Loop Quantization of the Schwarzschild Black Hole

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We quantize spherically symmetric vacuum gravity without gauge fixing the diffeomorphism constraint. Through a rescaling, we make the algebra of Hamiltonian constraints Abelian, and therefore the constraint algebra is a true Lie algebra. This allows the completion of the Dirac quantization procedure using loop quantum gravity techniques. We can construct explicitly the exact solutions of the physical Hilbert space annihilated by all constraints. New observables living in the bulk appear at the quantum level (analogous to spin in quantum mechanics) that are not present at the classical level and are associated with the discrete nature of the spin network states of loop quantum gravity. The resulting quantum space-times resolve the singularity present in the classical theory inside black holes.

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Spherically symmetric gravity in vacuum is perhaps one of the simplest symmetry reduced models to be studied where there is spatial dependence of the variables. In particular, it includes the interesting case of having a black hole present, with the challenge of its singularity. There have been previous investigations of the quantization of vacuum spherically symmetric gravity using complex Ashtekar variables by Thiemann and Kastrup [1], traditional metric variables by Kuchař [2], and using modern loop quantum gravity techniques by Campiglia *et al.* [3] and Tibrewala [4]. In all cases, the procedure started by choosing variables adapted to spherical symmetry. The resulting model has a diffeomorphism constraint associated with the symmetry under rescalings of the radial coordinate and a Hamiltonian constraint representing invariance under different foliations of space-time. Thiemann and Kastrup [1] were the first to complete the quantization of the model, remarkably, using Ashtekar's original complex variables. They noted that essentially there is only one degree of freedom, the Arnowitt-Deser-Misner (ADM) mass, that does not evolve in time. Using traditional variables and a suitable set of canonical transformations, Kuchař [2] reaches the same result. Bojowald and Swiderski [5] studied the model in terms of modern, real, Ashtekar variables and encountered difficulties in performing a canonical quantization using standard [6] loop quantum gravity techniques. Based on that work, a loop quantization was achieved by Campiglia *et al.* partially fixing the gauge, which eliminates the diffeomorphism constraint. Again, the only degree of freedom left is the ADM mass. Wave functions are functions of the ADM mass, and if one reconstructs the metric back from them, one has a singularity where the classical theory had one, the quantization being equivalent to the one found by Thiemann and Kastrup, and Kuchař. However, a later treatment using the semiclassical equations resulting from loop quantum gravity and covering both the interior and exterior of the black hole suggested that the singularity could be eliminated [7]. Studies of the quantization of black hole interiors using the isometry with the Kantowski-Sachs space-time also suggested that the singularity is eliminated by loop quantum gravity [8].

In this Letter, we would like to show that one can proceed to quantize these models without further gauge fixing. In principle, that would be problematic because the constraint algebra of general relativity, even in this simple $(1 + 1)$-dimensional example, is not a Lie algebra and that leads to problems implementing the Dirac quantization procedure. We will show, however, that through a simple rescaling of the Hamiltonian constraint without changing the canonical variables, one ends up with a true Lie algebra and can complete the quantization. In particular, one can find exactly the space of physical states annihilated by the constraints. Using the type of measures [9] common in loop quantization, one can show that the singularity is eliminated and one ends up with a regular space-time that tunnels through where the classical singularity used to be into another universe, akin to what happens classically in the Reissner-Nordström space-time but without singularities. The quantum theory has more observables than the ADM mass of the space-time, related to the fact that at the Planck scale one has structure when one introduces the types of measures one uses in loop quantum gravity. These types of degrees of freedom associated to the bulk suggest that it is possible to have loss of information either via the region of high curvature that replaces the singularity or through the bulk observables.

The treatment of spherically symmetric space-times with Ashtekar-type variables was pioneered by Bengtsson [10] and in more modern language discussed in detail by Bojowald and Swiderski [5]. We will follow here the notation of our previous paper [3], and we refer the reader to them and to Bojowald and Swiderski for more details.

