B-Meson Decay Constants from Improved Lattice Nonrelativistic QCD with Physical u, d, s, and c Quarks

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We present the first lattice QCD calculation of the decay constants f_B and f_{B_s} with physical light quark masses. We use configurations generated by the MILC Collaboration including the effect of u, d, s, and c highly improved staggered quarks in the sea at three lattice spacings and with three u/d quark mass values going down to the physical value. We use improved nonrelativistic QCD (NRQCD) for the valence b quarks. Our results are $f_B = 0.186(4)$ GeV, $f_{B_s} = 0.224(4)$ GeV, $f_{B_s}/f_B = 1.205(7)$, and $M_{B_s} - M_B = 85(2)$ MeV, superseding earlier results with NRQCD b quarks. We discuss the implications of our results for the standard model rates for $B_{(s)} \to \mu^+\mu^-$ and $B \to \tau\nu$.

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Introduction.—The B and B_s decay constants are key hadronic parameters in the standard model (SM) rate for $B_{(s)} \to \mu^+\mu^-$ and B/B_s oscillations, with the B-meson decay constant also determining the rate for $B \to \tau\nu$. The combination of experiment and theory for these processes provides important constraints on Cabibbo-Kobayashi-Maskawa unitarity [1] and the search for new physics, but the strength of the constraints is typically limited by the errors on the hadronic parameters.

The decay constants can only be determined accurately from lattice QCD calculations. Several methods have been developed for this [2], with errors decreasing over the years as calculations have improved. Here we provide a step change in this process, giving the first results for f_B and f_{B_s} that include physical u/d quark masses, obviating the need for a chiral extrapolation. As a result of this and other improvements described below, we have significantly improved the accuracy on f_{B_s}/f_B over previous calculations. The implications of our result are discussed in the conclusions section of this Letter.

Lattice calculation.—We use eight ensembles of "second-generation" gluon field configurations recently generated by the MILC Collaboration [3,4], with $N_f = 2 + 1 + 1$ highly improved staggered quarks (HISQ) [5] in the sea. To control discretization effects, we use three lattice spacings ranging from 0.15 to 0.09 fm and light to strange mass ratios of $m_l/m_s \sim 0.2, 0.1, 0.037$. Details of the ensembles are shown in Table I. The lattice spacings of five of the ensembles were determined using the $\Upsilon(2S - 1S)$ splitting in Ref. [6] where details, including a discussion of the systematic errors, can be found. The lattice spacing values of the additional ensembles (sets 3, 6 and 8) are determined in the same way. The valence part of the calculation uses lattice nonrelativistic QCD (NRQCD) [7–9] for the b quarks; the action is described in detail in Ref. [6]. It includes a number of improvements over earlier calculations, in particular one-loop radiative corrections (beyond tadpole improvement) to most of the coefficients of the $\mathcal{O}(v_b^4)$ relativistic correction terms. This action has been shown to give excellent agreement with experiment in recent calculations of the bottomonium [6,10] and B-meson spectrum [11]. We are now building on previous calculations with the tree-level NRQCD action [12–14] to extend this to B-meson decay constants. The b quark mass

TABLE I. Details of the gauge ensembles used in this calculation. β is the gauge coupling and a_Υ is the lattice spacing as determined by the $\Upsilon(2S - 1S)$ splitting in Ref. [6], where the three errors are statistics, NRQCD systematics, and experiment. am_l, am_s, and am_c are the sea quark masses, $L \times T$ gives the spatial and temporal extent of the lattices and $n_{\rm cfg}$ is the number of configurations in each ensemble. The ensembles 1, 2, and 3 will be referred to as "very coarse"; 4, 5, and 6 as "coarse"; and 7 and 8 as "fine."

