

Self-Consistent Dynamics of a Josephson Junction in the Presence of an Arbitrary Environment

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We derive microscopically the dynamics associated with the dc Josephson effect in a superconducting tunnel junction interacting with an arbitrary electromagnetic environment. To do so, we extend to superconducting junctions the so-called $P(E)$ theory (see, e.g., Ingold and Nazarov, [arXiv:cond-mat/0508728](https://arxiv.org/abs/cond-mat/0508728)) that accurately describes the interaction of a nonsuperconducting tunnel junction with its environment. We show the dynamics of this system is described by a small set of coupled correlation functions that take into account both Cooper pair and quasiparticle tunneling. When the phase fluctuations are small the problem is fully solved self-consistently, using and providing the exact linear admittance $Y(\omega)$ of the interacting junction.

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Fifty years ago Josephson stunned the community when he published [1] the equations that govern the behavior of superconducting tunnel junctions. These Josephson relations, as they became known, link the voltage V and the superconducting phase difference φ across the junction, and the current I through it:

$$I = I_0 \sin \varphi, \quad V = \frac{\hbar}{2e} \frac{d\varphi}{dt}. \quad (1)$$

If φ is static, $V = 0$, and a nondissipative current I flows through the junction, bounded by $|I| \leq I_0$. This maximum supercurrent I_0 (or the corresponding Josephson coupling energy $E_J = I_0 \hbar / 2e$) was originally predicted to be an intrinsic property of the tunnel junction, depending only on its resistance in the normal state and the superconducting gap of its electrodes [2], but not on other details such as the junction's geometry, or its fabrication process. Along the years, Josephson junctions (JJs) have proved invaluable electronic components forming exquisitely sensitive sensors (e.g., squid magnetometers, quantum-limited amplifiers), metrological Volt standard devices, or quantum bits and gates.

It is important to note that the first Josephson relation was derived assuming that the phase φ has negligible quantum fluctuations, and it is not obvious why it would be generally valid beyond this situation. Because the Josephson effect has, among others, metrological applications, the effect of phase fluctuations on Josephson tunneling were thoroughly investigated in the 1980s, mostly using path integral formalism [3–5]. It was concluded that in most practical experimental situations a JJ can indeed be described using the effective Josephson Hamiltonian $H_J = -E_J \cos \varphi$ that directly corresponds to the first Josephson relation, with, however, small corrections due to phase fluctuations that originate in its *electromagnetic environment* (i.e., the circuit connected to the junction). This was checked for instance in the so-called Macroscopic Quantum Tunneling experiments [4–6]. More recently, JJ-based quantum logic circuits were

also shown to be accurately described using the effective Josephson Hamiltonian [7], with their electromagnetic environment partly responsible for their decoherence [8]. Note, however, that some environmental decoherence mechanisms in JJ qubits were recently identified that cannot be captured within only the effective Josephson Hamiltonian model [9–11].

On the other hand, the environment of a JJ can have a more dramatic effect: the phase fluctuations generated by an impedance larger than the resistance quantum $R_Q = h/4e^2 \sim 6.5 \text{ k}\Omega$ are expected to suppress the superconducting character of a JJ [4], and some experiments have confirmed this prediction [12]. Presently several groups are actively developing nondissipative high impedance environments using 1D arrays of JJs in the search for coherent quantum phase slips [13,14], or to achieve engineering of quantum phase fluctuations [15,16]. Given the goal, it is questionable whether using the effective Josephson Hamiltonian is still fully relevant to model these arrays. Moreover, such JJ arrays implement impedances having several plasma mode resonances that are not readily handled by the available theory.

In this Letter we provide a general derivation of the Josephson coupling in the presence of phase fluctuations generated by an arbitrary electromagnetic environment. Our derivation starts from a microscopic description of the tunneling of individual electrons between the superconducting electrodes, and applies the machinery of the so-called $P(E)$ theory (PoET) [17–19]. This theory was developed in the 1990s to explain a reduction of differential conductance at low voltage (also called “zero-bias anomaly”) in nonsuperconducting sub- μm tunnel junctions, a phenomenon that is now often referred to as dynamical Coulomb blockade. In its original form this theory evaluates the incoherent tunneling rate of electrons properly taking into account the probability $P(E)$ that the environment absorbs an energy E during a tunnel event.

