Observing Geometric Frustration with Thousands of Coupled Lasers

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Geometric frustration, the inability of an ordered system to find a unique ground state plays a key role in a wide range of systems. We present a new experimental approach to observe large-scale geometric frustration with 1500 negatively coupled lasers arranged in a kagome lattice. We show how dissipation drives the lasers into a phase-locked state that directly maps to the classical XY spin Hamiltonian ground state. In our system, frustration is manifested by the lack of long range phase ordering. Finally, we show how next-nearest-neighbor coupling removes frustration and restores order.

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Geometric frustration may arise in ordered systems when competing interactions forbid simultaneous minimization of all interaction energies. Geometric frustration in either quantum or classical systems arises from the same source: a macroscopic degeneracy of the ground state of the Hamiltonian, and will therefore arise for both in the same geometry, resulting in finite entropy even at zero temperature [1–5]. Geometric frustration is manifested, in certain synthetic exotic materials, by the lack of magnetic ordering even when cooled to very low temperatures [4–7]. Recently, frustrated spin systems were simulated by a small model system of three trapped ions [8], and are currently being pursued in optical lattices, where antiferromagnetic spin systems have already been simulated [9–11] but not yet with frustrated geometries [12]. All these systems are externally cooled to their ground state so as to reveal frustration.

Here, we demonstrate a new experimental approach to couple and phase lock thousands of lasers in any 2D geometry. We observe geometric frustration with 1500 negatively coupled lasers arranged in a kagome lattice with nearest neighbor interactions. In laser systems, coupling by means of mutual light injection from one laser to another introduces losses which depend on the relative phase between the lasers and can lead to phase locking [13–16]. Such dissipative coupling [14] drives the system to a steady state solution that can be directly mapped to the ground state of the classical XY spin Hamiltonian [17]. Mapping of the phase of each laser to the angular orientation of a planar spin means that the highly degenerate and frustrated spin ground state can be simulated even when operating the lasers at room temperature. Photonic devices such as mode locked lasers [18], high Fresnel number laser cavities [19,20], and coupled photonic crystal waveguides [21,22] have been used in the past for observing the dynamics of interacting systems and thermodynamic phenomena. We believe that coupled laser systems offer a wide range of tunability and can be useful experimental tools to simulate interactions of large-scale systems using photonic devices.

Our experimental system is based on a unique degenerate laser cavity design which is presented schematically in Fig. 1(a). It includes a 10 mm wide Nd-Yag gain medium, a high reflecting mirror and a 90% reflectivity output coupler (O.C.), and two lenses in a $4f$ telescope that image one end of the cavity (O.C. plane) to the other end of the cavity (mirror plane). This assures that any field distribution will be reimaged on itself after each cavity round trip; hence, any field is an eigenmode of this high Fresnel number degenerate laser cavity [19,20]. A mask of apertures, placed adjacent to the O.C., forms independent near-Gaussian laser channels arranged in a desired two-dimensional lattice [Fig. 1(b)]. The $4f$ telescope eliminates diffraction from one laser to another, ensuring the lasers remain well localized and uncoupled.

Phase independence among the uncoupled lasers is verified by comparing the far-field intensity of a single laser to that of 1700 lasers [Fig. 1(c)]. As evident, the lack of interference fringes and essentially identical distributions indicate no phase correlation between different lasers in the lattice, as expected for the degenerate cavity [16]. Coupling between the different lasers is introduced gradually by displacing the O.C. further away from the mask plane so light diffracted from each laser is now coupled into its neighboring lasers. As the coupling strength depends on the overlapping Gaussian tails of the free space diffracting laser beams [23], sufficiently strong nearest-neighbor (NN) coupling and negligible next-nearest-neighbor (NNN) coupling can occur concomitantly.

