# Exact Distributions of the Number of Distinct and Common Sites Visited by $N$ Independent Random Walkers

Anupam Kundu, Satya N. Majumdar, and Grégory Schehr
*Laboratoire de Physique Théorique et Modèles Statistiques (UMR 8626 du CNRS), Université Paris-Sud, Bâtiment 100, 91405 Orsay Cedex, France*

(Received 11 February 2013; published 29 May 2013)

We study the number of distinct sites $S_N(t)$ and common sites $W_N(t)$ visited by $N$ independent one dimensional random walkers, all starting at the origin, after $t$ time steps. We show that these two random variables can be mapped onto extreme value quantities associated with $N$ independent random walkers. Using this mapping, we compute exactly their probability distributions $P_N^d(S,t)$ and $P_N^c(W,t)$ for any value of $N$ in the limit of large time $t$, where the random walkers can be described by Brownian motions. In the large $N$ limit one finds that $S_N(t)/\sqrt{t} \propto 2\sqrt{\log N} + \tilde{s}/(2\sqrt{\log N})$ and $W_N(t)/\sqrt{t} \propto \tilde{w}/N$ where $\tilde{s}$ and $\tilde{w}$ are random variables whose probability density functions are computed exactly and are found to be nontrivial. We verify our results through direct numerical simulations.

DOI: 10.1103/PhysRevLett.110.220602 PACS numbers: 05.40.Fb, 02.50.−r, 05.40.Jc

In elementary set theory, two fundamental concepts are the union and the intersection of a number of $N$ sets. While the union consists of all distinct elements of the collection of sets, the intersection consists of common elements of all the sets. These two notions appear naturally in everyday life: for example the area of common knowledge or the whole range of different interests amongst the members of a society would define respectively its stability and activity. In an habitat of $N$ animals, the union of the territories covered by different animals sets the geographical range of the habitat, while the intersection refers to the common area (e.g., a water body) frequented by all animals. In statistical physics, these two objects are modeled respectively by the number of distinct and common sites visited by $N$ random walkers (RWs). The knowledge about the number of distinct sites has applications ranging from the annealing of defects in crystals [1,2] and relaxation processes [3–6] to the spread of populations in ecology [7,8] or to the dynamics of web annotation systems [9].

Dvoretzky and Erdös [10] first studied the average number of distinct sites $\langle S_1(t)\rangle$ visited by a single $t$-step RW in $d$ dimensions, subsequently studied in [11–13]. Larralde *et al.* generalized this to $N$ independent $t$-step walkers moving on a $d$-dimensional lattice [14]. They found three regimes of growth (early, intermediate, and late) for the average number of distinct sites $\langle S_N(t)\rangle$ as a function of time. These three regimes are separated by two $N$-dependent times scales [14]. In particular they showed that in $d = 1$ and $t \gg \sqrt{\log N}$, $\langle S_N(t)\rangle \propto \sqrt{4Dt\log N}$, where $D$ is the diffusion constant of a single walker. Recently Majumdar and Tamm [15] studied the complementary quantity, the number of common sites $W_N(t)$ visited by $N$ walkers, each of $t$ steps, and found analytically a rich asymptotic late time growth of the average $\langle W_N(t)\rangle$. In the $(N-d)$ plane they found three distinct phases separated by two critical lines $d = 2$ and $d_c(N) = 2N/(N-1)$, with $\langle W_N(t)\rangle \sim t^\nu$ at late times where $\nu = d/2$ (for $d < 2$), $\nu = N - d(N-1)/2$ [for $2 < d < d_c(N)$] and $\nu = 0$ [for $d > d_c(N)$] (see also [16]). In particular, in $d = 1$, $\langle W_N(t)\rangle \sim \sqrt{4Dt}$ with a $N$-dependent prefactor. However, most of these studies were limited to the average number of distinct or common sites, and there exists virtually no information about their full probability distributions, e.g., the probabilities $P_N^d(S,t)$ that $S_N(t) = S$ and $P_N^c(W,t)$ that $W_N(t) = W$.

