Redundancies in Nambu-Goldstone Bosons

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We propose a simple criterion to identify when Nambu-Goldstone bosons for different symmetries are redundant. It solves an old mystery why crystals have phonons for spontaneously broken translations but no gapless excitations for equally spontaneously broken rotations. Similarly for a superfluid, the Nambu-Goldstone boson for spontaneously broken Galilean symmetry is redundant with phonons. The most nontrivial example is Tkachenko mode for a vortex lattice in a superfluid, where phonons are redundant to the Tkachenko mode which is identified as the Boboliubov mode.

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Introduction.—In many areas of physics, it is important to study consequences of microscopic physics on macroscopic behaviors, sometimes called *emergent* phenomena. One of the best examples in this category is the existence of gapless excitations, called Nambu-Goldstone bosons (NGBs), when global continuous symmetries are spontaneously broken [1].

For spontaneously broken internal symmetries in Lorentz-invariant systems, the symmetries dictate the number (n_{NGB}), dispersion relation, and interactions of NGBs completely. The present authors have generalized this well-known result to systems without Lorentz invariance, and proved a general formula [2]

$$n_{\mathrm{NGB}} = \dim(G/H) - \frac{1}{2}\mathrm{rank}\rho, \tag{1}$$

$$\rho_{ab} = -i\lim_{B\downarrow 0}\lim_{V\uparrow\infty}\frac{1}{V}\langle 0|[Q_a, Q_b]|0\rangle. \tag{2}$$

Here, $|0\rangle$ is a ground state of the finite volume system in the presence of external fields B_α that favor order parameters. Note that the symmetry breaking pattern itself is not sufficient to fix the number of NGBs and the additional information on the ground state, ρ, is required [3]. Here and hereafter, whenever we refer to broken generators Q_a, we mean suitable large-volume limits $\lim_{V\uparrow\infty}\int_V d^dx j_a^0(x)$, where $j_a^0(x)$ is the Noether charge density and d is the spatial dimension.

In the case of spacetime symmetries, however, the counting of NGBs is more subtle. Even for Lorentz-invariant systems, some examples elude the above rule for internal symmetries, e.g., spontaneously broken conformal and scale invariance [4]. There is an empirical prescription called *inverse Higgs mechanism* that allows one to identify possible constraints that *can* be imposed among NGBs [5], while it does not dictate if they *should* be imposed. Little is known for theories without Lorentz invariance.

In this Letter, we propose a simple criterion to determine what redundancies exist among NGBs in a given system. Redundancies can arise for two separate reasons: (1) special property of the ground state annihilated by a linear combination of symmetry generators, and (2) identities among Noether charge densities. It is complementary to the inverse Higgs mechanism because our criterion *requires* redundancies.

This result was inspired by the work by Low and Manohar [6], which pointed out that a *local* transformation of different symmetries may lead to the same field configurations. But they did not clearly distinguish the classical field configurations and quantum states and operators, and restricted themselves to Lorentz-invariant systems. We need to generalize their intuition and formulate it more concretely.

Noether constraints.—A symmetry is spontaneously broken if its generator Q_a ($a = 1, \ldots, n_{\mathrm{BG}}$) has an order parameter $\langle 0|[Q_a, \Phi_b(y)]|0\rangle \neq 0$. By inserting a complete set of states, one finds the existence of gapless states $|\pi_\beta\rangle$ such that $\langle\pi_\beta(\vec{p})|j_a^0(x)|0\rangle \neq 0$ for some a and $\lim_{\vec{p}\to 0}E_\beta(\vec{p}) = 0$.