Set	β	a_Υ (fm)	am_l	am_s	am_c	$L \times T$	$n_{\rm cfg}$
1	5.8	0.1474(5)(14)(2)	0.013	0.065	0.838	16×48	1020
2	5.8	0.1463(3)(14)(2)	0.0064	0.064	0.828	24×48	1000
3	5.8	0.1450(3)(14)(2)	0.002 35	0.0647	0.831	32×48	1000
4	6.0	0.1219(2)(9)(2)	0.0102	0.0509	0.635	24×64	1052
5	6.0	0.1195(3)(9)(2)	0.005 07	0.0507	0.628	32×64	1000
6	6.0	0.1189(2)(9)(2)	0.001 84	0.0507	0.628	48×64	1000
7	6.3	0.0884(3)(5)(1)	0.0074	0.037	0.440	32×96	1008
8	6.3	0.0873(2)(5)(1)	0.0012	0.0363	0.432	64×96	621

TABLE II. Parameters used for the valence quarks. am_b is the bare b quark mass in lattice units, u_{0L} is the Landau link value used for tadpole improvement, and $am_l^{\rm val}$, $am_s^{\rm val}$ are the HISQ light and strange quark masses.

Set	am_b	u_{0L}	$am_l^{\rm val}$	$am_s^{\rm val}$
1	3.297	0.8195	0.013	0.0641
2	3.263	0.820 15	0.0064	0.0636
3	3.25	0.819 467	0.002 35	0.0628
4	2.66	0.834	0.010 44	0.0522
5	2.62	0.8349	0.005 07	0.0505
6	2.62	0.834 083	0.001 84	0.0507
7	1.91	0.8525	0.0074	0.0364
8	1.89	0.851 805	0.0012	0.0360

is tuned, giving the values in Table II, by fixing the spin-averaged kinetic mass with the Υ/η_b masses.

The HISQ valence light quark masses are taken to be equal to the sea mass except on set 4 where there is a slight discrepancy. The s quark is tuned using the η_s meson [$M_{\eta_s} = 0.6893(12)$ GeV [6]]. Values very close to the sea s masses are found, meaning that partial quenching effects will be small.

To improve the statistical precision of the correlators, we take $U(1)$ random noise sources for the valence quarks using the methods developed in Ref. [13]. Along with the point source required for the matrix element, we include Gaussian smearing functions for the b quark source with two different widths. We include 16 time sources with b quarks propagating both forward and backward in time on each configuration. We checked the statistical independence of the results using a blocked autocorrelation function [6]. Even on the finer physical point ensembles, the correlations are very small between adjacent configurations and the integrated autocorrelation time is consistent with one.

The decay constant is defined from $\langle 0|A_0|B_q\rangle_{\rm QCD} = M_{B_q} f_{B_q}$, but the quantity that we extract directly from the amplitude of our correlator fits is $\Phi_{B_q} = \sqrt{M_{B_q}} f_{B_q}$; we convert to f_{B_q} at the end. For NRQCD, the full QCD matrix element is constructed from effective theory currents arranged in powers of $1/m_b$. For A_0 we consider the following currents, made from heavy quark Ψ_Q and light quark fields Ψ_q:

$$J_0^{(0)} = \bar{\Psi}_q \gamma_5 \gamma_0 \Psi_Q, \tag{1}$$

$$J_0^{(1)} = \frac{-1}{2m_b} \bar{\Psi}_q \gamma_5 \gamma_0 \gamma \cdot \nabla \Psi_Q, \tag{2}$$

$$J_0^{(2)} = \frac{-1}{2m_b} \bar{\Psi}_q \gamma \cdot \overleftarrow{\nabla} \gamma_5 \gamma_0 \Psi_Q. \tag{3}$$

These currents are related to the full QCD current through $\mathcal{O}(\alpha_s, \alpha_s \Lambda_{\rm QCD}/m_b)$ by

TABLE III. Coefficients for the perturbative matching of the axial vector current [Eq. (4)]. $z_0 = \rho_0 - \zeta_{10}$, $z_1 = \rho_1 - z_0$, $z_2 = \rho_2$ from Ref. [15].

