Kibble-Zurek Scaling and its Breakdown for Spontaneous Generation of Josephson Vortices in Bose-Einstein Condensates

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Atomic Bose-Einstein condensates confined to a dual-ring trap support Josephson vortices as topologically stable defects in the relative phase. We propose a test of the scaling laws for defect formation by quenching a Bose gas to degeneracy in this geometry. Stochastic Gross-Pitaevskii simulations reveal a $-1/4$ power-law scaling of defect number with quench time for fast quenches, consistent with the Kibble-Zurek mechanism. Slow quenches show stronger quench-time dependence that is explained by the stability properties of Josephson vortices, revealing the boundary of the Kibble-Zurek regime. Interference of the two atomic fields enables clear long-time measurement of stable defects and a direct test of the Kibble-Zurek mechanism in Bose-Einstein condensation.

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A possible mechanism for the formation of domain structures in the early Universe was proposed by Kibble [1]. He argued that the Universe cooled down after the hot big bang event and subsequently, passed through a symmetry breaking phase transition at a critical temperature T_c. Causally unconnected spatial domains settling into different vacua would lead to the formation of defects like domain walls, monopoles, strings, textures, etc. [2]. Due to thermal fluctuations thwarting the emerging order, it was postulated that the number of defects eventually settled at the so-called Ginzburg temperature $T_G < T_c$.

Later, Zurek [3] put forward an alternative argument focusing on the nonequilibrium aspect of the phase transition. The density of the defects is determined at the critical temperature instead and its number is scaled with the quench rate. The scaling exponent depends on the critical exponents of the underlying phase transition. This scenario, known as the Kibble-Zurek mechanism (KZM), should equally apply to condensed matter phase transitions accessible to laboratory experiments [4]. The KZM proved to be robust and was verified by a number of recent experiments on annular Josephson tunnel junctions [5–8] and theoretical research on Bose-Einstein condensates (BEC) [9–12]. It also extends to quantum phase transitions [13–15].

In the spirit of Kibble's argument, one might expect the KZM to fail in the limit of slow quenches where the time scale of other processes occurring in the system dominates over the quench time. Deviations from KZM predictions were observed in ^{4}He experiments [16] but the interpretation was controversial [17] and a manifestation of the Ginzburg temperature was ruled out in Ref. [18]. So far, the transition between the regime of KZ scaling and its breakdown has not been studied systematically.

In this Letter, we investigate the robustness of the KZ scaling in a system where departure from it can be understood in detail because the defects are easily quantified and are stable at the end of the quench. This avoids the difficulty of counting the decaying population of defects [12,19] or their remnants [20]. To this end, we study two linearly coupled quasi-1D atomic Bose gases in the ring configuration, as in Fig. 1. A quench through the Bose-Einstein condensation phase transition can generate Josephson vortices (JVs) confined between two BECs [21,22]. We show that the number of JVs obeys the KZ scaling law for fast quenches. On the contrary, for slow quenches, the predicted behavior deviates substantially, and we observe a much stronger quench-time dependence than expected for critical phenomena in our simulations.

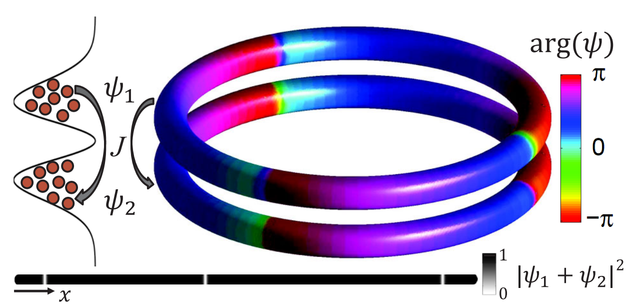

FIG. 1 (color online). Schematic of the two linearly coupled BECs. The isosurface shows the equilibrated condensate density profile and the color shows a phase profile with three Josephson vortices resulting from a quench. The trapping potential is visualized on the left. The interference pattern of the two atomic fields on the bottom shows clear evidence of the three Josephson vortices located at the low density regions.

This is due to decay processes occurring before the topological stability is established, in analogy to Kibble's arguments.

