aligned} \quad (4)$$

where $i, j = 1, 2, i \neq j$ (there is no sum over the indices i, j), we have defined the differential operator $\hat{\mathcal{D}} = \partial_{r_*}^2 + \omega^2 - F\ell(\ell+1)/r^2$, and r_* is the standard tortoise coordinate, such that $\partial_{r_*} r = F$. The functions Z_i^- and Z_i^+ are linear combinations of axial and polar variables, respectively, and they are also combinations of gravitational and electromagnetic perturbations.

The explicit form of the axial and polar potentials $V^{(i,\pm)}$ and $W^{(i,\pm)}$ is quite formidable. It will be presented in the accompanying paper [30] and in a publicly available MATHEMATICA notebook [31]. What matters is that Eq. (4) display the same symmetries as the master equations for a RN BH [22], and indeed they exactly reduce to the latter in the nonrotating case. In addition, Eq. (4) contain two first-order corrections in $\tilde{a}$. The first term is responsible for a Zeeman-like splitting of the eigenfrequencies, which breaks the degeneracy in the azimuthal index m. The second line in Eq. (4) is more interesting: this term couples the function Z_1^+ with the function Z_2^+, and the function Z_1^- with the function Z_2^-.

Once physically motivated boundary conditions are imposed, Eq. (4) defines an eigenvalue problem for the complex frequency $\omega = \omega_R + i\omega_I$. The boundary conditions for QNMs read simply

$$Z_j^\pm(r) \sim \begin{cases} e^{i\omega r_*}, & r \to \infty \\ e^{-i(\omega - m\Omega_H)r_*}, & r \to r_+ \end{cases}. \quad (5)$$

The near-horizon solution displays the typical frame-dragging effect occurring near spinning BHs, where

$$\Omega_H \sim \frac{\tilde{a}}{M(1 + \tilde{a}_{\max})^2} + \mathcal{O}(\tilde{a}^3) \quad (6)$$

is the angular velocity at the horizon of locally nonrotating observers, and $\tilde{a}_{\max} \equiv J_{\max}/M^2 = \sqrt{1 - (Q/M)^2}$ is the maximum spin parameter of a KN BH.

Numerical results.—The numerical solution of the axial and polar perturbation equations [Eq. (4)] is challenging, because their explicit form is very complicated [30,31]. Robust numerical methods to solve the coupled eigenvalue problem given by Eq. (4) with the boundary conditions [Eq. (5)] are reviewed in Ref. [32]. We have integrated the coupled system [Eq. (4)] and computed the corresponding eigenfrequencies using two independent methods: a highly efficient matrix-valued continued fraction technique and direct integration [30,32]. When both methods are applicable they validate each other, in the sense that the results agree within numerical accuracy.

For any given Q, our analysis allows us to extract the first-order corrections to the complex QNM frequencies:

$$\omega_{R,I} = \omega_{R,I}^{(0)} + \tilde{a} m \omega_{R,I}^{(1)} + \mathcal{O}(\tilde{a}^2), \tag{7}$$

where $\omega_{R,I}^{(i)}$ are functions of Q and of the multipolar index ℓ, whereas the m dependence has been factored out [24].

As a test of the slow-rotation approximation, we have computed the scalar QNMs of a KN BH to first order in $\tilde{a}$. These modes can be computed exactly in the Teukolsky formalism [33], so they give us the precious opportunity to estimate the errors introduced by the slow-rotation approximation. For any stationary and axisymmetric space-time, the scalar modes at first order in the angular momentum are governed by a master equation [24] whose corresponding eigenvalue problem can be solved with standard continued-fraction techniques [34,35].

In Fig. 1 we show the relative error of the slow-rotation approximation with respect to the "exact" result, computed by solving the scalar equation in a KN background via continued fractions [33]. In particular, the top (bottom) panels show the percentage deviation Δ for the real (imaginary) part of the fundamental $\ell = m = 1$ scalar mode at fixed values of Q. The slow-rotation approximation is accurate within one percent as long as $J/J_{\max} \lesssim 0.3$, and it is still accurate within 3% for $J/J_{\max} \lesssim 0.5$. Similar results also hold for other values of ℓ and m, and for the first few overtones [30]. Note the near-universal behavior of the percentage errors as functions of $J/J_{\max} = \tilde{a}/\tilde{a}_{\max}$ for all values of Q. Indeed the parameter $\tilde{a}_{\max}$, which appears explicitly in the QNM boundary conditions [Eq. (5)], plays a fundamental role in our perturbative scheme and the slow-rotation approximation is accurate only far from extremality, i.e., when $\tilde{a} \ll \tilde{a}_{\max} = \sqrt{1 - Q^2/M^2} < 1$.