Set	z_0	z_1	z_2
1	0.024(2)	0.024(3)	−1.108(4)
2	0.022(2)	0.024(3)	−1.083(4)
3	0.022(1)	0.024(2)	−1.074(4)
4	0.006(2)	0.007(3)	−0.698(4)
5	0.001(2)	0.007(3)	−0.690(4)
6	0.001(2)	0.007(2)	−0.690(4)
7	−0.007(2)	−0.031(4)	−0.325(4)
8	−0.007(2)	−0.031(4)	−0.318(4)

$$\langle A_0 \rangle = (1 + \alpha_s z_0)(\langle J_0^{(0)} \rangle + (1 + \alpha_s z_1)\langle J_0^{(1)} \rangle + \alpha_s z_2 \langle J_0^{(2)} \rangle). \tag{4}$$

One-loop coefficients were calculated in Ref. [15]. Here we reorder the perturbation series to make the process of renormalization clearer. The z_i depend on am_b and are given in Table III for the range of masses needed here. We see that the one-loop renormalization of the tree-level current $J_0^{(0)} + J_0^{(1)}$ is tiny [16]. z_0 includes the effect of mixing between $J_0^{(0)}$ and $J_1^{(1)}$ at one loop. We evaluate the renormalization of Eq. (4) using α_s in the V scheme at scale $q = 2/a$. Values for α_s are obtained by running down from $\alpha_s^{\overline{\rm MS}}(M_Z) = 0.1184$ [21] and range from 0.285 to 0.314.

Results.—We fit heavy-light meson correlators with both $J_0^{(0)}$ and $J_0^{(1)}$ operators at the sink simultaneously using a multiexponential Bayesian fitting procedure [22]. The B and B_s are fit separately; priors used in the fit are described in Ref. [11]. The amplitudes and energies from the fits are given in Tables IV and V. $a^{3/2}\Phi_q^{(0)}$ is the matrix element of the leading current $J_0^{(0)}$ and $a^{3/2}\Phi_q^{(1)}$ is that of $J_0^{(1)}$ and $J_0^{(2)}$, whose matrix elements are equal at zero meson momentum. Notice that the statistical errors in Φ do not increase on the physical point lattices, because they have such large volumes.

TABLE IV. Raw lattice amplitudes for B_s and B from each ensemble; errors are from statistics and fitting only. $a^{3/2}\Phi_q^{(0)}$ and $a^{3/2}\Phi_q^{(1)}$ are the leading amplitude and $1/m_b$ correction.

Set	$a^{3/2}\Phi_s^{(0)}$	$a^{3/2}\Phi_s^{(1)}$	$a^{3/2}\Phi^{(0)}$	$a^{3/2}\Phi^{(1)}$
1	0.3720(10)	−0.0300(3)	0.3220(19)	−0.0260(3)
2	0.3644(6)	−0.0291(3)	0.3093(11)	−0.0257(8)
3	0.3621(16)	−0.0288(2)	0.2986(17)	−0.0237(4)
4	0.2733(4)	−0.0234(2)	0.2373(9)	−0.0197(4)
5	0.2679(3)	−0.0234(1)	0.2272(7)	−0.0197(3)
6	0.2653(2)	−0.0229(1)	0.2193(8)	−0.0194(3)
7	0.1747(3)	−0.0170(1)	0.1525(8)	−0.0146(6)
8	0.1694(3)	−0.0167(0)	0.1386(5)	−0.0136(1)

TABLE V. Raw lattice energies from each ensemble; errors are from statistics and fitting only. aM_π are the pion masses used in the chiral fits; $aE(B_s)$ and $aE(B)$ are the energies of the B_s and B meson. Results on sets 3, 6, and 8 are new; others are given in Ref. [11].

Set	aM_π	$aE(B_s)$	$aE(B)$
3	0.10171(4)	0.6067(7)	0.5439(12)
6	0.08154(2)	0.5158(1)	0.4649(6)
8	0.05718(1)	0.4025(2)	0.3638(5)

We take two approaches to the analysis. The first is to perform a simultaneous chiral fit to all our results for Φ, Φ_s, Φ_s/Φ, and $M_{B_s} - M_B$ using $SU(2)$ chiral perturbation theory. The second is to study only the physical u/d mass results as a function of lattice spacing.