The system under study can be realized by crossing a vertical Gaussian-Laguerre laser beam and two horizontal sheet beams [23] to form an optical dipole trap or with rf dressing on an atom chip [24]. Another way is trapping the atoms with two hyperfine states coupled via Raman transitions [25] in a single ring trap [23]. Along the z axis, the trapping potential can be treated as a double-well potential as shown in Fig. 1. Assuming tight confinement, the transverse motion can be eliminated. The resulting coupled Gross-Pitaevskii equations for the order parameter ψ_1 and ψ_2 in each ring assume the dimensionless form

$$i\partial_t\psi_j = (\mathcal{L}_j - \mu)\psi_j - J\psi_{3-j}, \tag{1}$$

where $\mathcal{L}_j = -(1/2)\partial_{xx} + g|\psi_j|^2$ $(j = 1,\ 2)$, μ is the chemical potential, and J the tunneling energy. Length, time, and energy are scaled by $a_h = \sqrt{\hbar/m\omega}$, $1/\omega$, and $\hbar\omega$, respectively, where m is the atomic mass and ω is the transverse trapping frequency. Accordingly, the dimensionless nonlinear interaction strength g is related to the s wave scattering length a by $g = 2a/a_h$.

Equation (1) supports topological and nontopological defects in the form of the JV and the dark soliton (DS), respectively, $\tilde{\psi}_{1,2} = \sqrt{1+\nu}\tanh(p\tilde{x}) \pm iB\,\mathrm{sech}(p\tilde{x})$, where $\nu = J/\mu$, and the scaling $x = \sqrt{\mu}\tilde{x}$ and $\psi_j = \sqrt{\mu/g}\tilde{\psi}_j$ has been applied. Both the DS with $B = 0$ and $p = \sqrt{1+\nu}$ and the JV with $p = 2\sqrt{\nu}$ and $B = \sqrt{1-3\nu}$ for $\nu \leq 1/3$ are localized excitations on the length scale $a_h(\sqrt{\mu}p)^{-1}$ above the vacuum where $\psi_1 = \psi_2 = \mathrm{const}$ [21]. The DS, where both components have identical profiles, is nontopological because it can continuously deform to the vacuum by a family of moving "grey" solitons with decreasing energy [26]. Although they may be present transiently during quenches through the phase transition, DSs will thus not survive the final stage of cooling. Furthermore, for $\nu < 1/3$, DSs are dynamically unstable with respect to decay into JVs, which have lower energy [27]. The stability properties of the JV, on the other hand, depend on the dimensionless parameter ν and may change during the quench. The JV bifurcates from the DS at $\nu = 1/3$ as a time-reversal symmetry broken state (vortex and antivortex) with a characteristic phase winding of 2π around a point located between the two rings (see Fig. 1), and only exists for smaller values of ν. From numerical simulations, it is known that JVs can move with respect to the background BEC, although explicit solutions are unknown. For $1/5 < \nu < 1/3$, variational arguments indicate that the JV is energetically unstable [27]. For $\nu < 1/5$ where the JV resembles the Sine-Gordon soliton [21,22], the stationary solution is a metastable local energy minimum, since the energy increases with velocity. Thus, at sufficiently small ν, JVs are topologically stable, enabling experimental tests of the KZ scaling by counting the number of JVs at the end of quench in a dual-ring BEC. The defects would be immediately evident by the interference images of two expanding atomic fields. The situation is strikingly different from a single 1D BEC where the KZ scaling law was predicted to govern a transient population of eventually decaying DSs, which makes experimental detection more difficult [12].

The nonequilibrium dynamics during the thermal quenches can be described by the coupled stochastic Gross-Pitaevskii equations [28,29]:

$$d\psi_j = (i + \Gamma)[(\mu(t) - \mathcal{L}_j)\psi_j + J\psi_{3-j}]dt + dW_j, \tag{2}$$

where Γ is the growth rate and dW_j is the thermal noise satisfying the fluctuation-dissipation relation $\langle dW_j^*(x,t)dW_k(x',t)\rangle = 2\Gamma T\delta_{jk}\delta(x-x')dt$, with T being the temperature in units of $\hbar\omega/k_B$. At the mean field equilibrium level, the phase transition is described by the ground state of the energy $\mathcal{H} = \int dx[1/2|\partial_x\psi_1|^2 + 1/2|\partial_x\psi_2|^2 + V(\psi_1,\psi_2)]$, where we seek the minimum of the potential $V(\psi_1,\psi_2) \equiv \sum_{j=1,2}|\psi_j|^2[g/2|\psi_j|^2 - \mu] - J[\psi_1^*\psi_2 + \psi_2^*\psi_1]$ for $J > 0$. The symmetry $V(\psi_1,\psi_2) = V(\psi_2,\psi_1)$ imposes a common amplitude for the ground state fields. Taking $\psi_1 = \sqrt{n}e^{i\phi_1}$, $\psi_2 = \sqrt{n}e^{i\phi_2}$, and $\Delta = \phi_1 - \phi_2$, the minimum of $V = gn^2 - 2\mu n - 2Jn\cos\Delta$ occurs at $\Delta = 0$, $n = (\mu + J)/g$, for $\mu > -J$. At the critical point $\mu = -J$, the minimum is independent of Δ and each field breaks $U(1)$ symmetry.