In high dimensions, $d > 2$, the trajectories of the $N$ walkers hardly overlap, which renders the study of these distributions much simpler. For instance, from the central limit theorem, $P_N^d(S,t)$ converges, for large $d$ and $N$, to a Gaussian distribution. In contrast, in one dimension ($1d$) where the overlap between the trajectories is maximal, one would expect significant deviations from the Gaussian distribution. In this Letter, we provide a complete analytical characterization of this non-Gaussian statistics by revealing an interesting connection to extreme value statistics (EVS), a subject of much current interest. In addition, the $1d$ case is relevant to biological applications, in particular to the process of finding location of specific DNA sequences by proteins [17], and also to environmental sciences, for example, river pollution [18], as discussed below.

Single molecule experiments have evidenced [19] that search processes in the cell combine $1d$ diffusion along DNA (“sliding”) and $3d$ excursions (“hopping” or “jumping”): this is called facilitated diffusion. The $1d$ diffusion along DNA has been reported for various types of proteins [19] which, in many cases, behave as a collection of independent RWs in $1d$: they can indeed bypass one another by stepping or jumping [20]. Hence, for instance, for DNA-repairing proteins, $S_N(t)$ is directly related to the size of the DNA sequence which has been scanned. In the case of river pollution the number of distinct sites will give the range of pollution whereas the number of common points measures the range of highly polluted region by different polluting agents or dye diffusing in the river [18]. The study of the number of distinct and common points in $1d$ case is the main subject of this Letter.

In this case, we show that $S_N(t)$ and $W_N(t)$ can be mapped onto extreme values (nearest and furthest displacements) associated with $N$ independent walkers. This connection to EVS allows us to compute $P_N^d(S,t)$ and $P_N^c(W,t)$ exactly for $t \gg 1$ and arbitrary $N$. Indeed, although the $N$ walkers are independent, conditioning their trajectories to a given number of distinct (or common) visited sites introduces strong effective correlations between them. We show that the induced correlations between the walkers persist even for $N \to \infty$ where the limiting distributions are not given by EVS of independent random variables, as erroneously argued in the previous studies of $S_N(t)$ [14].

We consider $N$ independent and identical $t$-step RWs $x_1(\tau), x_2(\tau), \ldots, x_N(\tau)$ on a $1d$ lattice, all starting at the origin, each with the same diffusion constant $D$. Distinct sites are those that are visited at least once by at least one of the $N$ walkers [14], while common sites correspond to sites visited individually at least once by all the $N$ walkers [15]. We denote by $M_i$ and $m_i$ the maximum and the minimum displacements of the $i$th walker $x_i$ up to time $t$. The number of distinct sites visited, $S_N$ [21], is then the sum of the range on the positive (+ ve) side, $M_+$, and the range on the negative (− ve) side $m_-$ (see Fig. 1),

$$S_N = M_+ + m_-, \qquad M_+ = \max_{1\le i\le N} M_i,$$
$$m_- = -\min_{1\le i\le N} m_i. \tag{1}$$

Similarly, the number of common sites visited, $W_N$, is the common span on the +ve axis plus the common span $m_+$ on the −ve axis,

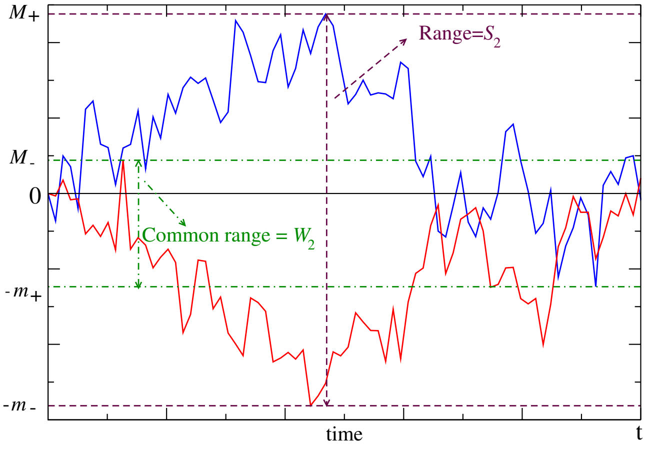

FIG. 1 (color online). Schematic diagram of two independent RWs, where $M_+$, $M_-$, $m_+$, $m_-$, and $S_2$, $W_2$ are shown.

$$W_N = M_- + m_+, \qquad M_- = \min_{1\le i\le N} M_i,$$
$$m_+ = -\max_{1\le i\le N} m_i. \tag{2}$$

Equations (1) and (2) establish a precise connection between $S_N$ and $W_N$ and the EVS of $N$ independent RWs.