While perturbative in tunneling, this theory is nonperturbative in the strength of coupling to the environment and it can deal with an arbitrary frequency-dependent linear electromagnetic environment. Note that it also applies to incoherent Cooper pair tunneling in JJs at finite sub-gap voltages. Its predictions were shown to be quantitative in a number of experiments, in particular when the environment consists of resonators [20,21]. Here, by generalizing PoET to the dc Josephson effect, a coherent flow of Cooper pairs through the junction, we obtain a unified nonperturbative treatment of arbitrary environmental effects in both normal and superconducting tunnel junctions. In this approach we show that one is lead naturally to introduce a self-consistent mean-field electrodynamic response of the junction, something that, as far as we know, has not been done explicitly previously for JJs. In this formulation the junction is systematically and properly combined with the rest of the circuit, resulting in an intuitive picture of the system. In the case when the phase fluctuations are small we work out the linear response of the junction and a simple iterative scheme to evaluate a renormalized I_0 and its admittance. As an illustrative example, we work out the self-consistency for a JJ in an Ohmic environment at zero temperature. In the conclusion we discuss the scope of our results and possible extensions.

The circuit we consider, shown in Fig. 1(a), consists of a pure tunnel element connected in parallel with the junction's geometric capacitor and in series with an arbitrary linear electromagnetic environment with impedance $Z(\omega)$. The Hamiltonian of the circuit is

$$H = H_L + H_R + H_{\text{env}} + H_T,$$

where H_{env} describes the voltage source and $Z(\omega)$ in the manner of Caldeira and Legget [5] and $H_{L,R}$ are the BCS Hamiltonians of the junction's electrodes. For the left electrode, for instance, we have

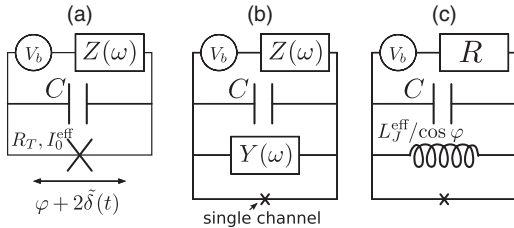


FIG. 1. (a) We consider a Josephson junction characterized by its normal state resistance R_T and derive its effective critical current I_0^{eff} taking into account both a static (φ) and a fluctuating phase difference $\tilde{\delta}(t)$ driven by the electromagnetic environment. (b) As seen from an individual tunnel channel, the environment consists of the impedance $Z(\omega)$ of the connecting circuit, of the junction's own capacitance C , and of the electromagnetic response due to tunneling in the other channels, here described by a linear admittance $Y(\omega)$, but which in the general case is a nonlinear element. (c) We solve the problem in the case of an Ohmic environment, retaining only the dominant inductive contribution in Y .

$$H_L = \sum_{\ell\sigma} \xi_{\ell} c_{\ell\sigma}^{\dagger} c_{\ell\sigma} - \Delta \sum_{\ell} c_{\ell\uparrow}^{\dagger} c_{\ell\downarrow}^{\dagger} + c_{\ell\downarrow} c_{\ell\uparrow},$$

where σ is the spin index, ℓ is a composite channel and momentum index for the electrons in the leads, and the overbar denotes the opposite-momentum state (H_R has the same form, with states indexed by r instead of ℓ). Finally, $H_T = \hat{T} + \hat{T}^{\dagger}$ is the tunneling Hamiltonian treated as a perturbation, where the operator $\hat{T} = e^{i\tilde{\delta}} \sum_{\ell,r,\sigma} t_{\ell r} c_{r\sigma}^{\dagger} c_{\ell\sigma}$ transfers an electron from the left to the right electrode. We work in a gauge where the electrodes have real BCS order parameters Δ (assumed identical in L and R) and, consistently, the $e^{i\tilde{\delta}}$ term here takes care of transferring the electronic charge e between the electrodes [10,19]. We restrict to zero dc voltage across the junction so that $\hat{\delta}(t) = \varphi/2 + \tilde{\delta}(t)$ with φ being the superconducting phase difference across the junction and $\tilde{\delta}(t)$ a zero-mean fluctuating phase operator driven by $Z(\omega)$. By introducing the standard Bogoliubons operators

$$\gamma_{1k} = u_k c_{k\uparrow} + v_k c_{k\downarrow}^{\dagger}; \quad \gamma_{0k} = -v_k c_{k\uparrow} + u_k c_{k\downarrow}^{\dagger}$$