The transition to the broken symmetry phase is simulated via Eq. (2) with time-dependent chemical potential

$$\mu(t) = t/\tau_Q, \tag{3}$$

where τ_Q is the quench time. The quench starts from a thermal gas with a chemical potential $-\mu_0 < 0$ and ceases in the Bose-condensed phase at $\mu_0 > 0$. Due to interring coupling, $\tilde{\mu}(t) = t/\tau_Q + J$ acts as the effective chemical potential; the precise location of the transition in a dynamical quench must be determined numerically. We evaluate the total number of JVs during the quench with $N_{JV} = \oint|d(\phi_1 - \phi_2)|/2\pi$. The net number $N_{JV,\mathrm{net}} = |\oint d\phi_1 - \oint d\phi_2|/2\pi$ is the difference between the number of clockwise and anticlockwise vortices.

The KZ theory applied to the BEC phase transition gives the relaxation time and healing length close to the critical point as

$$\tau = \tau_0|\tilde{\mu}|^{-1}, \qquad \xi = \xi_0|\tilde{\mu}|^{-1/2}, \tag{4}$$

where ξ_0 and τ_0 depend on the microscopic details of the system. Following Eq. (3) and the KZ scenario [30], we obtain the typical size of the domains after the quench

$$\hat{\xi} = \xi_0\left(\frac{\tau_Q}{\tau_0}\right)^{1/4}, \tag{5}$$

where for our system $\tau_0 = \Gamma^{-1}$. When $\mu(t)$ exceeds $-J$, localized phase domains start to grow in each ring.

Typically, a piece of the (anti)vortex will fall within a ξ-sized domain in which the phase is chosen randomly. Therefore, for small J, in a ring with circumference C, the total number of JVs is estimated to be

$$\langle N_{JV}\rangle \sim C/\hat{\xi} = C\xi_0^{-1}\left(\frac{\tau_Q}{\tau_0}\right)^{-1/4}, \tag{6}$$

and thus, obeys the $-1/4$ power-law scaling with quench time. The number of JVs, thus, shows a stronger quench-time dependence than the winding number of a single-ring BEC, which was predicted to scale with $\tau_Q^{-1/8}$ [11].

We consider a gas of ^{87}Rb atoms with a transverse confining frequency of $\omega = 2\pi \times 200$ Hz. We numerically integrate Eq. (2) with $C = 30$, $T = 10^{-3}$, and $g = 0.05$, which are realistic parameters with the setup of Ref. [31]. The scaling in Eq. (6) is verified by averaging N_{JV} over 500 trajectories for $J = 5$ and 25. The value of μ_0 is chosen to be sufficiently large so that the resulting defect number is independent of it. As shown in Fig. 2, the results for fast quenches compare favorably with the KZM prediction, yet the number of JVs deviates from the KZ scaling for slow quenches.

The stability of a JV depends on conditions that change during the quench. According to the KZM, two different regimes exist: For early and late times during the quench, relaxation is efficient and fluctuations in the Bose gas follow the changing chemical potential adiabatically. However, when the diverging relaxation time τ of Eq. (4) exceeds the time scale of the quench $\mu/\dot{\mu}$, fluctuations transiently freeze out and the system enters the *impulse* regime. This occurs when

$$\tau(\tilde{\mu}(\hat{t})) = |\tilde{\mu}/\dot{\tilde{\mu}}|_{t=\hat{t}} = \hat{t}, \tag{7}$$

giving the freeze-out time scale $\hat{t} = \sqrt{\tau_0\tau_Q}$. At the following impulse-adiabatic transition, the frozen fluctuations are imprinted onto the forming BEC. We, thus, expect that the stability properties of defects formed at this transition point determine their survival during the adiabatic phase of the quench.