We first point out that this general theorem immediately tells us the NGBs are redundant if a linear combination of Noether currents annihilate the ground state for nonzero coefficients c_a,

$$\int d^dx\sum_a c_a(x)j_a^0(x)|0\rangle = 0. \tag{3}$$

We call them Noether constraints. In general, the coefficients $c_a(x)$ are spacetime dependent. To see the redundancy, we use proof by contradiction. Suppose that all $|\pi_\beta\rangle$s are linearly independent to each other. If we multiply $\sum_\beta|\pi_\beta\rangle\langle\pi_\beta|$ on the above equation,

$$\int d^dx\sum_{a\beta} c_a(x)|\pi_\beta\rangle\langle\pi_\beta|j_a^0(x)|0\rangle = 0. \tag{4}$$

Because $\langle\pi_\beta|j_a^0(x)|0\rangle \neq 0$, this relation is nothing but the linear dependence among the NGB states and hence

contradicts with our assumption. One can eliminate one of these states by solving Eq. (4) and repeat the argument on the remaining NGB states and Noether constraints. This way, we can see that the would-be NGB states have redundancies by the number of independent Noether constraints Eq. (3).

The rest of the discussion is how such Noether constraints arise in two general categories.

Internal symmetries.—Let us first look at a simple example: Heisenberg ferromagnet. Our argument on internal symmetry assumes the translational invariance of the ground state. Knowing that the order parameter is the uniform magnetization, we consider the Hamiltonian $\tilde{H} = J\sum_{\langle ij\rangle}\vec{s}_i \cdot \vec{s}_j - \mu B_z \sum_i s_{zi}$ and its ground state in which all spins are aligned in the positive z direction. For any $B_z > 0$, the raising operator $S_+ = \sum_i (s_{xi} + is_{yi})$ must annihilate the ground state, since otherwise $S_+|0\rangle$ has a lower energy $(-\mu B_z)$ than the ground state. By taking the thermodynamic limit and successively turning off the magnetic field $B_z \downarrow 0$, we obtain the Noether constraint,

$$\sum_i (s_{xi} + is_{yi})|0\rangle = 0. \tag{5}$$

The states created by two broken charges, S_x and S_y are hence not independent. Indeed, it is known that there is only one magnon (quantized spin wave) state, consistent with Eq. (2).

In general, it is common to introduce an external field for each order parameter so that the selected ground state has a proper thermodynamic limit. Therefore, for antiferromagnets, we should consider the limit of vanishing staggered magnetic field to get such a ground state. As a result, Eq. (5) is not satisfied and we can see S_x and S_y excite independent NGBs.

A similar phenomenon has been discussed in a relativistic field theory with a chemical potential [7]. The model consists of a two-component scaler field $\psi(x)$ with U(2) global symmetry generated by the Pauli matrices τ_i and the identity matrix τ_0. The field acquires the vacuum expectation value, say, $\langle 0|\psi(x)|0\rangle = (v, 0)^T$, breaking generators Q_1, Q_2, and $Q_3 + Q_0$ spontaneously. Because of the chemical potential, $(j_3^0 + j_0^0)(x)$ also develops a nonzero expectation value. By applying an external field to this density, we can derive the Noether constraint $\int d^d x (j_1^0 + ij_2^0)(x)|0\rangle = 0$, resulting in one less NGB than the number of broken generators [7].

In order to generalize our argument to an arbitrary internal symmetry group, let us consider a simple Lie group G. (Since all Lie groups can be decomposed into simple groups and U(1) factors, extension to the most general case is straightforward.) We choose the basis of generators in such a way that only $j_1^0(x)$ may have a nonvanishing expectation value [8]. Let Q_α $(\alpha = 1, \ldots, \mathrm{rank}G)$ be Cartan generators of G and $Q_{\pm,\sigma} = Q_{\sigma R} \pm iQ_{\sigma I}$ be raising and lowering operators such that $[Q_1, Q_{\pm,\sigma}] = \pm q_\sigma Q_{\pm,\sigma}$ and $q_\sigma > 0$ $(\sigma = 1, \ldots, m)$. If $\langle 0|j_1^0(x)|0\rangle$ is nonzero, it serves as an order parameter of spontaneously broken generators $Q_{\sigma R,I}$. Hence, it is legitimate to introduce an external field B_1 as $\tilde{H} = H - B_1 Q_1$, in addition to other external fields, if necessary. Assuming the commutativity in taking vanishing limits for each external field, we obtain Noether constraints

$$\int d^d x (j_{\sigma R}^0 + ij_{\sigma I}^0)(x)|0\rangle = 0 \tag{6}$$

for $\sigma = 1, \ldots, m$. As a consequence, n_{NGB} reduces by the number of constraints m.