In general, lasers have many degrees of freedom such as phase, amplitude, lasing frequency, polarization and gain. In an effort to reduce amplitude and frequency variations as well as spontaneous emission noise, the lasers operate well above lasing threshold and with coupling strengths well above the critical coupling value [24,25]. The use of the same two flat cavity mirrors assured that all lasers have similar cavity length and thereby near common lasing frequencies. Additional efforts to reduce disorder were made (see Ref. [26]). These efforts allowed us to consider

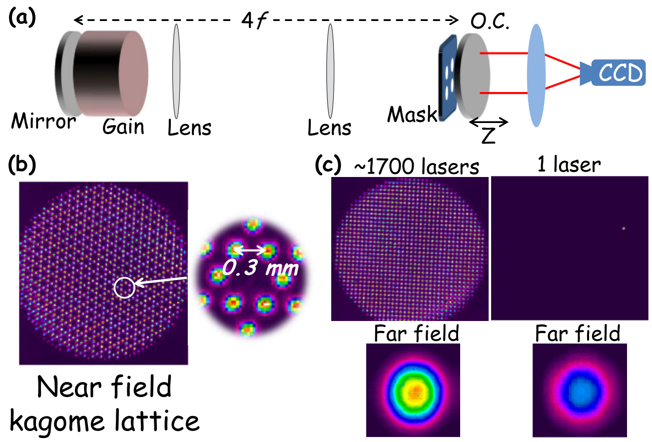

FIG. 1 (color online). Experimental arrangement for coupling more than a thousand independent lasers. (a) An arrangement of a degenerate cavity that supports many independent lasers. It is comprised of gain medium, a back mirror and an O.C. placed at either ends of the cavity, a mask of apertures that forms the many laser channels and two lenses in a $4f$ telescope arrangement that images the mask plane to the back mirror plane. A CCD camera positioned at the focal plane of a focusing lens detects the far-field interference pattern, from which the relative phase between the lasers is determined. (b) An example of the experimental near-field intensity pattern of ~1500 individual Gaussian laser beams arranged in 2D kagome lattice. (c) Phase independence among the different lasers is experimentally verified by comparing the far-field intensity distribution from a single laser to that from 1700 lasers.

a simplified model whereby only the lasers phases are taken into account.

Assuming equal amplitudes and negligible disorder, the equations that describe the lasers phase dynamics $\varphi_i(t)$, reduce to a simple form of the Kuramoto model [17] which describes a general class of oscillators

$$\frac{d\varphi_i}{dt} = \kappa \sum_{j=(i)_{\rm NN}} \sin(\varphi_j - \varphi_i), \tag{1}$$

where κ is the coupling coefficient and the sum $j = (i)_{\rm NN}$ is over laser i's NN. This dynamics describe an over damped motion under a potential flow $\dot{\vec{\varphi}} = -\vec{\nabla}V(\vec{\varphi})$ where,

$$V(\varphi) = -\kappa \sum_{(i,j)_{\rm NN}} \cos(\varphi_j - \varphi_i). \tag{2}$$

Thus, the steady state solution of Eq. (1) occurs when the potential $V(\vec{\varphi})$ is minimal and exactly equals the ground state solution to the classical Hamiltonian for the XY model given by

$$\mathcal{H} = -J \sum_{(i,j)_{\rm NN}} \cos(\varphi_j - \varphi_i), \tag{3}$$

where J is the interaction energy term and φ_i describes the orientation of the classical planar spin $\vec{\sigma}_i = (\cos\varphi_i, \sin\varphi_i)$. Hence, by mapping the phase of each laser to the orientation of a classical planar spin [26], the ground state of the classical XY Hamiltonian is simulated by simply observing the steady state of the coupled lasers [17]. Intuitively, this result reflects that fact that mode competition in each laser, which favors states with minimum losses, drives their dissipative motion to these minimal loss phase locked states [14].

In our experiments we place the output coupler near the quarter Talbot distance from the mask so that coupling between NN lasers becomes negative [23], thus driving the relative phase between them towards π [27,28]. The negative coupling ($\kappa < 0$) is mapped to an antiferromagnetic interactions ($J < 0$), which for some frustrated geometries, has a large accidental degeneracy that is not related to the symmetry of the Hamiltonian [4,5,29,30].