For the chiral analysis we use the same formula and priors for $M_{B_s} - M_B$ as in Ref. [11]. Pion masses used in the fits are listed in Table V and the chiral logarithms $l(M_\pi^2)$ include the finite volume corrections computed in Ref. [23] which have negligible effect on the fit. For the decay constants the chiral formulas, including analytic terms up to M_π^2 and the leading logarithmic behavior, are (see, e.g., Ref. [24])

$$\Phi_s = \Phi_{s0}(1.0 + b_s M_\pi^2/\Lambda_\chi^2), \tag{5}$$

$$\Phi = \Phi_0\left(1.0 + b_l \frac{M_\pi^2}{\Lambda_\chi^2} + \frac{1+3g^2}{2\Lambda_\chi^2}\left(-\frac{3}{2} l(M_\pi^2)\right)\right). \tag{6}$$

The coefficients of the analytic terms b_s, b_l are given priors 0.0(1.0), g has prior 0.5(5), and Φ_0, Φ_{s0} have 0.5(5). To allow for discretization errors each fit formula is multiplied by $(1.0 + d_1(\Lambda a)^2 + d_2(\Lambda a)^4)$, with $\Lambda = 0.4$ GeV. We expect discretization effects to be very similar for Φ and Φ_s and so we take the d_i to be the same, but differing from the d_i used in the $M_{B_s} - M_B$ fit. Since all actions used here are accurate through a^2 at tree level, the prior on d_1 is taken to be 0.0(3) whereas d_2 is 0.0(1.0). The d_i are allowed to have mild m_b dependence as in Ref. [11]. The ratio Φ_s/Φ is allowed additional light quark mass dependent discretization errors that could arise, for example, from staggered taste splittings. For comparison, we have fit the results using $SU(2)$ heavy meson staggered chiral perturbation theory [17,25] which changes the results by less than 1 sigma. We have tested that the fit is stable with respect to changes to the priors for g, b_l, b_s, d_i and adding or removing discretization corrections.

The results of the decay constant chiral fits are plotted in Figs. 1 and 2. Extrapolating to the physical point appropriate to $m_l = (m_u + m_d)/2$ in the absence of electromagnetism, i.e., $M_\pi = M_{\pi^0}$, we find $\Phi_{B_s} = 0.519(10)$ GeV$^{3/2}$, $\Phi_B = 0.427(9)$ GeV$^{3/2}$, $\Phi_{B_s}/\Phi_B = 1.215(7)$. For $M_{B_s} - M_B$ we obtain 86(1) MeV, in agreement with the result of Ref. [11].

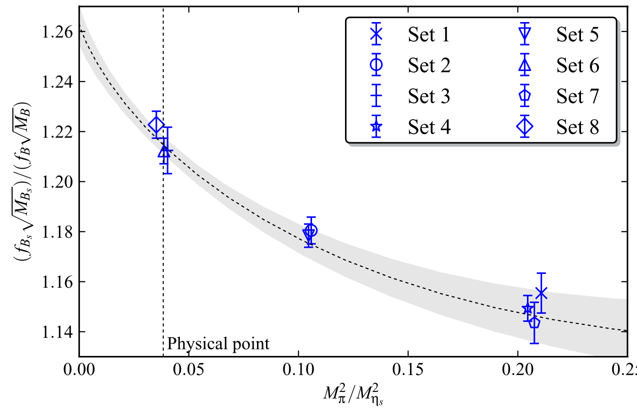

FIG. 1 (color online). Fit to the decay constant ratio Φ_{B_s}/Φ_B. The fit result is shown in gray and errors include statistics, and chiral or continuum fitting.

Figures 3 and 4 show the results of fitting $M_{B_s} - M_B$ and the decay constants from the physical point ensembles only, and allowing only the mass dependent discretization terms above. The results are $\Phi_{B_s} = 0.521(8)$ GeV$^{3/2}$, $\Phi_B = 0.428(7)$ GeV$^{3/2}$, $\Phi_{B_s}/\Phi_B = 1.216(7)$, and $M_{B_s} - M_B = 87(1)$ MeV. Results and errors agree well between the two methods and we take the central values from the chiral fit as this allows us to interpolate to the correct pion mass.