($k = \ell, r$) with the usual BCS coherence factors u_k, v_k we can diagonalize $H_{L,R}$, whereas H_T becomes

$$\begin{aligned} H_T = & \sum_{\ell,r} t_{\ell r} [\gamma_{0r}^{\dagger} \gamma_{1\ell} (-e^{-i\tilde{\delta}} u_{\ell} v_r - e^{-i\tilde{\delta}} u_r v_{\ell}) \\ & + \gamma_{1r}^{\dagger} \gamma_{0\ell} (-e^{-i\tilde{\delta}} u_{\ell} v_r - e^{i\tilde{\delta}} u_r v_{\ell}) \\ & + \gamma_{1r}^{\dagger} \gamma_{1\ell} (e^{i\tilde{\delta}} u_r u_{\ell} - e^{-i\tilde{\delta}} v_r v_{\ell}) \\ & + \gamma_{0r}^{\dagger} \gamma_{0\ell} (-e^{-i\tilde{\delta}} u_r u_{\ell} + e^{i\tilde{\delta}} v_r v_{\ell})] + (\ell \rightleftharpoons r)^{\dagger}. \end{aligned}$$

In thermal equilibrium situations the supercurrent through the junction is given by the thermodynamic relation

$$I = \frac{2e}{\hbar} \frac{dF}{d\varphi}, \quad (2)$$

where F is the free energy. To lowest order in perturbation theory the change of F due to H_T can be cast as

$$\Delta F = \frac{1}{\hbar} \int_0^{+\infty} dt \text{Im} S_{H_T}(t), \quad (3)$$

with $S_{H_T}(t) = \langle H_T(t) H_T(0) \rangle$ where the angular brackets denote averaging over the unperturbed quasiparticle and environment states that act as bath degrees of freedom whose time evolution is the unperturbed one. A straightforward algebraic calculation gives

$$\begin{aligned} S_{H_T}(t) = & \sum_{\ell,r,\eta=\pm} |t_{\ell r}|^2 [u_r v_r u_{\ell} v_{\ell} (A_{\eta}(t) - B_{\eta}(t)) C_{\eta\eta}(t) e^{i\eta\varphi} \\ & + ((u_{\ell}^2 v_r^2 + u_r^2 v_{\ell}^2) A_{\eta}(t) \\ & + (u_{\ell}^2 u_r^2 + v_{\ell}^2 v_r^2) B_{\eta}(t)) C_{\eta-\eta}(t)] \end{aligned} \quad (4)$$

with

$$\begin{aligned}
A_{\eta=\pm}(t) &= \langle \gamma_{0\ell}^{\eta}(t) \gamma_{0\ell}^{-\eta} \gamma_{1r}^{-\eta}(t) \gamma_{1r}^{\eta} + \gamma_{1\ell}^{-\eta}(t) \gamma_{1\ell}^{\eta} \gamma_{0r}^{\eta}(t) \gamma_{0r}^{-\eta} \rangle \\
B_{\eta}(t) &= \langle \gamma_{0\ell}^{\eta}(t) \gamma_{0\ell}^{-\eta} \gamma_{0r}^{-\eta}(t) \gamma_{0r}^{\eta} + \gamma_{1\ell}^{-\eta}(t) \gamma_{1\ell}^{\eta} \gamma_{1r}^{\eta}(t) \gamma_{1r}^{-\eta} \rangle \\
C_{\eta\eta'}(t) &= \langle e^{i\eta\delta(t)} e^{i\eta'\delta(0)} \rangle,
\end{aligned}$$