Using Ashtekar-like variables adapted to the symmetry of the problem, one is left with two pairs of canonical variables E^φ, K_φ and E^x, K_x that are related to the traditional canonical variables in spherical symmetry $ds^2 = \Lambda^2 dx^2 + R^2 d\Omega^2$ by $\Lambda = E^\varphi/\sqrt{|E^x|}$, $P_\Lambda = -\sqrt{|E^x|}K_\varphi$, $R = \sqrt{|E^x|}$, and $P_R = -2\sqrt{|E^x|}K_x - E^\varphi K_\varphi/\sqrt{|E^x|}$ where P_Λ, P_R are the momenta canonically conjugate to Λ and R, respectively, x is the radial coordinate, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. We are taking the Immirzi parameter equal to 1. For most of the Letter, we will analyze the region with $E^x > 0$, and we will therefore drop the absolute value signs inside the square roots; if one wishes to analyze other regions, the absolute value signs should be reinstated.

The total Hamiltonian density for the theory is given by

$$H_T = N\left[\frac{((E^x)')^2}{8\sqrt{E^x}E^\varphi} - \frac{E^\varphi}{2\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}} - \frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{E^x}(E^x)''}{2E^\varphi}\right] - N_r[(E^x)'K_x - E^\varphi K_\varphi']. \quad (1)$$

We proceed to rescale the Lagrange multipliers, $N_r^{\rm old} = N_r^{\rm new} - 2N^{\rm old}(K_\varphi\sqrt{E^x})/(E^x)'$ and $N^{\rm old} = N^{\rm new}(E^x)'/E^\varphi$, and from now on we will drop the "new" subscripts. This leads to a total Hamiltonian that, after an integration by parts, reads

$$H_T = \int dx\left[-N'\left(-\sqrt{E^x}(1+K_\varphi^2) + \frac{((E^x)')^2\sqrt{E^x}}{4(E^\varphi)^2} + 2GM\right) + N_r[-(E^x)'K_x + E^\varphi K_\varphi']\right]. \quad (2)$$

We are not including contributions at the boundary for simplicity. The constant of integration $2GM$ is obtained imposing the boundary conditions for the lapse. The discussion in detail is present in Ref. [2]. A remarkable fact is that this rescaling of the constraints makes the Hamiltonian constraint have an Abelian algebra with itself and the usual algebra with the diffeomorphism constraint. We had already noted this in Ref. [3], but after gauge fixing the diffeomorphism constraint, here we point out that it is true even without gauge fixing the diffeomorphism constraint.

We now proceed to quantize. We start by recalling the basis of spin network states in one dimension (see Ref. [3] for details). One has graphs g consisting of a collection of edges e_j connecting the vertices v_j. It is natural to associate the variable K_x with edges in the graph and the variable K_φ with vertices of the graph. For bookkeeping purposes, we will associate each edge with the vertex to its left. One then constructs a standard holonomy for K_x and a "point holonomy" for K_φ (since it behaves as a scalar),

$$T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \langle K_x, K_\varphi | g, \vec{k}, \vec{\mu}\rangle = \prod_{e_j\in g}\exp\left(\frac{i}{2}k_j\int_{e_j}K_x(x)dx\right)\prod_{v_j\in g}\exp\left(\frac{i}{2}\mu_j K_\varphi(v_j)\right) \quad (3)$$

with e_j the edges of the spin network, g and v_j its vertices, and the integer k_j is the "color" associated with the edge e_j, and the real number μ_j the "color" associated with the vertex v_j. On these states, the triads act multiplicatively,

$$\hat{E}^x(x)T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \ell_{\rm Planck}^2 k_i(x)T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi), \quad (4)$$

$$\hat{E}^\varphi(x)T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \ell_{\rm Planck}^2\sum_{v_i\in g}\delta(x - x(v_i))\mu_i T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi), \quad (5)$$

where $k_i(x)$ is the color of the edge including the point x. If the latter is at a vertex, it is the edge to the right of it. $x(v_i)$ is the position of the vertex v_i.