If we rearrange the broken generators as $Q_a = (Q_{1R}, Q_{1I}, \ldots, Q_{mR}, Q_{mI}, \ldots)$, the matrix ρ defined in Eq. (2) takes the block diagonal form where each 2 by 2 blocks reads $\lim_{B\downarrow 0}\lim_{V\uparrow\infty}\langle 0|j_1^0(0)|0\rangle i(q_\sigma/2)\tau_y$. Therefore, the rank of ρ is precisely $2m$, as predicted by the counting rule Eq. (1).

Spacetime symmetries.—Another reason for redundancies is when the Noether charge densities are linearly dependent. Namely, $\sum_a c_a(x) j_a^0(x) = 0$ as an operator identity, and the redundancy is obviously independent of the property of the ground state.

To illustrate the point, let us consider a simple crystal. The Lagrangian or Hamiltonian is both translationally and rotationally invariant, with six generators in three spatial dimensions. A crystal spontaneously breaks all six symmetries. However, it is well known that there are three gapless phonon excitations (two transverse and one longitudinal), but no more. We are not aware of satisfactory explanation for the lack of NGBs for rotational symmetries in the literature.

The crucial observation is that the Noether charge densities for translation T^{0i} and rotation R^{0i} are related by

$$R^{0i} = \epsilon_{ijk} x^j T^{0k}. \tag{7}$$

Therefore, what could have been NGBs for spontaneously broken rotational symmetries are redundant with those for spontaneously broken translational symmetries, hence only three NGBs. Note that x^i are *parameters* and not *operators* in quantum field theories. The NGB in helimagnets [9] with the Dzyaloshinskii-Moriya interaction can be understood in a similar manner.

A more nontrivial example is a superfluid. The matter field changes its phase under the particle-number symmetry U(1) as $\psi(\vec{x}, t) \to e^{i\theta}\psi(\vec{x}, t)$, while it changes both its argument and the phase under the Galilean boost by velocity $\vec{v}$, $\psi(\vec{x}, t) \to e^{i(m\vec{v}\cdot\vec{x} - (1/2)m\vec{v}^2 t)}\psi(\vec{x} - \vec{v}t, t)$ (we set $\hbar = 1$). The order parameter $\langle 0|\psi(\vec{x}, t)|0\rangle = \psi_0$ hence breaks one phase symmetry and three boost symmetries. However, there is only one gapless excitation, namely, the Bogoliubov mode. Recall that consideration of the

spontaneously broken Galilean invariance is crucial to the Landau's criterion for superfluidity.

The lack of independent NGBs for Galilean symmetry again can be seen in the operator identity that the Noether current for the Galilean boost $B^{i\mu}$ is related to the U(1) current as

$$B^{i\mu} = tT^{i\mu} - mx^i j^\mu. \tag{8}$$

Here and hereafter, the Greek index μ refers to the spacetime index, $0 = t, 1 = x, 2 = y, 3 = z$. It is straightforward to derive this identity from the Lagrangian density $\mathcal{L} = i\psi^\dagger\dot{\psi} - (1/2m)\vec{\nabla}\psi^\dagger\cdot\vec{\nabla}\psi - V(\psi^\dagger\psi)$. Since the translational invariance is not broken in the superfluid, T^{i0} does not create a gapless excitation, while those created by B^{i0} and j^0 are linearly dependent, hence redundant.