where a fermion operator with a minus exponent means an annihilation operator. The $e^{\pm i\varphi}$ terms in Eq. (4) are each related to the transfer of two spin-conjugate electrons in a given direction, i.e., a whole Cooper pair with charge $2e$, they thus correspond to the Josephson effect. Note also that they come with the $u_r v_r$ and $u_\ell v_\ell$ factors that correspond to the anomalous Green's function of the electrodes, carrying the essence of superconductivity. The φ -independent terms, on the contrary, are related to a back-and-forth transfer of an electron and correspond to ordinary quasiparticle tunneling, the only processes remaining in the normal state. These processes do not transfer a net charge through the junction but they still couple to the phase fluctuations and contribute to the dynamics of the JJ. While these processes are obviously disregarded when JJs are modeled using only the effective Josephson Hamiltonian (e.g., most JJ-based qubit literature), the full Ambegaokar-Eckern-Schön effective action for the JJ [3] [whose form is closely related to Eq. (4)] allows accounting for them in path integral formalism. In the present approach we handle these terms using only two-point real-time correlators and sparing the use of path integrals. The correlators $C_{+-}(t)$, $C_{-+}(t)$ that accompany quasiparticle tunneling are those encountered in the standard PoET [specifically, $\langle e^{i\delta(t)} e^{-i\delta(0)} \rangle = \int dt e^{-iEt/\hbar} P(E)$ is the inverse Fourier transform of $P(E)$], while the Cooper pair tunneling comes with distinct correlators $C_{++}(t)$, $C_{--}(t)$. For simplicity we here assume phase fluctuations are symmetric; i.e., $C_{++} = C_{--}$ and $C_{+-} = C_{-+}$ (we discuss the limit of validity of this assumption in the Supplemental Material [22]). Going to a continuum of states in the electrodes, from Eq. (4) we obtain the exact result at lowest order in tunneling

$$\begin{aligned}
S_{H_T}(t) &= \frac{2R_Q}{\pi^2 R_T} [(p(t)^2 - q(t)^2) C_{+-}(t) \\
&\quad + m(t)^2 C_{++}(t) \cos\varphi], \quad (5)
\end{aligned}$$

where R_T is the normal state tunnel resistance of the junction and $m(t)$, $p(t)$, $q(t)$ are, respectively, the inverse Fourier transforms of $\mathcal{M}(\varepsilon) = -\Delta f(-\varepsilon)\rho(\varepsilon)/\varepsilon$, $\mathcal{P}(\varepsilon) = f(-\varepsilon)\rho(\varepsilon)$, $\mathcal{Q}(\varepsilon) = -f(-\varepsilon)\theta(\varepsilon^2 - \Delta^2)\text{sgn}(\varepsilon)$ with $\rho(\varepsilon) = |\varepsilon|\text{Re}(\varepsilon^2 - \Delta^2)^{-1/2}$ the BCS density of states, θ the Heavyside step, and $f(\varepsilon)$ the occupation probability of the Bogoliubov quasiparticles, which need not be thermal. Here both electrodes are assumed identical but the general case could also be handled. Note that in principle the gap Δ of the electrodes should be self-consistently evaluated from $f(\varepsilon)$, an effect which becomes important at temperatures comparable to the critical temperature or in strong non-equilibrium. If we first ignore a possible φ dependence of

$C_{\pm\pm}$, then, by combining Eqs. (2), (3), and (5) one obtains a generalization of the first Josephson relation with an effective critical current

$$I_0^{\text{eff}} = \frac{2}{\pi e \hbar R_T} \left| \int_0^{+\infty} dt \text{Im}[m(t)^2 C_{++}(t)] \right|, \quad (6)$$

which remains valid beyond thermal equilibrium. This expression generalizes PoET in real-time formulation [23,24]. In the case where phase fluctuations are negligible $C_{++}(t) \equiv 1$, and one recovers all standard results on JJ, such as, e.g., the temperature dependence of the critical current [2]. Hence C_{++} is a kernel giving a renormalization of the critical current with respect to the standard Ambegaokar-Baratoff value [2]. We will see below that $C_{\pm\pm}$ should in principle depend on φ (albeit weakly in usual cases), thus yielding additional terms that cause a departure from the purely sinusoidal current-phase relation predicted by Josephson.

We now consider finite phase fluctuations and first assume that the degrees of freedom generating these fluctuations can be regarded as a linear impedance Z_{eff} as in the usual PoET [19]. Such fluctuations are then Gaussian and consequently $C_{\pm\pm}$ can be expressed in terms of only the two-point correlator $S_{\delta}(t) = \langle \tilde{\delta}(t) \tilde{\delta}(0) \rangle$. As a consequence of the fluctuation-dissipation theorem, $S_{\delta}(t)$ can in turn be evaluated from the spectral density of the environment. Namely

$$\begin{aligned}
C_{+-} &= e^{S_{\delta}(t) - S_{\delta}(0)} = e^{J(t)} \\
C_{++} &= e^{-S_{\delta}(t) - S_{\delta}(0)} = e^{-J(t) - 2S_{\delta}(0)} \quad (7)
\end{aligned}$$

$$S_{\delta}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \frac{\text{Re}Z_{\text{eff}}(\omega)}{2R_Q} \frac{e^{-i\omega t}}{1 - e^{-\beta\hbar\omega}}. \quad (8)$$

Here we have also introduced the usual PoET notation $J(t) = S_{\delta}(t) - S_{\delta}(0)$ [19]. Replacing C_{++} in Eq. (6) we can pull out of the integral the renormalization factor $\lambda = e^{-2S_{\delta}(0)}$, which plays a major role in the following.