To deal with the Hamiltonian constraint, we follow the steps usual in loop quantum cosmology and replace $K_\varphi \to \sin(\rho K_\varphi)/\rho$ in order to have a well-defined operator on the kinematical Hilbert space (some authors choose $\rho = 1$ [6]). We also choose a factor ordering, and it is convenient to rescale H and take a square root to simplify solving it,

$$\hat{H}(N) = \int dx N(x)\left(2\left\{\sqrt{\sqrt{\hat{E}^x}[1+\sin^2(\rho\hat{K}_\varphi)/\rho^2] - 2GM}\right\}\hat{E}^\varphi - \sqrt[4]{\hat{E}^x}(\hat{E}^x)'\right), \quad (6)$$

and the quantum constraint is also Abelian free of anomalies. We have defined the action of the relevant operators involved in the constraint on the $T_{g,\vec{k},\vec{\mu}}$ basis. Let us recall that K_φ is not well defined as an operator, only its exponentiation, so we had to polymerize with ρ the polymerization parameter. Acting on a quantum state, we have that

$$\hat{H}(N)T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \sum_{v_i\in g}N(v_i)(k_i\ell_{\rm Planck}^2)^{1/4}\left[2\sqrt{1 + \frac{\sin^2(\rho K_\varphi(v_i))}{\rho^2} - \frac{2GM}{\sqrt{k_i\ell_{\rm Planck}^2}}}\,\ell_{\rm Planck}^2\mu_i - (k_i - k_{i-1})\ell_{\rm Planck}^2\right]T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi). \quad (7)$$

Seeking, for simplicity, a solution of the form $\Psi(K_\varphi, K_x, g, \vec{k}, M) = \sum_{v\in g}\sum_{\mu(v)} T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi)\Psi[\mu(v), M]$ (one could also consider superpositions in $\vec{k}$), the equation $\hat{H}(N)\Psi = 0$ can be solved and leads to

$$\Psi(K_\varphi, K_x, g, \vec{k}, M) = \exp(f(K_\varphi, g, \vec{k}, M))\Pi_{e_j\in g}\exp\left(\frac{i}{2}k_j\int_{e_j}K_x(x)dx\right), \quad (8)$$

with f given by $f = \sum_{v_j\in g} -(i/2)\Delta K_j m_j \times F[\sin((\rho K_\varphi(v_j), im_j)]$, with $\Delta K_j = K_\varphi(v_j) - K_\varphi(v_{j-1})$, $m_j = [\rho\sqrt{1 - 2GM/\sqrt{k_j}\ell_{\rm Planck}}]^{-1}$, and $F(\phi, m) = \int_0^\phi (1 - m^2\sin^2 t)^{-1/2}dt$ the Jacobi (sometimes also known as Legendre) elliptic function of the first kind. Although m_j is purely imaginary inside the horizon, one can show that $\exp(f)$ is a pure phase factor for any value of k_j, no matter if it corresponds to the black hole interior or not.

The above solution of the Hamiltonian constraint is not invariant under diffeomorphisms. That can be readily corrected via standard group averaging in which one superposes a family of states related by diffeomorphisms [11]. For reasons of space, we do not show it explicitly since the construction is standard. One ends up with states that are superpositions of spin networks with vertices in all possible positions along the radial line, preserving the order of them (this last point will play a crucial role in the appearance of new quantum observables). In higher dimensions, such order could be associated with the diffeoinvariant nontrivial knotting of the spin networks. The resulting state is a functional of K_x, K_φ labeled by a diffeomorphism-related class of graphs $\tilde{g}$, the colors for each edge $\vec{k}$, and the ADM mass M. We denote them as $|\vec{k}, \tilde{g}\rangle$, omitting the dependence on M for simplicity. These vectors define a basis for the physical space of states $\mathcal{H}_{\rm phys}$.

The graph g is based on an integer number V of vertices located at $x(v_1), \ldots, x(v_V)$. Since the elements of the basis of the states of $\mathcal{H}_{\rm phys}$ have a well-defined number of vertices, one can construct an Dirac observable $\hat{V}$ that acting on $|\vec{k}, \tilde{g}\rangle$ has as eigenvalue the integer number V. This is an observable that has no classical counterpart.