Since, for large $t \gg 1$, the lattice RWs converge to Brownian motions (BMs), the probability distributions $P_N^d(S,t)$ and $P_N^c(W,t)$ take the scaling form

$$P_N^d(S,t) = \frac{1}{\sqrt{4Dt}} p_N^d\left(\frac{S}{\sqrt{4Dt}}\right), \quad P_N^c(W,t) = \frac{1}{\sqrt{4Dt}} p_N^c\left(\frac{W}{\sqrt{4Dt}}\right), \tag{3}$$

where $p_N^d(s)$ is the probability density function (PDF) of the span or range, $s = S/\sqrt{4Dt}$, and $p_N^c(w)$ is the PDF of the common span or common range, $w = W/\sqrt{4Dt}$, for $N$ independent BMs (see Fig. 1) on the unit time interval and diffusion constant $D = 1/4$. The rescaled quantities $S_N/\sqrt{4Dt}$ and $W_N/\sqrt{4Dt}$ in (3) are given by (1) and (2), where $M_\pm$, $m_\pm$ are replaced by their counterparts $\tilde{M}_\pm = M_\pm/\sqrt{4Dt}$ and $\tilde{m}_\pm = m_\pm/\sqrt{4Dt}$ corresponding to $N$ independent BMs on the unit time interval.

It is useful to summarize our main results. We obtain exactly, for any $N$, the PDFs $p_N^d(s)$ and $p_N^c(w)$ as presented in (12) and (15) along with (8) and (9). The moments can also be computed explicitly [22]. The tails of the PDFs can be derived explicitly,

$$p_N^d(s) \sim \begin{cases} a_N s^{-5} \exp[-N\pi^2/(4s^2)], & s \to 0, \\ b_N \exp(-s^2/2), & s \to \infty, \end{cases} \tag{4}$$

and

$$p_N^c(w) \sim \begin{cases} c_N w, & w \to 0 \\ d_N w^{1-N} \exp(-N w^2), & w \to \infty, \end{cases} \tag{5}$$

where $a_N$, $b_N$, $c_N$ and $d_N$ are computable constants (see below). For $N \to \infty$, one finds that both PDFs approach a nontrivial limiting form,

$$p_N^d(s) \sim 2\sqrt{\log N}\,\mathcal{D}(2\sqrt{\log N}(s - 2\sqrt{\log N})),$$
$$\mathcal{D}(\tilde{s}) = 2e^{-\tilde{s}} K_0(2e^{-\tilde{s}/2}), \tag{6}$$

where $K_n(x)$ denote the modified Bessel functions, and

$$p_N^c(w) = N\mathcal{C}(Nw), \qquad \mathcal{C}(\tilde{w}) = \frac{4}{\pi}\tilde{w}e^{-\frac{2}{\sqrt{\pi}}\tilde{w}}, \qquad \tilde{w} > 0. \tag{7}$$

Note that $\mathcal{D}(\tilde{s})$ (6) is not the Gumbel distribution, as it was initially argued in [14]. Remarkably the same distribution $\mathcal{D}(\tilde{s})$ also appears as the limiting distribution of the maximum of a large collection of logarithmically correlated random variables on a circle [23–25]. We check indeed $\int_{-\infty}^{\tilde{s}} \mathcal{D}(\tilde{s}')d\tilde{s}' = 2e^{-\tilde{s}/2}K_1(2e^{-\tilde{s}/2})$, as in [23]. As a byproduct of our computation, we show that $\mathcal{D}(\tilde{s})$ is the convolution of two independent Gumbel distributions.