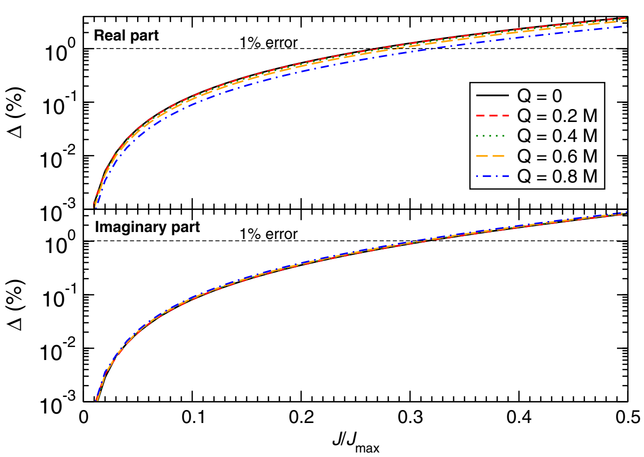

FIG. 1 (color online). Top panel: percentage error $\Delta \equiv 10^2|1 - \omega_{\rm slow}/\omega_{\rm exact}|$ induced by the slow-rotation approximation in the real part of the fundamental $\ell = m = 1$ scalar mode. Bottom panel: percentage error for the imaginary part of the same mode. The errors are only mildly sensitive to Q if plotted as functions of $J/J_{\max}$, where $J_{\max}$ is defined below Eq. (6). Similar results hold for other scalar modes [30].

Figure 2 shows our main numerical results for the fundamental gravitoelectromagnetic perturbations with $\ell = 2$, the most relevant for gravitational-wave emission (see, e.g., Ref. [36]). In each panel we show four curves, corresponding to the axial and polar "gravitational" and "electromagnetic" modes (as defined in the decoupled limit, $Q \to 0$). The zeroth-order terms shown in the left panels are simply RN QNMs; they agree with continued-fraction solutions [34] of the equations first derived by Zerilli [37].

We carried out a more extensive QNM calculation working in the axial case, where our results can be verified using two independent methods. By virtue of the isospectrality between axial and polar modes visible in Fig. 2 and discussed below, these results cover the whole QNM spectrum of slowly rotating KN BHs. We found that the zeroth- and first-order quantities shown in Fig. 2 (plus analogous calculations for $\ell > 2$ and the first overtones [30]) are well fitted by functions of the form

$$\omega_{R,I}^{(0,1)} = \sum_{k=0}^{4} f_k y^k. \tag{8}$$

Here we have defined a parameter $y = 1 - \tilde{a}_{\max}$, which is in one-to-one correspondence with Q/M, but is better suited for fitting. The coefficients f_i for the fundamental $\ell = 2$ gravitoelectromagnetic modes are listed in Table I.

Stability.—None of our numerical searches (for $0 < Q < M$, $J \ll J_{\max}$, and $\ell = 2, 3, 4$) returned exponentially growing modes. This confirms early arguments by Mashhoon in favor of the stability of the KN metric [38].

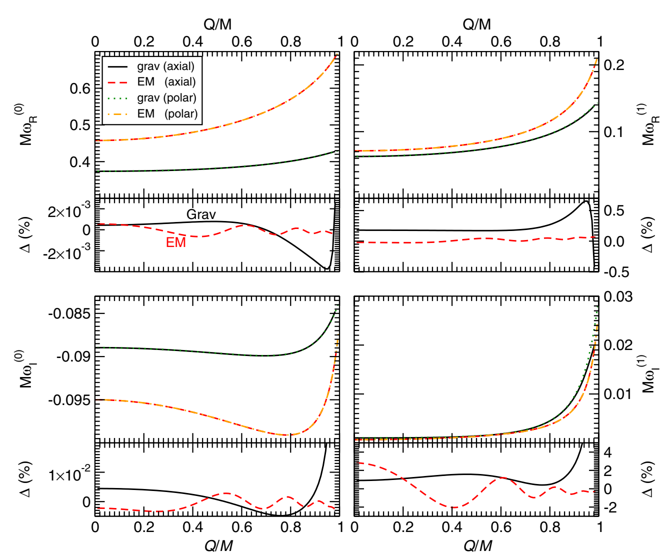

FIG. 2 (color online). Zeroth-order (left panels) and first-order (right panels) terms of the slow-rotation expansion of the KN QNM frequencies [cf. Eq. (7)]. All quantities are plotted as a function of Q/M, and they refer to the fundamental mode ($n = 0$) with $\ell = 2$. The lower part of each panel shows the percentage difference between axial and polar quantities: our results are consistent with isospectrality to $\mathcal{O}(0.1\%)$ for the real part and to $\mathcal{O}(1\%)$ for the imaginary part of these modes.