Even more interesting is the observable associated with the sequence of monotonically growing integers $\vec{k}$ that characterize the sequence of characteristic radii of black holes. The Hamiltonian constraint does not change the values of $\vec{k}$ and neither does the diffeomorphism constraint. In the classical theory, the radial coordinate E^x is diffeomorphic to x^2, but since it can only be a monotonically growing function, this restricts the types of transformations allowed. To yield a nonmonotonic function, one would have to consider diffeomorphisms that are not invertible. That restriction is what in the quantum theory ends up yielding a new observable. If one were in more than one dimension, there are similar, more complex restrictions arising from the knotting of spin networks.

An operator associated with the sequence $\vec{k}$ that is a Dirac observable acting on the physical space of states is $O(z)$ with $z\in[0,1]$, $\hat{O}(z)|\vec{k}, \tilde{g}\rangle_{\rm phys} = \ell^2_{\rm Planck}k_{{\rm Int}(Vz)}|\vec{k}, \tilde{g}\rangle_{\rm phys}$, where ${\rm Int}(Vz)$ is the integer part of Vz, and V is the number of vertices. These quantum operators characterize the quantum geometry, and as we shall see, they may have profound physical implications. On the physical Hilbert space $\mathcal{H}_{\rm phys}$, M is a Dirac observable but $E^x(x)$ is not. However, given an arbitrary monotonic function from the interval of the radial direction we are studying (for instance the origin and an asymptotic boundary $[0, x_+]$) to the interval [0, 1], which we call $z(x)$, one has that $\hat{E}^x(x)|\vec{k}, \tilde{g}\rangle_{\rm phys} = \hat{O}(z(x))|\vec{k}, \tilde{g}\rangle_{\rm phys}$. The function $z(x)$ characterizes the gauge freedom in E^x. Recall that the eigenvalues of $\hat{E}^x$ in the kinematical Hilbert space can only take the values $\ell^2_{\rm Planck}k_i$. Another way of understanding this is that we are defining an evolving constant of the motion associated with E^x that is a function of a "parameter" given by the function $z(x)$ and the observable $\hat{O}(z)$.

In a similar fashion, one can define evolving constants of the motion that represent the metric of space-time acting on $\mathcal{H}_{\rm phys}$. The parameters of the evolving constant are K_φ subject to suitable boundary conditions and $z(x)$. For instance, the classical expression of the g_{tx} component of the metric is

$$g_{tx} = g_{xx}N_r = -\frac{(E^x)' K_\varphi}{2\sqrt{E^x}\sqrt{1 + K_\varphi^2 - \frac{2GM}{\sqrt{E^x}}}}, \quad (9)$$

with similar expressions for g_{tt} and g_{xx}. This can be derived choosing a gauge in which K_φ and E^x are given functions of space and preserving the gauge fixing conditions in time through the determination of the lapse and the shift. For instance, this would lead to the usual form of the Schwarzschild metric, provided one chooses $K_\varphi^0 = 0$ and $E_0^x = x^2$. We can proceed to promote it to a quantum operator on $H_{\rm phys}$ in terms of $\hat{O}(z)$, $\hat{M}$ and parametrized by K_φ and $z(x)$ (the expression needs to be made well defined first by introducing holonomies),

$$\hat{g}_{tx} = -\frac{(\hat{E}^x)'\sin(\rho K_\varphi)}{2\rho\sqrt{\hat{E}^x}\sqrt{1 + \frac{\sin^2(K_\varphi)}{\rho^2} - \frac{2G\hat{M}}{\sqrt{\hat{E}^x}}}}. \quad (10)$$