We start by computing the joint cumulative distribution functions (JCDF) $\mathbf{P}_d(l_1, l_2) = \mathrm{Pr.}(\tilde{M}_+ \leq l_1, \tilde{m}_- \leq l_2)$, relevant for $p_N^d(s)$ and the JCDF $\mathbf{P}_c(j_1, j_2) = \mathrm{Pr.}(\tilde{M}_- \geq j_1, \tilde{m}_+ \geq j_2)$ relevant for $p_N^c(w)$. We remind that the diffusion constant is set to $D = 1/4$. Since the $N$ BMs are identical and independent, $\mathbf{P}_d(l_1, l_2) = g^N(l_1, l_2)$, where $g(l_1, l_2) = \mathrm{Pr.}(\tilde{M} \leq l_1, \tilde{m} \geq -l_2)$ is the JCDF of the maximum $\tilde{M}$ and the minimum $\tilde{m}$ for a single BM on the unit time interval. It can be computed by the standard method of images [26],

$$g(l_1, l_2) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}} \sin\left(\frac{(2n+1)\pi l_2}{l_1 + l_2}\right) e^{-\left(\frac{(2n+1)\pi}{2(l_1+l_2)}\right)^2}. \tag{8}$$

Similarly, $\mathbf{P}_c(j_1, j_2) = h^N(j_1, j_2)$ where $h(j_1, j_2) = \mathrm{Pr.}(\tilde{M} \geq j_1, \tilde{m} \leq -j_2)$ reads

$$h(j_1, j_2) = 1 - \mathrm{erf}(j_1) - \mathrm{erf}(j_2) + g(j_1, j_2), \tag{9}$$

where $\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-y^2} dy$, $\mathrm{erf}(j_1) = \mathrm{Pr.}(\tilde{M} \leq j_1)$, and $\mathrm{erf}(j_2) = \mathrm{Prob}(\tilde{m} \geq -j_2)$. From the joint PDF $\partial^2 \mathbf{P}_d(l_1, l_2)/\partial l_1 \partial l_2$ and using (1), we obtain

$$p_N^d(s) = \int_0^\infty dl_1 \int_0^\infty dl_2 \delta(s - l_1 - l_2) \frac{\partial^2 g^N}{\partial l_1 \partial l_2} \tag{10}$$

with $g \equiv g(l_1, l_2)$. Similarly, from the joint PDF $\partial^2 \mathbf{P}_c(j_1, j_2)/\partial j_1 \partial j_2$ and using (2) we obtain

$$p_N^c(w) = \int_0^\infty dj_1 \int_0^\infty dj_2 \delta(w - j_1 - j_2) \frac{\partial^2 h^N}{\partial j_1 \partial j_2} \tag{11}$$

with $h \equiv h(j_1, j_2)$. For small values of $N$, the double integrals in (10) and (11) can be performed explicitly and numerical simulations confirm these exact results [22]. Below we provide a physical interpretation of these formulas (10) and (11), and perform their asymptotic analysis both for small and large arguments. We also analyze their limiting form for $N \to \infty$.

*Distinct sites.*—To find the tails of $p_N^d(s)$ at small and large $s$ for finite $N$, we rewrite (10) as

$$p_N^d(s) = \int_0^s dl_2 \Psi_d(s - l_2, l_2), \quad \text{where}$$

$$\Psi_d(l_1, l_2) = N g^{N-1} \frac{\partial^2 g}{\partial l_1 \partial l_2} + N(N-1) g^{N-2} \frac{\partial g}{\partial l_1} \frac{\partial g}{\partial l_2}. \tag{12}$$

We interpret the two contributions in $\Psi_d(l_1, l_2)$ as follows [22]: the first term corresponds to a configuration where one particle explores a region $[-l_2, s - l_2]$ (we call it a box) of size $s$ in unit time interval, such that its maximum is at $s - l_2$ and its minimum is at $-l_2$, while all the other $(N-1)$ particles stay inside this box. On the other hand, the second term corresponds to a configuration where two particles create, in a different way, the same box $[-l_2, s - l_2]$ of size $s$: one of the two particles has its maximum at $s - l_2$ and minimum larger than $-l_2$, while the second particle has its minimum at $-l_2$ and maximum below $s - l_2$, and all other $(N-2)$ particles stay strictly inside this box.