Vortex lattice.—Perhaps the most nontrivial example is the redundancy among NGBs in a vortex lattice. Rotating superfluids and atomic BEC form a triangular lattice of quantized vortices [10], spontaneously breaking the translational symmetry. It is known that the system supports a soft collective oscillation with a quadratic dispersion, so-called Tkachenko mode [11–14]. Since the Tkachenko mode is often associated with an elliptically polarized lattice vibration, one may naively expect the existence of the usual (Bogoliubov) phonon, which corresponds to the fluctuation of the superfluid phase. Until today, all prior works on the collective modes in the system have been based on the hydrodynamic theory. Although they seem to imply the absence of such a gapless mode, the reason for the missing mode has been left unclear.

To clarify the low-energy structure of the system, here we construct an effective Lagrangian. In order to discuss it from the symmetry-breaking point of view, we do not take into account the inhomogeneity due to trapping potential or the centrifugal potential. In other words, we focus on the region where the trapping potential almost cancels the centrifugal potential but still retains a finite particle density. Our system thus can be rephrased as bosons which couple to an effective uniform magnetic field $B_{\rm eff} = 2m\Omega/e_{\rm eff}$ as if they have a charge $e_{\rm eff}$. The effective Lagrangian for vortices in superfluids has been discussed in several papers [15], but they did not discuss the vortex lattice configuration. They also introduced several fields in addition to NG degrees of freedom, which is not suitable for our purpose.

Let us start with the standard Lagrangian [16],

$$\mathcal{L} = \frac{i}{2}(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) - \frac{1}{2m}|\vec{\nabla}\psi|^2 - V_{\rm trap}(\vec{x})\psi^\dagger\psi - \frac{1}{2}g(\psi^\dagger\psi)^2. \tag{9}$$

We restrict ourselves to two spatial dimensions and the zero temperature. To go to the corotating frame with the angular frequency $\vec{\Omega} = \Omega\hat{z}$, one makes the substitution $\partial_t \to \partial_t - \vec{\Omega}\times\vec{x}\cdot\vec{\nabla}$. Assuming a Bose-Einstein condensate, we plug $\psi = \sqrt{n}e^{-i\theta_{\rm tot}}$ into the Lagrangian and obtain

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) - \frac{1}{2m}|(\vec{\nabla} - im\vec{\Omega}\times\vec{x})\psi|^2 \\ &\quad - V_{\rm eff}(\vec{x})\psi^\dagger\psi - \frac{1}{2}g(\psi^\dagger\psi)^2 \\ &= n\mu - \frac{(\vec{\nabla}n)^2}{8mn} - V_{\rm eff}(\vec{x})n - \frac{1}{2}gn^2 \\ &\simeq \frac{1}{2g}[\mu - V_{\rm eff}(\vec{x})]^2,\end{aligned} \tag{10}$$

where $V_{\rm eff}(\vec{x}) \equiv V_{\rm trap}(\vec{x}) - (m/2)\Omega^2x^2$ and

$$\mu \equiv \dot{\theta}_{\rm tot} - \frac{1}{2m}(\vec{\nabla}\theta_{\rm tot} + m\vec{\Omega}\times\vec{x})^2.$$

In the third line (10), we integrated n out, keeping only the leading term in the derivative expansion [17].

If we neglect the effective potential $V_{\rm eff}(\vec{x})$, as we do so for the rest of the Letter, the Lagrangian possesses the *magnetic* translational symmetry,

$$\psi'(\vec{x} + \vec{a}, t) = \psi(\vec{x}, t)e^{im\vec{x}\cdot\vec{\Omega}\times\vec{a}}. \tag{11}$$

Because of the lack of Galilean invariance, the energy momentum tensor no longer satisfies $T^{0i} = mj^i$. Instead,

$$T^{0i} = mj^i - 2m\Omega\epsilon^{ij}x^j j^0. \tag{12}$$

In the vortex lattice system, both $P^i \equiv \int d^dx T^{0i}$ and $N \equiv \int d^dx j^0$ are spontaneously broken. However, according to our general criterion, the operator identity Eq. (12) suggests that T^{0i} and j^0 do not produce independent NGBs. We will explicitly verify this claim in the following.