Our error budget is given in Table VI. The errors that are estimated directly from the chiral and continuum fit are those from statistics, the lattice spacing and g, and other chiral fit parameters. The two remaining sources of error in the decay constant are missing higher order corrections in the operator matching and relativistic corrections to the current. We estimate the operator matching error by allowing in our fits for an am_b-dependent α_s^2 correction to the renormalization in Eq. (4) with prior on the coefficient of 0.0(2), i.e., ten times the size of the one-loop correction z_0. This error cancels in the ratio f_{B_s}/f_B. We also allow for α_s^2

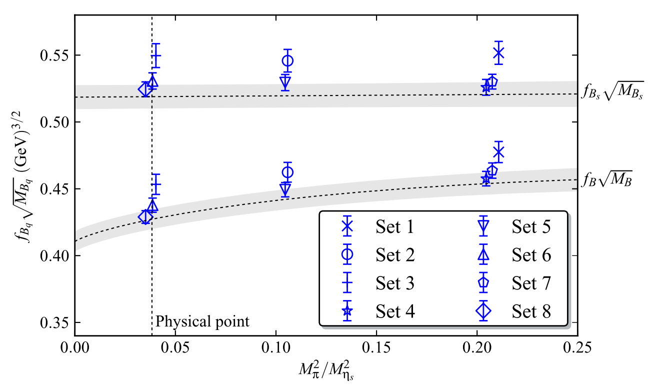

FIG. 2 (color online). Fit to the decay constants Φ_{B_s} and Φ_B. Errors on the data points include statistics and scale only. The fit error, in gray, includes chiral or continuum fitting and perturbative errors.

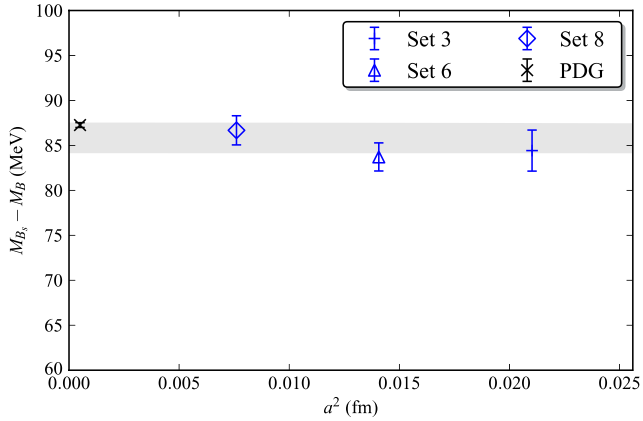

FIG. 3 (color online). Fit to the mass difference $M_{B_s} - M_B$ on the three physical point ensembles only. Errors on data points include statistics and scale; the fit error is shown in gray. An electromagnetic correction of $-1(1)$ MeV has been applied to the lattice results and the fit to allow comparison with experiment.

corrections multiplying $J_0^{(1,2)}$ with coefficient 0.0(1.0). The matrix element of $J_0^{(1)}$ is about 10% of $J_0^{(0)}$ from Table IV. Missing current corrections at the next order in $1/m_b$ will be of size $(\Lambda_{QCD}/m_b)^2 \simeq 0.01$ which we take as an error. Finally, we estimated in Ref. [11] that to correct for missing electromagnetic effects, $M_{B_s} - M_B$ should be shifted by $-1(1)$ MeV.

Using the Particle Data Group masses $M_{B_l} = (M_{B^0} + M_{B^\pm})/2 = 5.27942(12)$ GeV and $M_{B_s} = 5.36668(24)$ GeV [26] to convert Φ_q to f_{B_q} our final results are

$$f_B = 0.186(4)\ \text{GeV}, \qquad f_{B_s} = 0.224(4)\ \text{GeV},$$
$$f_{B_s}/f_B = 1.205(7), \qquad M_{B_s} - M_B = 85(2)\ \text{MeV}. \tag{7}$$

For the B-meson decay constant we need to distinguish between f_{B_d} and f_{B_u}. Since sea quark mass effects are much smaller than valence mass effects we simply do