When $s \to 0$ in (12), one can replace $g(l_1, l_2)$ (8) by its asymptotic behavior when $l_1$, $l_2 \to 0$, where $g(l_1, l_2) \sim (4/\pi) \sin(\pi l_2/(l_1 + l_2)) e^{-(\pi^2/4(l_1+l_2)^2)}$. Inserting it in (12), we see that both terms in (12) contribute equally. After integration over $l_2$, one then obtains the result announced in (4) for $s \to 0$ with $a_N = 4\pi^{3/2} N(N-1)(4/\pi)^{N-2} \times (\Gamma((N-1)/2)/\Gamma(N/2))$, where $\Gamma(x)$ is the gamma function. To perform the large $s$ asymptotic of $p_N^d(s)$, we rewrite the sum in $g(l_1, l_2)$ (8) using the Poisson summation formula. This allows us to show that the first term in (12), which corresponds to create a box $[-l_2, s - l_2]$ with one particle, decreases as $e^{-(s+l_2)^2} e^{-l_2^2}$, whereas the second term where the same box is created by two particles decreases as $e^{-(s-l_2)^2} e^{-l_2^2}$. Since $l_2$ is always +ve, the two-particles term wins over the one-particle term when $s \to \infty$. This is physically understandable because creating a very large span with two particles is more likely than creating the same one with a single particle. It also follows from this analysis that the integral over $l_2$ in (12) is dominated by $l_2 \sim \mathcal{O}(s)$, which yields finally the large $s$ behavior announced in (4) with $b_N = 2N(N-1)/\sqrt{\pi}$. In Fig. 2 we verify that the small and large $s$ asymptotics of $p_N^d(s)$ given in (4), for $N = 10$, describe fairly well, without any fitting parameter, the distribution obtained from direct simulation.

What happens for large $N$? The typical scale of the fluctuations of $S_N/\sqrt{t}$ can be estimated from the relations with EVS (1). The variables $\tilde{M}_i$'s which are the maxima of the $i$th BM on the unit interval, are independent and identically distributed variables. Their common PDF is a half-Gaussian, $p(M) = (2/\sqrt{\pi}) e^{-M^2}$, $M > 0$. The same holds for the variables $-\tilde{m}_i$'s. Hence, for large $N$, standard results of EVS [27] state that the typical value of

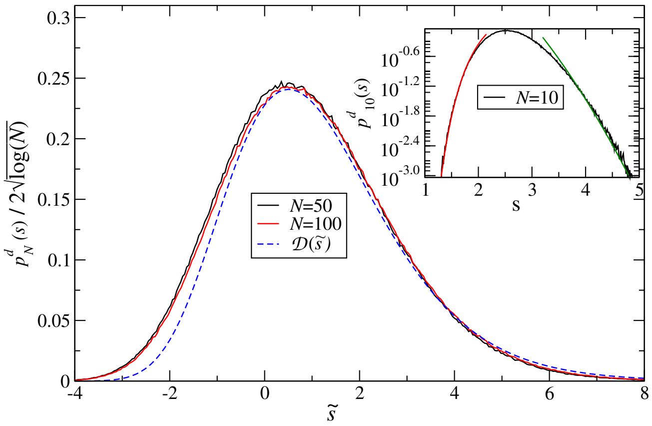

FIG. 2 (color online). Plot of $p_N^d(s)/(2\sqrt{\log N})$ as a function of $\tilde{s} = 2\sqrt{\log N}(s - 2\sqrt{\log N})$. The dashed line indicates the exact asymptotic results for $N \to \infty$, $\mathcal{D}(\tilde{s})$ in (6). *Inset:* Plot of $p_{10}^d(s)$, obtained from simulation, compared with its asymptotic behavior (4).