The square root that appears in $\hat{g}_{tx}$ leads to the following inequality in order to get a self-adjoint operator (notice that there are no factor ordering issues): $1 + (\sin\rho K_\varphi/\rho)^2 - (2GM/\sqrt{E^x/\epsilon}) \geq 0$. The most unfavorable point is when the eigenvalues of E^x become small. The most favorable gauge choice from the point of view of keeping the expression positive at that point is $K_{\varphi,0} = \pi/(2\rho)$, where $K_{\varphi,0}$ is K_φ evaluated at $x = 0$ (since E^x is monotonic, the worst case happens at $x = 0$). Therefore, the gauge condition for the square root that appears in the metric to be real and therefore the metric self-adjoint is, in terms of the eigenvalues of $\hat{E}^x$, given by

$$k_0 > \left(\frac{2GM}{\ell_{\text{Planck}}(1+\frac{1}{\rho^2})}\right)^2.$$

As a consequence, small values of k_0 are excluded in order to have a self-adjoint metric operator, and as a consequence, the singularity is avoided. The region exterior to the horizon is covered for any gauge K_φ since the last term in the first inequality is less or equal to 1 outside the horizon. Notice that there exist gauge choices that would make the metric singular. Those correspond to coordinate singularities and loop quantum gravity correctly does not eliminate them.

Although these general considerations about the geometry are true for any state, obviously not all states lead to semiclassical geometries. To begin with, the operators corresponding to the metric are distributional, having support only at vertices of the spin net. One can envision them approximating smooth geometries for spin nets with densely packed vertices and some additional conditions: (a) One would need to consider a superposition of ADM masses with a weight $\Psi(M)$ that are peaked functions around a given value M_0; (b) one also needs to require some smoothness in the radial coordinate by limiting the jumps of the eigenvalues from a vertex to the next, i.e., $\Delta k_i < \Delta_0$ with Δ_0 controlling the level of smoothness. One could, in particular, approximate a smooth Schwarzschild geometry with quantum corrections. Also notice that for simplicity we have also kept the discussion in terms of states that are eigenstates of $\hat{O}(z)$. In reality, one will have states that will involve superpositions of values of $\vec{k}$ as well.

The analysis can be extended to the interval $[-x_+, x_+]$ with a trivial extension of $O(z)$ to $z \in [-1, 1]$. The expectation value of the determinant of the space-time metric can be explicitly calculated in a suitable gauge, and it goes through a maximum value and starts decreasing for negative values of x. One can view this as a generalization of the Kruskal extension including a new region that tunnels through the singularity.

We have performed a loop quantization of the vacuum spherically symmetric space-times. Apart from using variables adapted to spherical symmetry, we did not perform any additional gauge fixing. Through a rescaling of the Hamiltonian constraint, the constraint algebra was turned into a Lie algebra. We were able to exactly solve the constraints and find the space of physical states. We encountered that in addition to the ADM mass and its canonically conjugate momentum, other Dirac observables arose in the quantum theory associated with the bulk of the space-time. The metric of the space-time can be analyzed as an operator in the physical space of state viewing it as an evolving constant of the motion written in terms of the Dirac observables and free parameters that represent the coordinate freedom. One sees that the singularity that arises in the classical theory is eliminated and is replaced by a region of high curvature through which the space-time could be extended, yielding a global structure similar to that of the Reissner-Nordström space-time but without singularities, as had been anticipated in a previous treatment using the effective semiclassical theory [7].

The existence of the new quantum observables and the associated degrees of freedom may have some relevance for the recent discussion of "firewalls" in black hole evaporation. Almheiri *et al.* [12] (and earlier, Braunstein *et al.* [13]) showed that in order to preserve the unitarity of the S matrix during black hole evaporation, drastic changes in the usual picture were needed, like surrounding the black hole with a firewall. This follows from fundamental hypotheses, like the existence of a unitary S matrix that describes the evolution of the incoming pure state that forms the black hole and the outgoing Hawking radiation. From the perspective of our analysis, this hypothesis is not obvious since in principle there could be part of the information lost when falling into the black hole interior tunneling into another region or into the new local degrees of freedom we discussed. Our analysis is at the moment limited to the vacuum case. However, from the form of the Hamiltonian constraint coupled to matter, one can see that the bulk observables persist in that case, suggesting that the analysis of the information issue made could be carried out explicitly in the case of an evaporating black hole.

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