TABLE I. Coefficients of the fit [Eq. (8)] for the real and imaginary part of the fundamental ($n = 0$) gravitoelectromagnetic modes with $\ell = 2$. We denote by $s = 1$ and $s = 2$ the modes that in the decoupled limit $Q \to 0$ are electromagnetic and gravitational, respectively. The fits [Eq. (8)] reproduce the data to within 1% for $\omega_I^{(1)}$, and to within 0.1% for the other quantities, for any $Q \lesssim 0.95M$. Similar fits have comparable accuracy also for $\ell > 2$ and for the first overtone [30].

	(ℓ,n,s)	f_0	f_1	f_2	f_3	f_4
$\omega_R^{(0)}$	(2,0,1)	0.4576	0.2659	0.0118	0.1228	−0.1382
$\omega_R^{(1)}$	(2,0,1)	0.0712	0.0769	0.0596	0.0727	−0.0216
$\omega_I^{(0)}$	(2,0,1)	−0.0950	−0.0184	0.0137	0.0132	0.0107
$\omega_I^{(1)}$	(2,0,1)	0.0007	0.0043	0.0060	−0.0089	0.0366
$\omega_R^{(0)}$	(2,0,2)	0.3737	0.0525	0.0607	−0.0463	−0.0070
$\omega_R^{(1)}$	(2,0,2)	0.0628	0.0676	0.0209	0.0823	−0.0810
$\omega_I^{(0)}$	(2,0,2)	−0.0890	−0.0055	0.0024	0.0214	−0.0084
$\omega_I^{(1)}$	(2,0,2)	0.0010	0.0014	0.0091	0.0174	0.0145

Mashhoon's results apply only to the eikonal limit ($\ell \gg 1$) and they rely on a somewhat heuristic geodesic analogy, rather than on a self-consistent treatment of the perturbation equations. In this sense, our QNM calculations provide the first, fully consistent numerical evidence for the stability of the KN space-time.

Isospectrality.—Gravitoelectromagnetic perturbations of Schwarzschild and RN BHs in general relativity have a noteworthy property, first proved by Chandrasekhar [22]: even though the polar and axial sectors of the perturbations are described by completely different potentials, their QNM spectra are identical [17]. Mathematically, this happens because the polar and axial potentials can be written in terms of a single "superpotential" (cf. Secs. 26 and 43 of Ref. [22]). This property can be interpreted as being due to "supersymmetry," in the sense of nonrelativistic quantum mechanics [39–41].

Isospectrality is easily broken: e.g., it does not hold if the cosmological constant is nonzero [42–44], if the underlying theory is not general relativity [45,46], or in higher dimensions (cf. Appendix A of Ref. [17]). The left panels of Fig. 2 (which refer to the RN limit) show that polar and axial modes are isospectral within our numerical accuracy. Given the complex form of the polar equations, this is a nontrivial check of our numerical techniques.

A priori, there is no reason why such a remarkable and fragile symmetry should hold true also for rotating (KN) BHs. A tantalizing result of our numerical study is strong evidence that the axial and polar sectors of KN gravitoelectromagnetic perturbations are indeed isospectral to first order in the BH spin. The left panels of Fig. 2 show that the linear corrections $\omega_{R,I}^{(1)}$ are identical functions of Q for axial and polar modes within the numerical errors (which are dominated by uncertainties in the direct integration used to compute polar modes). Various arguments can be made to support the claim that the observed deviations from isospectrality are of a purely numerical nature. (1) Isospectrality is verified to a higher level of accuracy far from extremality: this is consistent with the fact that QNMs are more challenging to compute in the extremal limit. (2) The deviations from isospectrality shown in Fig. 2 are roughly constant or decreasing functions of Q (at least for $Q \lesssim 0.8M$) and they are affected by a small residual error even when $Q = 0$, where isospectrality must hold exactly. (3) The direct integration method is more accurate as ℓ grows and, correspondingly, the deviations between axial and polar modes decrease. (4) Finally, we verified that the error can be reduced by increasing the accuracy of the integrator.