In Fig. 3, the impulse-adiabatic transition can be clearly observed from the particle number, $N_{1,2}(t) = \int|\psi_{1,2}(x,t)|^2dx$. A rapid increase of particle number takes place at $\tilde{\mu} = f\hat{t}/\tau_Q$, with $f \approx 4.7$. The value of f appears to depend weakly on the details of the system, including the parameters J and Γ, consistent with the theoretical argument that $\tilde{\mu}$ is relevant for the quench dynamics. While the particle number is small in the impulse regime, it follows the dashed linear time dependence in the adiabatic regime. Similar behavior was observed for a single ring BEC [11]. Therefore, we can predict the chemical potential at the impulse-adiabatic transition as:

$$\hat{\mu} = \mu(f\hat{t} - \tau_Q J) = f\sqrt{\frac{\tau_0}{\tau_Q}} - J. \tag{8}$$

We denote the critical ratio of tunneling to chemical potential for JV stability by $\nu_c = J/\mu_c$, where μ_c is the stabilizing chemical potential for given J. As shown in Fig. 4(a), the defects are frozen in until $\hat{\mu} > \mu_c$ for fast

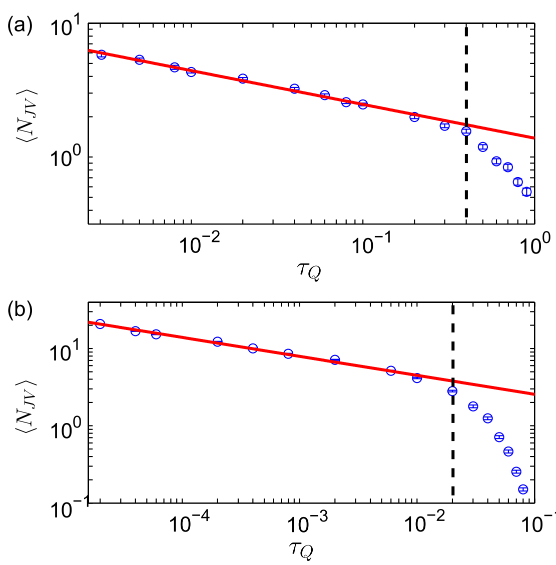

FIG. 2 (color online). Scaling of the total number of JVs with respect to τ_Q at $J = 5$ in (a) and $J = 25$ in (b) averaged over 500 trajectories of Eq. (2). The error bars indicate the standard deviations. The red lines show the best power-law fit for fast quenches with exponents -0.2523 ± 0.0128 in (a) and -0.2456 ± 0.0131 in (b), which agree with the KZM prediction of $-1/4$. The dashed lines indicate the critical quench time $\tau_Q^{\rm crit}$ of Eq. (9) for the breakdown of the KZ scaling law.

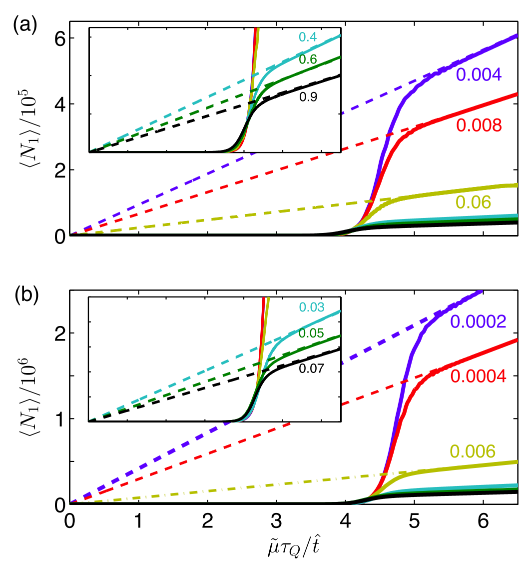

FIG. 3 (color online). Particle number of component ψ_1 as a function of time for $J = 5$ in (a) and $J = 25$ in (b). The vertical scale of the inset is magnified by a factor of 10 and reveal details for slow quenches. The color-coded labels show different τ_Q. Quenches with vastly different τ_Q show a knee structure characteristic of the impulse-adiabatic transition with a rapid particle number increase around $\tilde{\mu}\tau_Q/\hat{t} = 4.7$.