$\tilde{M}_+ = \max_{1\le i\le N}\tilde{M}_i$ is $\mathcal{O}(\sqrt{\log N})$, while its fluctuations are of order $1/\sqrt{\log N}$ and governed by a Gumbel distribution. The same holds for $\tilde{m}_- = -\min_{1\le i\le N}\tilde{m}_i$. For large $N$, these two extremes become uncorrelated as the global maximum and global minimum are most likely reached by two independent walkers. Hence one gets

$$g^N\left[\mu_N + \frac{\tilde{l}_1}{2\mu_N}, \mu_N + \frac{\tilde{l}_2}{2\mu_N}\right] \xrightarrow{N\to+\infty} e^{-e^{-\tilde{l}_1}} e^{-e^{-\tilde{l}_2}} \qquad (13)$$

with $\mu_N = \sqrt{\log N}$. Inserting (13) in (10) with $\tilde{s} = 2\mu_N(s - 2\mu_N)$, one finds

$$p_N^d(s) \sim 2\sqrt{\log N}\int_{-\infty}^{\infty} d\tilde{l}_2 e^{-\tilde{s}} e^{-e^{-\tilde{l}_2}} e^{-e^{-(\tilde{s}-\tilde{l}_2)}}, \qquad (14)$$

which can be evaluated explicitly to give (6). In Fig. 2 we plot $p_N^d(s)/2\sqrt{\log N}$ against $\tilde{s}$ for $N = 50$ and 100. They show a relatively good agreement with the exact result $\mathcal{D}(\tilde{s})$ after an overall shift of order $\mathcal{O}(1/\log N)$ along the $x$ axis, thus revealing, as expected [28], a slow convergence towards the asymptotic result. In [14] the authors argued that the limiting distribution should be a Gumbel distribution, overlooking the fact that it is actually the convolution of two Gumbel distributions, as in (14). In particular, for large $\tilde{s}$, $\mathcal{D}(\tilde{s}) \sim \tilde{s}e^{-\tilde{s}}$, while the Gumbel distribution decays as a pure exponential.

*Common sites.*—To find the small and large $w$ asymptotics of $P_N^c(w)$, we write (11) as

$$p_N^c(w) = \int_0^w dj_2 \Psi_c(w - j_2, j_2) \quad \text{where}$$
$$\Psi_c(j_1, j_2) = Nh^{N-1}\frac{\partial^2 h}{\partial j_1 \partial j_2} + N(N-1)h^{N-2}\frac{\partial h}{\partial j_1}\frac{\partial h}{\partial j_2}. \qquad (15)$$

In (15), one interprets the first term as one single particle creating a common span $[-j_2, w - j_2]$ of size $w$ and the second term as two particles collaboratively creating the same common span [22]. In both cases, the remaining particles are such that their maxima are above $w - j_2$ and their minima are below $-j_2$. When $w \to 0$ in (15), $h(j_1, j_2)$ can be replaced by its asymptotic behavior for small $j_1, j_2$: $h(j_1, j_2) \sim (1 - (2/\sqrt{\pi})(j_1 + j_2))$. Integrating then over $j_2$ in (15) yields the small $w$ behavior in (5) with $c_N = 4N(N-1)/\pi$. Note that for $w \ll 1$, it is much more likely to create a box of size smaller than $w$ with two particles (which occurs with a probability $\propto w^2$) than with a single one [which occurs with probability $\propto \exp(-\pi^2/4w^2)$]. The former configurations thus dominate for small $w$.

To get the large $w$ behavior of $p_N^c(w)$, we estimate $h(j_1, j_2)$ for large $j_1$ (15). This is conveniently done by using the Poisson formula, which yields $h(j_1, j_2) \sim \operatorname{erfc}(2j_1 + j_2) + \operatorname{erfc}(j_1 + 2j_2)$. This shows that for $w \gg \sqrt{\log N}$, the second term in (15) becomes subdominant compared to the first one. Hence for very large $w$ the leading contribution comes from the first term where we replace $h^{(N-1)}(w - j_2, j_2) \sim [\operatorname{erfc}(w + j_2) + \operatorname{erfc}(2w - j_2)]^{N-1}$ by $\operatorname{erfc}^{(N-1)}(w)$, as one can show that the integral over $j_2$ in (15) is dominated by the vicinity of $j_2 = 0$ [22]. This leads to the large $w$ behavior in (5) with $d_N = 8N/\pi^{N/2}$. The asymptotic behaviors of $p_N^c(w)$ (5) have been verified numerically for $N = 3$ in Fig. 3.