For a triangle of three lasers with negative coupling, a state where all pairs of NN lasers phase lock with a relative π phase cannot exist. Thus, the minimal-loss steady-state solution has the relative $\pm 2\pi/3$ phase between neighboring lasers, as for spins with antiferromagnetic interactions (top left of Fig. 2). This yields a twofold Z_2 discrete degeneracy in addition to the continuous U(1) degeneracy of a uniform rotation of all phase [24]. In one degenerate state, the phase winds clockwise along the triangular plaquette whereas in the other it winds counter clockwise, giving rise to either positive or negative chirality. For a triangle lattice, the $\mathrm{U}(1) \times Z_2$ degeneracy extends throughout the entire lattice, where plaquettes with positive and negative chiralities regularly alternate with each other, forming long range phase ordering [Fig. 2(a)]. The experimental results for the near and far field patterns obtained for a triangular lattice of above 2000 lasers are shown in Fig. 2(a). The high contrast far-field interference pattern indicates that a high degree of phase locking occurs between the lasers. Specifically, the dark center and six sharp intense "Bragg" peaks along the edges of the first Brillioun zone indicate antiferromagnetic phase ordering throughout the entire system [26].

A kagome lattice is formed by removing 1/4 of the sites from a triangular lattice such that any two triangular plaquettes share at most only one common site [Fig. 2(b)]. Here again, each plaquette in the steady state solution has either positive or negative chirality. However, since the plaquettes now interact via only a single site, the ordering constraint of the chiralities is substantially less restrictive than in the triangular lattice. This leads to a highly frustrated system where, in addition to the continuous U(1) degeneracy, an exponentially large discreet degeneracy exists, resulting in a finite residual entropy per site [3–5,29,30]. Consequently, the number of phase-locked states with the same minimal losses exponentially increases with the lattice size [3]. Specifically, even when all lasers are phase locked, long range phase ordering does not occur. The experimental results for the near and far

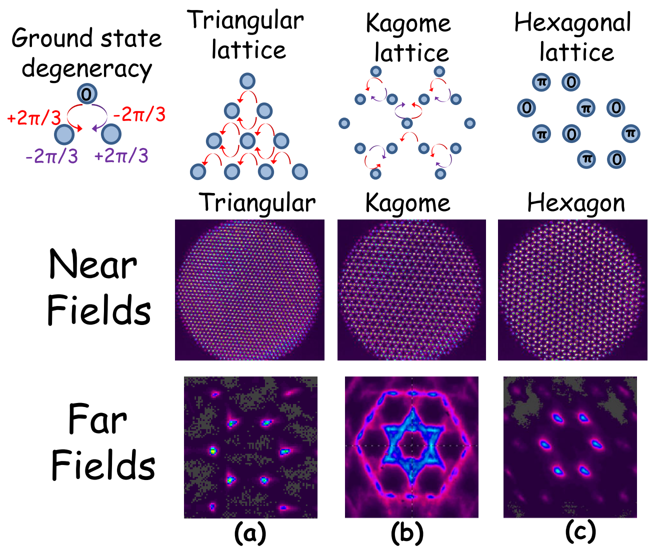

FIG. 2 (color online). Near and far field experimental results for negatively coupled lasers arranged in a triangular, kagome, and hexagonal lattices. The top left corner schematically shows the two degenerate phase-locking minimal-loss states that occur in each triangle of laser. In one, the phase increases by $2\pi/3$ in the clockwise direction whereas in the other it increases by $2\pi/3$ in the counter clockwise direction, giving rise to a vortex with either positive or negative chirality. (a) Experimental results for a triangular lattice of ~2000 lasers where phase locking occurs with triangular plaquettes with regularly alternating positive and negative chiralities. This causes long range phase ordering as observed by the sharp intense "Bragg" peaks in the far-field pattern. (b) Experimental results for a kagome lattice of ~1500 lasers where phase locking occurs with the chirality of one triangular plaquette not dictated by the chirality of another. This causes frustration resulting in a lack of long range phase ordering as observed by the lacking of sharp peaks in the far-field pattern. The "bowties" observed in the far field pattern are a signature of a frustrated ground state of a kagome lattice similar to the one in (5). (c) Experimental results for a hexagonal lattice of ~1300 lasers where phase locking occurs with neighboring lasers locking to a relative π phase shift. This causes long range phase ordering as is observed by the sharp peaks in the far-field pattern.

field patterns obtained for a kagome lattice of ~1500 lasers are shown in Fig. 2(b). Here again, the center of the far-field pattern is dark indicating near perfect phase locking within each plaquette. However, unlike for the triangular lattice, here the light is spread out over a large area along the edges of the first Brillouin zone and is no longer concentrated in sharp peaks, a clear signature of the lack of long range phase ordering caused by the frustration. This lack of long range phase ordering was verified by directly measuring the phase correlations between pairs of lasers at different distances (see Ref. [26]).