Note that, unless $\text{Re}Z_{\text{eff}}(\omega \sim 0) = \mathcal{O}(\omega^2)$ or smaller, $S_{\delta}(0) = \infty$ (signaling thermal or quantum phase diffusion), yielding $\lambda = 0$ and thus $I_0^{\text{eff}} = 0$. This might seem surprising since in most cases when one measures a JJ, it is connected to a circuit that contains normal metal at room temperature (with finite dc resistance), but its critical current is nevertheless measured finite. The apparent paradox is resolved when one considers the JJ as being part of its own electromagnetic environment [see Fig. 1(b)]: a superconducting JJ perfectly shunts the rest of the circuit at zero frequency, preventing phase diffusion and the divergence of $S_{\delta}(0)$. More importantly, doing so is actually the only way to enforce an amplitude and a dynamics of the phase fluctuations in the system that are actually consistent with the presence of the junction, unlike in standard PoET [25]. This inclusion of the junction in its own environment can also be justified microscopically: a typical metallic tunnel

junction contains a very large number N of independent Landauer channels that only interact through their common phase. Thus, as seen from each individual channel, the other channels form a (*a priori* nonlinear) bath whose response is that of the full junction (up to corrections of order $1/N$) and which we treat like the rest of the environment. Let us stress also that in typical tunnel junctions even if the junction's conductance is large, its individual channels remain very weakly transmissive. Hence, lowest order perturbation in tunneling is sufficient and all the complications in the behavior of the JJ arise solely from the electromagnetic interaction among the channels and with the environment, which are treated here in a self-consistent mean-field manner. Such a self-consistent mean-field approach of PoET has been successfully checked experimentally in low-resistance normal-state junctions [26], and, in that case, when the junction is described as a linear element (see below), this was shown to correspond to a self-consistent harmonic approximation that minimizes the free energy in the path integral description of the system [27]. Let us finally remark that in this mean-field approach the superconducting character of the JJ gives rise to a chicken-and-egg situation that requires a self-consistent solution, much like for the value of Δ in BCS theory itself.

We now close the loop by working out the self-consistency in the linear regime assumed in this part. Within this hypothesis, the response of the junction can be obtained from a generalized fluctuation-dissipation relation [28] and is expressed as an admittance

$$Y(\omega) = \frac{\cos\varphi}{iL_J^{\text{eff}}\omega} + 2 \int_0^\infty dt i \text{Im}S_I(t) \frac{e^{i\omega t} - 1}{\hbar\omega} \quad (9)$$

that is exact at lowest order in perturbation [22]. In this expression $L_J^{\text{eff}} = ((2e/\hbar)I_0^{\text{eff}})^{-1}$ is the effective Josephson inductance and $S_I(t) = \langle \hat{I}(t)\hat{I}(0) \rangle$ is the correlator of the current operator $\hat{I} = (2e/\hbar)(\partial H_T/\partial \varphi)$ through the junction. This latter definition implies that $S_I(t, \varphi) = (e/\hbar)^2 S_{H_T}(t, \varphi + \pi)$, readily obtained from Eq. (5). In the self-consistent approach we discuss here we shall then replace

$$Z_{\text{eff}}(\omega) = [Y(\omega) + iC\omega + Z^{-1}(\omega)]^{-1} \quad (10)$$

in Eq. (8), where C is the junction capacitance and Z the impedance of the external circuit as seen from the junction [see Fig. 1(b)]. Thus we are able to obtain the full dynamics of the system (and I_0^{eff} as a by-product) by solving the self-consistency defined by Eqs. (5), (9), (10), (8), and (7). This can, for instance, be done by iterating from an initial guess such as $Y(\omega) = \cos\varphi/iL_J^0\omega$, L_J^0 being the Josephson inductance in the absence of environment. In order to be valid the iterated solution must be consistent with the assumption of linear behavior of the effective environment, i.e.,

$$\sqrt{S_\delta(0)} \ll 2\pi, \quad (11)$$

so that phase fluctuations do not feel the nonlinearity of the JJ. In practice this means I_0^{eff} should not be reduced more than a few percent with respect to I_0 for this linear approach to be valid. If this later criterion is fulfilled, then the solution obtained is essentially the exact dynamics of the junction at lowest order in tunneling.