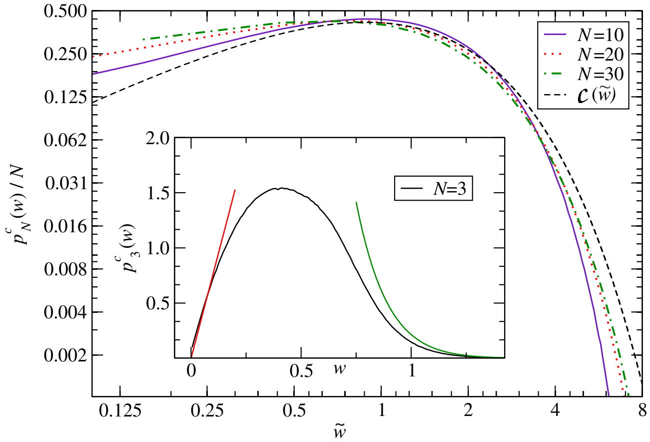

FIG. 3 (color online). Log-log plot of $p_N^c(w)/N$ as a function of $\tilde{w} = Nw$. The dashed line indicates the exact asymptotic results for $N \to \infty$, $\mathcal{C}(\tilde{w})$ in (7). *Inset:* Plot of $p_3^d(w)$, obtained from simulation, compared with its asymptotic behavior (5).

To obtain the typical scale of $W_N/\sqrt{t}$ for large $N$, we use its relation to EVS (2). From EVS for independent and identically distributed random variables [27], we know that $\tilde{M}_- = \min_{1\le i\le N} M_i$, where $M_i \ge 0$ and distributed according to a half-Gaussian, is of order $\mathcal{O}(N^{-1})$. Its PDF is given by a Weibull law, which is here an exponential distribution [27]. Indeed one has $\Pr.(N\tilde{M}_- \ge x) = e^{-(2/\sqrt{\pi}x)}$, $x > 0$, as $N \to \infty$. The same holds for $\tilde{m}_+$, which for large $N$ becomes independent of $\tilde{M}_-$ as both of them are reached by two independent walkers. Hence, from (2), $NW_N/\sqrt{2t}$ is given by the convolution of two exponential laws,

$$p_N^c(w) \sim N^2(4/\pi)e^{-(2/\sqrt{\pi})Nw}\int_0^w dk \sim N\mathcal{C}(Nw), \qquad (16)$$

with $\mathcal{C}(\tilde{w})$ as announced in (7). We have also obtained this result by a direct large $N$ expansion of (15). In Fig. 3 we plot $p_N^c(w)/N$ against $\tilde{w} = Nw$ for $N = 10$, 20 and 30 and see that they tend to coincide with the function $\mathcal{C}(\tilde{w})$, although the convergence is slow (a common fact for EVS [28]) and nonuniform. The best agreement is seen around $\tilde{w} \sim 1$, where $p_N^c(w)$ is peaked. This is expected as $\mathcal{C}(\tilde{w})$ describes the typical fluctuations of $W_N(t)$. It would be interesting to study the large deviations of $p_N^c(w)$ away from the peak, for finite but large $N$.

*Conclusion.*—We have achieved a complete analytic description of the PDFs of the number of distinct and common sites visited by $N$ independent RWs after $t$ time steps, for large $t$. Our results could be relevant for proteins diffusing independently on DNA [19] or polluting agents diffusing in rivers [18]. One may wonder about the effects of interactions between the walkers. For instance, one can study nonintersecting (vicious) RWs [29], which would be relevant for proteins sliding on DNA without the possibility to bypass [20]. In view of recent results for EVS for vicious walkers [30,31], it will be interesting to study the statistics of $S_N(t)$ and $W_N(t)$ in this interacting case.