The removal of sites from a triangular lattice does not always lead to frustration. On the contrary, in some cases removal of sites leads to a bipartied lattice that has only one ground state. The hexagonal, (graphene) lattice [Fig. 2(c)], is constructed by removing 1/3 of the sites from a triangular lattice [22]. The system occupies a phase distribution state that has NN lasers phase locked with a relative π phase resulting in antiferromagnetic long range phase ordering, as verified by the experimental results [Fig. 2(c)].

In general, the technique most commonly used for determining the structure factor S_K of a lattice is to measure the angular dependence of its scattering cross section. Using a Monte Carlo simulation [3,29], we calculated the ground state structure factor of the XY model on a frustrated kagome lattice of 42 spins [Fig. 3(a) right inset], and compared it to the experimental far-field interference pattern of the lasers [Fig. 3(a) left inset]. Figure 3(a) presents the cross section along a closed trajectory of points Γ, K, and M appearing in the insets. Error bars denote the standard deviation of the cross sections calculated along the other five lobes of the far field intensity pattern (for a detailed account of the contribution factors to the errors see Ref. [26]). The blue solid line is the structure factor calculated from the Monte Carlo simulations and the red circles are along the lasers far field intensity profile. The good agreement indicates that our lasers indeed simulate a state that is very near a true zero temperature ground state of the XY model [26].

Finally, we considered the interactions between NNN lasers, which were predicted to have a significant effect on frustrated magnetism in a kagome lattice [29,30]. As NNN coupling adds another interaction term between neighboring plaquettes, the number of possible degenerate chirality configurations is substantially reduced leading to $U(1) \times Z_2$ degeneracy and restoring long range phase ordering. Moreover, phase transitions that are associated with either proliferations of domain walls or with unbinding of vortex pairs are expected to occur with both ferromagnetic and anti-ferromagnetic NNN interactions [24,25]. The NNN coupling is readily implemented in our lasers system by further displacing the output coupler, thereby increasing the amount of free space diffraction and consequently the overlap of the diffracted Gaussian tails in a well controlled and calibrated manner [23]. To quantitatively determine the transition into long range phase ordering, we calculated the widths of the Bragg peaks in the far field interference pattern, which indicate the range of phase ordering. Figure 3(b) presents the inverse average widths of the four narrowest Bragg peaks as a function of the coupling strength between NNN lasers. The Bragg peak widths are normalized by the Brillouin zone width π/d, where $d = 300\ \mu$m is the distance between lasers. Error bars denote the errors of the measured width arising from the finite pixel size in our CCD camera, and are thus more substantial for the widths of the narrow peaks. For NNN coupling of $\kappa = 0.008$, frustration leads to Bragg peaks widths that are only slightly narrower than the width of the Brillouin zone, indicating that phase ordering occurs only between neighboring lasers. For NNN coupling of $\kappa = 0.014$, the