Simplifying approximations can be made or not depending on the value of the “plasma frequency” $\omega_p = (\cos\varphi/L_J^{\text{eff}}C)^{1/2}$ defined as the resonance frequency of the purely inductive first term of Eq. (9) with the junction's capacitance C . If ω_p is significantly smaller than $\omega_{\text{Gap}} = 2\Delta/\hbar$, then at low temperature it is a good approximation to keep in $Y(\omega)$ only the inductive term, that precisely suppresses the divergence of $S_\delta(0)$. This is justified because the integral in Eq. (9) has only a slight capacitive contribution at frequencies $\omega \lesssim \omega_{\text{Gap}} = 2\Delta/\hbar$ with dissipation setting in only at frequencies close to or above ω_{Gap} . With this simplification Z_{eff} reduces to the impedance of an LC oscillator resonating at ω_p damped by the external impedance $Z(\omega)$. Furthermore, still in the case when $\omega_p < \omega_{\text{Gap}}$, the characteristic time scale of phase fluctuations (ω_p^{-1}) is significantly longer than that of $m^2(t)$, which is ω_{Gap}^{-1} . Then, in Eqs. (6) and (7) we can take the short-time limit $J(t \rightarrow 0) = 0$, yielding the simple renormalization $I_0^{\text{eff}} = \lambda I_0$. A similar renormalization of the Josephson coupling was obtained at $\varphi = 0$ in Refs. [4,29]. We see here that this is valid only when ω_{Gap} is the fastest dynamics in the problem and that the opposite situation cannot be treated correctly in approaches starting from the effective Josephson Hamiltonian.

Let us now fully work out an example in the above simplifying assumption $\omega_p < \omega_{\text{Gap}}$, $I_0^{\text{eff}} = \lambda I_0$, and further restricting to the “Ohmic” case where $Z(\omega) = R$ [Fig. 1(c)] and zero temperature. Then the effective environment reduces to an RLC circuit with impedance $Z_{\text{eff}}(\omega) = (\lambda \cos\varphi/i\omega L_J^0 + iC\omega + R)^{-1}$ for which $S_\delta(0)$ can be calculated analytically and from which we derive the self-consistency equation

$$\lambda = \exp - \frac{R}{2R_Q} \frac{\tanh^{-1}\left(\frac{1-2\lambda q^2 \cos\varphi}{\sqrt{1-4\lambda q^2 \cos\varphi}}\right) + i\frac{\pi}{2}}{\sqrt{1-4\lambda q^2 \cos\varphi}}, \quad (12)$$

where $q = R\sqrt{C/L_J^0}$ would be the quality factor of the plasma oscillation at $\varphi = 0$, in absence of renormalization. Again, valid solutions must satisfy Eq. (11), that is, $1 - \lambda \ll 1$. However, this always fails at $\varphi = (\pi/2) \bmod \pi$ where ω_p vanishes and where a treatment beyond linear response is needed. When the approximation is valid (away from the pathological points) we predict that the renormalization of I_0 is different at $\varphi = 0$ and $\varphi = \pi$, leading to a slightly anharmonic current-phase relation.

This anharmonicity is a generic feature in the self-consistent approach because it causes $C_{+\pm}(t)$ in Eq. (5) to have a φ dependence through the dynamical response of the JJ.

In conclusion, we have extended the framework of the PoET to address the effect of an arbitrary electromagnetic environment on the Josephson effect in metallic tunnel junctions. Doing so, we reached a self-consistent description of the Josephson effect, shedding new light on the interaction of a JJ with its environment, including its dynamics. This notably predicts that the celebrated first Josephson relation generically departs from a sinusoid when the impedance of its environment is increased, a fact that should be verifiable experimentally. For strictly dc Josephson effect and small phase fluctuations, the self-consistency is fully worked out using the exact linear admittance of the interacting JJ, a quantity that is accessible to measurements and that should be useful for quantum circuit engineering. We think more work in this direction could extend this approach to non-dc situations and non-Gaussian phase fluctuations [22]. This would provide the general “circuit laws” for Josephson junctions, a quantum nonlinear generalization of the classical “impedance combination laws.”

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