We thank A. Perret for a useful discussion. This research was supported by ANR Grant No. 2011-BS04-013-01 WALKMAT and in part by the Indo-French Centre for the Promotion of Advanced Research under Project No. 4604−3.

[1] J. R. Beeler, Jr. and J. A. Delaney, Phys. Rev. **130**, 962 (1963).
[2] J. R. Beeler, Jr., Phys. Rev. **134**, A1396 (1964).
[3] A. Blumen, J. Klafter, and G. Zumofen, in *Optical Spectroscopy of Glasses*, edited by I. Zschokke (Reidel, New York, 1986), p. 199.
[4] R. Czech, J. Chem. Phys. **91**, 2498 (1989).
[5] P. Bordewijk, Chem. Phys. Lett. **32**, 592 (1975).
[6] C. A. Condat, Phys. Rev. A **41**, 3365 (1990).
[7] L. Edelstein-Keshet, *Mathematical Models in Biology* (Random House, New York, 1988).
[8] E. C. Pielou, *An Introduction to Mathematical Ecology* (Wiley-Interscience, New York, 1969).
[9] C. Cattuto, A. Barrat, A. Baldassarri, G. Schehr, and V. Loreto, Proc. Natl. Acad. Sci. U.S.A. **106**, 10 511 (2009).
[10] A. Dvoretzky and P. Erdös, in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (University of California, Berkeley, 1951).
[11] G. H. Vineyard, J. Math. Phys. (N.Y.) **4**, 1191 (1963).
[12] E. W. Montroll and G. H. Weiss, J. Math. Phys. (N.Y.) **6**, 167 (1965).
[13] F. van Wijland and H. J. Hilhorst, J. Stat. Phys. **89**, 119 (1997).
[14] H. Larralde, P. Trunfio, S. Havlin, H. E. Stanley, and G. H. Weiss, Nature (London) **355**, 423 (1992); Phys. Rev. A **45**, 7128 (1992).
[15] S. N. Majumdar and M. V. Tamm, Phys. Rev. E **86**, 021135 (2012).
[16] L. Turban, arXiv:1209.2527.
[17] O. Bénichou, C. Loverdo, M. Moreau, and R. Voituriez, Rev. Mod. Phys. **83**, 81 (2011).
[18] S. A. Socolofsky and G. H. Jirka, *Environmental Fluid Mechanics 1: Mixing and Transport Processes in the Environment*, (Texas A&M University, College Station, 2005), https://ceprofs.civil.tamu.edu/ssocolofsky/cven489/Book/Book.htm.
[19] J. Gorman and E. C. Greene, Nat. Struct. Mol. Biol. **15**, 768 (2008).
[20] J. Gorman, A. J. Plys, M. L. Visnapuu, E. Alani, and E. C. Greene, Nat. Struct. Mol. Biol. **17**, 932 (2010).
[21] From now on, we drop the explicit $t$ dependence on $S_N$, $W_N$, $M_i$, $m_i$.
[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.220602 for details of calculations and numerical simulations.
[23] Y. V. Fyodorov and J.-Ph. Bouchaud, J. Phys. A **41**, 372001 (2008).
[24] D. Carpentier and P. Le Doussal, Phys. Rev. E **63**, 026110 (2001).
[25] Y. V. Fyodorov, P. Le Doussal and A. Rosso, J. Stat. Mech. (2009) P10005.
[26] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, England, 2001).
[27] E. J. Gumbel, *Statistics of Extremes* (Dover, New York, 1958).
[28] G. Györgyi, N. R. Moloney, K. Ozogány, Z. Rácz, Phys. Rev. Lett. **100**, 210601 (2008).
[29] M. E. Fisher, J. Stat. Phys. **34**, 667 (1984).
[30] G. Schehr, S. N. Majumdar, A. Comtet and J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008).
[31] P. J. Forrester, S. N. Majumdar, and G. Schehr, Nucl. Phys. **B844**, 500 (2011); **B857**, 424(E) (2012).