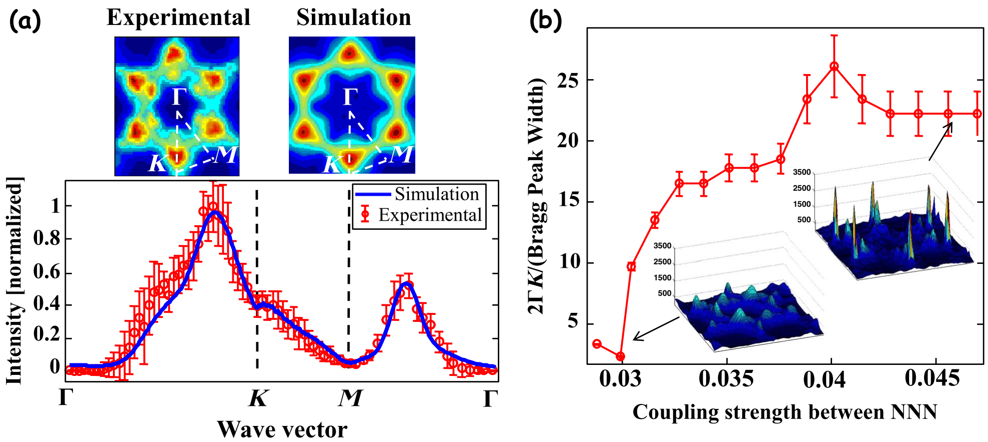

FIG. 3 (color online). Quantifying the phase locking state on a kagome lattice and the effect of next nearest neighbor interactions. (a) Top: the calculated magnetic structure factor for zero temperature ground states of the XY model using Monte Carlo simulations, and the experimental far field interference pattern of the lasers. Bottom: cross sections of the calculated structure factor along a closed trajectory of points Γ, K, and M (blue solid line), in good agreement with the experimental far-field intensity profile of the lasers (red circles). Error bars denote the standard deviation of the cross sections of all six lobes in the far-field intensity pattern. (b) The inverse widths (at $1/e$) of the Bragg peaks in the experimental far field interference pattern, indicating the range of phase ordering, as a function of the calculated NNN coupling strength. The widths are normalized by the Brillouin zone width, (twice the reciprocal lattice vector $\pi/2d$ where d is the distance between neighboring lasers). The left insets show the wide Bragg peaks for a frustrated kagome lattice with NNN coupling of $\kappa = 0.008$. The right inset shows the narrow Bragg peaks that occur when NNN coupling is $\kappa = 0.014$, indicating phase ordering throughout the entire system.

kagome lattice exhibits Bragg peaks widths that are >20 times narrower than the width of the Brillouin zone. Such a narrowing indicates that there is no frustration and now phase ordering occurs between lasers positioned >20 sites apart; i.e., phase ordering occurs throughout the entire system.

The wide range of tunability and simplicity offered by coupled lasers makes them an ideal and highly versatile platform for probing additional aspects of geometric frustration as well as other more general phenomena of quasi-lattices and disorder. For example, in a kagome lattice, coupled lasers can be exploited for determining how the NNN interaction sign affects the transition temperature to an ordered state, an unresolved issue is still disputed among theoreticians [29,30]. Imaginary coupling [23] could be used to implement the Dzyaloshinsky-Moriya interactions thereby giving rise to many intriguing investigations related to symmetry breaking [31,32]. The effects of defects on frustrated systems and their influence on the glass transition could now be explored on the single-particle level [33,34]. The effects of quantum noise in large scale systems can be explored by operating the lasers near the lasing threshold [24,25]. Finally, by removing the mask, it is possible to investigate the interactions between large numbers of modes in a high Fresnel number laser cavity [19,20].

The ability to control the coupling range offers new opportunities to address open questions. For example, resolve whether a second order phase transition to a phase-locked state would occur as predicted by the Kuramoto model [17] or whether strong chaotic amplitude fluctuations [35] would lead to a first order phase transition [36]. Global coupling can also help to investigate the intriguing millennium bridge synchronization [37]. Furthermore, by resorting to an intracavity spatial filter tailored to achieve a desired complex coupling pattern, it should be possible to implement the RKKY [38] interactions and observe the Edwards-Anderson model for the glassy phase [39]. Finally, phase locking a very large number of lasers has important implications for laser beam combining as a means to obtain high brightness laser sources [40].

To conclude, a new experimental approach to observe large-scale geometric frustrated magnetism with 1500 negatively coupled lasers in a kagome lattice was implemented. We showed that dissipation drives the coupled lasers into a phase-locked state whereby frustration is manifested by the lack of long range phase ordering as directly observed from the far field interference pattern. In addition, we demonstrated how the introduction of next-nearest-neighbor interaction removes the frustration and restores long range phase ordering throughout the entire system.

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