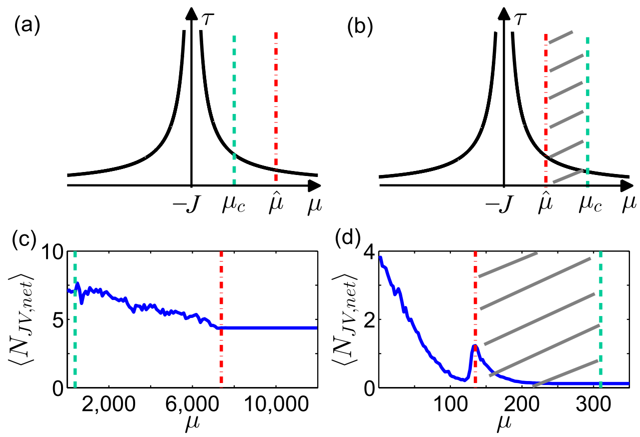

FIG. 4 (color online). Panels (a) and (b) show schematic plots of the relaxation time vs chemical potential for fast and slow quenches, respectively. The KZ scaling is unaffected by the stability of JVs in (a), while the resulting number of defects is affected by the decay happening in the shaded area in (b). Panels (c) and (d) show the net number of JVs for $\tau_Q = 5 \times 10^{-5}$ (fast) and 0.08 (slow) for $J = 25$, respectively. In (c) and (d), the locations of μ_c and $\hat{\mu}$ are obtained from $\nu_c = J/\mu_c$ and by reading the knee structure of the particle number [Fig. 3(b)], respectively. For the fast quench in (c), the net number stabilizes right after $\hat{\mu}$, while for the slow quench it decays in the shaded region shown in (d).

quenches ensuring the topological protection of JVs and hence, the KZM signature. However, for slow quenches the impulse regime terminates earlier with $\hat{\mu} < \mu_c$, which causes the decay of the JVs in the shaded region in Fig. 4 until the topological stability of JV is established at $\mu(t) = \mu_c$. Although the critical ν_c for a moving JV at finite temperatures is unknown, we can estimate ν_c from the numerical simulations, at the point where KZ scaling breaks down. From Eq. (8), we obtain the criterion for obtaining stable JVs

$$\tau_Q < \tau_Q^{\rm crit} = \tau_0 f^2 (\nu_c/J)^2 (1 + \nu_c)^{-2}. \quad (9)$$

The value $\nu_c \simeq 0.0813$ is obtained from the data for $J = 5$, which suggests $\tau_Q^{\rm crit} = 0.02$ for $J = 25$, as shown by the vertical dashed line plotted in Fig. 2(b). This prediction agrees with the numerical data very well. The critical quench time depends on the growth rate through τ_0, and we have also verified the prediction of Eq. (9) at different growth rates. In the slow quench regime, as seen in Fig. 2, the defect number falls off more rapidly with quench time than expected from the KZM. Since the variation is far from linear, we do not expect to enter a new regime of power-law scaling for slow quenches. Note that slow quenches show the same knee structure characterizing the impulse-adiabatic transition as the fast quenches that lead to KZ scaling (Fig. 3). We have also verified that $\langle N_{JV}\rangle$ continues to satisfy the KZ scaling for slow quenches (solid line in Fig. 2), when counted immediately after the impulse-adiabatic transition at $\hat{\mu}$. This supports our argument that the reduced defect number is due to thermal decay processes happening after the transition, and that defect formation is unaffected by thermal fluctuations during freeze-out. Moreover, by varying the circumference, we verify that the KZM departure is not due to the finite-size effects discussed in Refs. [30,32,33].

For JVs, the slow quench regime is similar to Kibble's idea [1], where thermal fluctuations suffice to destroy the emerging order before the system reaches the Ginzburg temperature. We observe a Ginzburg-like regime where thermal effects destroy the pattern of symmetry breaking inherited from criticality during the interval $\hat{\mu} < \mu < \mu_c$, shown in the shaded region of Fig. 4(b). This scenario is consistent with the evolution of the net number of vortices during a quench shown in Figs. 4(c) and 4(d). This measure is an indicator of the stability of individual JVs, unlike the total number that is affected by their pairwise annihilation (to which KZ scaling is immune).

The absence of any clear cut evidence of cosmological nature and the difficulty in observing the KZ scaling in condensed-matter experiments is usually not attributed to the failure of the mechanism but may be explained by the decay of defects in the post-quench era [12,19,20]. This is circumvented if the formed defects are topologically protected. The defects observed in the successful experiments of Refs. [5–8] have this property, as do the JVs that are the subject of this Letter. While cosmological defects protected by topology may still survive in dark matter or dark energy, their detection is difficult [34].

Our work paves the way for a direct test of KZM in the Bose-Einstein condensation phase transition, by eliminating post-quench decay of defects [11]. The quench of μ could be supplanted by a controlled sweep of J, providing another knob for varying the quench time.

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