Outlook.—It is tempting to conjecture that the isospectrality we found at linear order may in fact hold exactly, at all orders in rotation. In order to verify this hypothesis it will be crucial to include effects of second order in the spin—a formidable undertaking. At second order the causal structure of a spinning metric starts differing from the nonspinning case, and parity-mixing terms appear in the perturbation equations [24]. If isospectrality were to hold true also at second order, there would be no fundamental reason to believe that it should be broken at higher orders. However, let us stress that isospectrality is a highly nontrivial property even at linear order in rotation, in view of the mixing of gravitational and electromagnetic perturbations. Hopefully our work will stimulate further studies to verify whether isospectrality is an exact property of the KN space-time. Besides brute-force extensions of our work to higher orders in rotation, other possible means of studying this problem include numerical time evolutions (cf. Refs. [47–49]) or analytical work, perhaps along the lines of Ref. [41].

Other interesting extensions of this work concern asymptotically anti-de Sitter (AdS) space-times. Our approach can be easily applied to compute the QNMs of (slowly rotating) KN-AdS BHs. This would be useful in the context of the AdS/CFT correspondence [50], which predicts that these solutions are dual to thermal states of a CFT living in a rotating Einstein universe [51,52].

Such an extension would also be interesting in the context of supergravity. To clarify this point, let us recall that the QNMs of asymptotically flat, extremal RN BHs have a remarkable property: electromagnetic perturbations with angular index ℓ are isospectral with gravitational perturbations with index $\ell + 1$ [53]. This is related to the fact that, when embedded in $N = 2$, four-dimensional supergravity, these solutions preserve part of the supersymmetry. Using this property, it is possible to prove that the one-loop corrections to the BH entropy cancel [54]. In the case of rotating, asymptotically flat BHs this reasoning does not apply, since these solutions are not supersymmetric. However, certain KN-AdS BHs embedded in $N = 2$, four-dimensional supergravity preserve half of the supersymmetry [55,56]. Since these solutions can be slowly rotating, the techniques developed in this Letter could be used to extend the arguments of Ref. [54] to supersymmetric KN-AdS BHs.

Last but not least, it would be interesting to extend our calculation of KN QNMs in the context of the KN/CFT conjecture [13,14], which predicts that the QNMs of the near-horizon KN geometry correspond to the poles of the retarded Green's function of the dual chiral CFT [57].