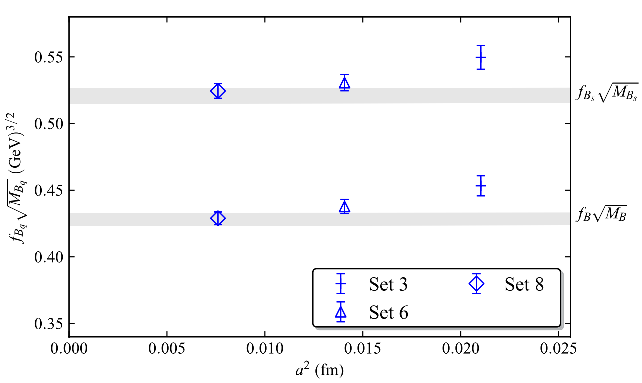

FIG. 4 (color online). Fit to the decay constants Φ_{B_s} and Φ_B on the three physical point ensembles only. Errors on the data points include statistics and scale only. The fit error includes chiral and continuum fitting and perturbative errors.

this by extrapolating Φ_{B_s} and Φ_B to values of M_π^2 corresponding to fictitious mesons made purely of u or d quarks using $m_u/m_d = 0.48(10)$ [26]. This gives

$$f_{B_s}/f_{B^+} = 1.217(8), \qquad f_{B_s}/f_{B^0} = 1.194(7),$$
$$f_{B^+} = 0.184(4)\ \text{GeV}, \qquad f_{B^0} = 0.188(4)\ \text{GeV}. \tag{8}$$

Conclusions.—Our results agree with but improve substantially on two earlier results using nonrelativistic approaches for the b quark and multiple lattice spacing values on $N_f = 2 + 1$ ensembles using asqtad sea quarks. These were $f_{B_s} = 228(10)$ MeV, $f_{B_s}/f_B = 1.188(18)$ (NRQCD and HISQ) [14], and $f_{B_s} = 242.0(9.5)$ MeV and $f_{B_s}/f_{B^+} = 1.229(26)$ (Fermilab/asqtad) [17]. We also agree well (within the 2% errors) with a previous result for f_{B_s} of 225(4) MeV obtained using a relativistic (HISQ) approach to b quarks on very fine $N_f = 2 + 1$ lattices [27]. Our simultaneous determination of $M_{B_s} - M_B$ to 2% agrees with experiment [87.4(3) MeV [26]].

We can determine new lattice "world-average" error-weighted values by combining our results in Eq. (7) with the independent results of Refs. [17,27] since effects from c sea quarks, which they do not include, should be negligible [28]. The world averages are then $f_{B_s} = 225(3)$ MeV and $f_{B_s}/f_{B^+} = 1.218(8)$ giving $f_{B^+} = 185(3)$ MeV.

These allow for significant improvements in predictions for SM rates. For example, updating Ref. [29] with the world average for f_{B_s} above and our result for f_{B^0} [Eq. (8)] we obtain

$$\text{Br}(B_s \to \mu^+\mu^-) = 3.17 \pm 0.15 \pm 0.09 \times 10^{-9},$$
$$\text{Br}(B_d \to \mu^+\mu^-) = 1.05 \pm 0.05 \pm 0.05 \times 10^{-10}, \tag{9}$$

where the second error from f_{B_q} has been halved and is no longer larger than other sources of error such as $V_{tb}^* V_{tq}$. Note that this is the flavor-averaged branching fraction at $t = 0$; the time-integrated result would be increased by 10% in the B_s case [to $3.47(19) \times 10^{-9}$] to allow for the width difference of the two eigenstates [30,31]. The current experimental results [32] for $B_s \to \mu^+\mu^-$ agree with this prediction.

TABLE VI. Full error budget from the chiral fit as a percentage of the final answer.

Error %	Φ_{B_s}/Φ_B	$M_{B_s} - M_B$	Φ_{B_s}	Φ_B
Electromagnetism	0.0	1.2	0.0	0.0
a dependence	0.01	0.9	0.9	0.9
Chiral	0.01	0.2	0.04	0.04
g	0.01	0.1	0.0	0.01
Statistics and scale	0.30	1.2	0.7	0.7
Operator	0.0	0.0	1.3	1.3
Relativistic	0.5	0.5	1.0	1.0
Total	0.6	2.0	2.0	2.0

From the world-average f_{B^+} above we also obtain the standard model rate

$$\frac{1}{|V_{ub}|^2}\,\mathrm{Br}(B^+ \to \tau\nu) = 6.05(20), \qquad (10)$$

with 3% accuracy. Calculations of matrix elements for B_s/B mixing with physical u/d quarks are now underway.

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