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[1] S. Hawking, Commun. Math. Phys. **25**, 152 (1972).
[2] P. T. Chrusciel, J. L. Costa, and M. Heusler, Living Rev. Relativity **15**, 7 (2012); arXiv:1205.6112.
[3] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. (N.Y.) **6**, 918 (1965).
[4] B. Carter, Phys. Rev. **174**, 1559 (1968).
[5] C. L. Pekeris and K. Frankowski, Phys. Rev. A **39**, 518 (1989).
[6] G. Gibbons, Commun. Math. Phys. **44**, 245 (1975).
[7] R. Blandford and R. Znajek, Mon. Not. R. Astron. Soc. **179**, 433 (1977).
[8] N. Dadhich and Z. Y. Turakulov, Classical Quantum Gravity **19**, 2765 (2002); arXiv:gr-qc/0112031.
[9] W. Unruh, Phys. Rev. Lett. **31**, 1265 (1973).
[10] S. Chandrasekhar, Proc. R. Soc. A **349**, 571 (1976).
[11] D. N. Page, Phys. Rev. D **14**, 1509 (1976).
[12] G. F. Torres del Castillo and G. Silva-Ortigoza, Phys. Rev. D **42**, 4082 (1990).
[13] T. Hartman, K. Murata, T. Nishioka, and A. Strominger, J. High Energy Phys. 04 (2009) 019.
[14] T. Hartman, W. Song, and A. Strominger, J. High Energy Phys. 03 (2010) 118.
[15] K. D. Kokkotas and B. G. Schmidt, Living Rev. Relativity **2**, 2 (1999).
[16] H.-P. Nollert, Classical Quantum Gravity **16**, R159 (1999).
[17] E. Berti, V. Cardoso, and A. O. Starinets, Classical Quantum Gravity **26**, 163001 (2009); arXiv:0905.2975.
[18] R. Konoplya and A. Zhidenko, Rev. Mod. Phys. **83**, 793 (2011); arXiv:1102.4014.
[19] A. L. Dudley and J. D. Finley, Phys. Rev. Lett. **38**, 1505 (1977).
[20] A. L. Dudley and J. D. Finley, J. Math. Phys. (N.Y.) **20**, 311 (1979).
[21] V. Bellezza and V. Ferrari, J. Math. Phys. (N.Y.) **25**, 1985 (1984).
[22] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, 1983).
[23] P. Pani, V. Cardoso, L. Gualtieri, E. Berti, and A. Ishibashi, Phys. Rev. Lett. **109**, 131102 (2012); arXiv:1209.0465.
[24] P. Pani, V. Cardoso, L. Gualtieri, E. Berti, and A. Ishibashi, Phys. Rev. D **86**, 104017 (2012); arXiv:1209.0773.
[25] Y. Kojima, Phys. Rev. D **46**, 4289 (1992).
[26] Y. Kojima, Astrophys. J. **414**, 247 (1993).
[27] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. A **433**, 423 (1991).
[28] V. Ferrari, L. Gualtieri, and S. Marassi, Phys. Rev. D **76**, 104033 (2007).
[29] Y. Kojima, Prog. Theor. Phys. **90**, 977 (1993).
[30] P. Pani, E. Berti, and L. Gualtieri (to be published).
[31] http://blackholes.ist.utl.pt/?page=Files; http://www.phy.olemiss.edu/ berti/qnms.html.
[32] P. Pani, arXiv:1305.6759.
[33] E. Berti and K. D. Kokkotas, Phys. Rev. D **71**, 124008 (2005).
[34] E. W. Leaver, Phys. Rev. D **41**, 2986 (1990).
[35] E. Berti, AIP Conf. Proc. C **0405132**, 145 (2004); arXiv:gr-qc/0411025.
[36] E. Berti, V. Cardoso, and C. M. Will, Phys. Rev. D **73**, 064030 (2006).
[37] F. Zerilli, Phys. Rev. D **9**, 860 (1974).
[38] B. Mashhoon, Phys. Rev. D **31**, 290 (1985).
[39] E. Witten, Nucl. Phys. **B188**, 513 (1981).
[40] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
[41] P. T. Leung, A. Maassen van den Brink, W. M. Suen, C. W. Wong, and K. Young, J. Math. Phys. (N.Y.) **42**, 4802 (2001); J. Math. Phys. (N.Y.) **42**, 4802 (2001).
[42] F. Mellor and I. Moss, Phys. Rev. D **41**, 403 (1990).
[43] V. Cardoso and J. P. S. Lemos, Phys. Rev. D **64**, 084017 (2001).
[44] E. Berti and K. D. Kokkotas, Phys. Rev. D **67**, 064020 (2003).
[45] V. Cardoso and L. Gualtieri, Phys. Rev. D **80**, 064008 (2009).
[46] C. Molina, P. Pani, V. Cardoso, and L. Gualtieri, Phys. Rev. D **81**, 124021 (2010); arXiv:1004.4007.
[47] E. N. Dorband, E. Berti, P. Diener, E. Schnetter, and M. Tiglio, Phys. Rev. D **74**, 084028 (2006).
[48] H. Witek, V. Cardoso, A. Ishibashi, and U. Sperhake, Phys. Rev. D **87**, 043513 (2013); arXiv:1212.0551.
[49] S. R. Dolan, arXiv:1212.1477.
[50] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[51] M. M. Caldarelli, G. Cognola, and D. Klemm, Classical Quantum Gravity **17**, 399 (2000).
[52] S. W. Hawking and H. S. Reall, Phys. Rev. D **61**, 024014 (1999).
[53] H. Onozawa, T. Mishima, T. Okamura, and H. Ishihara, Phys. Rev. D **53**, 7033 (1996).
[54] R. Kallosh, J. Rahmfeld, and W. K. Wong, Phys. Rev. D **57**, 1063 (1998).
[55] V. A. Kostelecky and M. J. Perry, Phys. Lett. B **371**, 191 (1996).
[56] M. M. Caldarelli and D. Klemm, Nucl. Phys. **B545**, 434 (1999).
[57] B. Chen and C.-S. Chu, J. High Energy Phys. 05 (2010) 004.