In the presence of vortices, the phase $\theta_{\rm tot}$ contains singularities. We decompose $\theta_{\rm tot}$ into the regular part $\theta_{\rm reg}$ and the vortex part $\theta_{\rm sing}$; i.e., $\theta_{\rm tot} = \theta_{\rm reg} + \theta_{\rm sing}$. Since $\theta_{\rm sing}$ is only defined up to a smooth function, this decomposition is not unique and we will fix the ambiguity later. Because of the singularity, $\theta_{\rm sing}$ does no longer satisfy $d^2\theta_{\rm sing} = 0$. In fact, $(1/2\pi) * d(d\theta_{\rm sing})$ ($*$ is the Hodge dual) can be identified as the vortex current $j_{\rm vortex}$ [$j^\mu_{\rm vortex} = (1/2\pi)\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta_{\rm sing}$] that automatically satisfies the topological conservation law $d * j_{\rm vortex} = \partial_\mu j^\mu_{\rm vortex} = 0$.

Now let us introduce a continuum description of the vortex dynamics. Because the crystalline order breaks the magnetic translation, we introduce fields X^a that specify the position of the vortices. Here, we follow the notation in Ref. [18]: X^a is the Lagrangian coordinate frozen on the lattice, while x^i is the Eulerian coordinate. We fix the relation between X^a and x^i in such a way that $\vec{u}(\vec{x}, t) \equiv \vec{x} - \vec{X}(\vec{x}, t)$ represents the displacement from the equilibrium position $\vec{x}$. The vortex current in the continuum description can be

expressed as $j_{\text{vortex}} = *(1/2)m_0\epsilon_{ab}dX^a \wedge dX^b$ ($j^\mu_{\text{vortex}} = (1/2)m_0\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_\nu X^a \partial_\lambda X^b$) [18] where $m_0 = -(m\Omega/\pi)$ [16] is the number density of the vortices in the equilibrium.

By equating these two expressions for the topological current, we have $d(d\theta_{\text{sing}}) = -m\Omega\epsilon_{ab}dX^a \wedge dX^b$, which gives

$$d\theta_{\text{sing}} = -m\Omega\epsilon_{ab}X^a dX^b + d\chi. \tag{13}$$

A smooth function χ corresponds to the ambiguity mentioned above. We choose $\chi = m\Omega\epsilon_{jk}x^j X^k$ so that the explicit coordinate dependence drops from the Lagrangian. Assuming the triangular lattice and adding the corresponding elastic energy $E_{\text{el}}(\partial\vec{u}) \equiv (2C_1 + C_2)(\vec{\nabla}\cdot\vec{u})^2 + C_2(\vec{\nabla}\times\vec{u})^2$ (in the notation of Ref. [13]), we arrive at our effective Lagrangian,

$$\mathcal{L}_{\text{eff}} = \frac{1}{g}\mu^2 - E_{\text{el}}(\partial\vec{u}), \tag{14}$$

$$\mu = \dot{\theta}_{\text{reg}} - m\vec{\Omega}\cdot\vec{u}\times\dot{\vec{u}} - \frac{1}{2m}(\vec{\nabla}\theta_{\text{reg}} + 2m\vec{\Omega}\times\vec{u} - m\Omega\epsilon_{kl}u^k\vec{\nabla}u^l)^2. \tag{15}$$

The ground state of $H - \mu_0 N$ (N is the total number of particles) is characterized as $\theta_{\text{reg}} = \mu_0 t$ and $\vec{u} = \vec{0}$. $\mathcal{L}_{\text{eff}}$ describes the dynamics of fluctuation $\varphi \equiv \mu_0 t - \theta_{\text{reg}}$ and $\vec{u} = \vec{x} - \vec{X}$. Similar expressions can be found in Ref. [19] that discusses the vortex lattice in superconductors in a different context, but its derivation is empirical in contrast to ours based on symmetry and derivative expansion.

As a nontrivial test, let us derive hydrodynamic equations as the Euler-Lagrange equations of the effective Lagrangian. Variation with respect to θ_{reg} gives the continuity equation $\partial_\mu j^\mu = \partial_t n + \vec{\nabla}\cdot(n\vec{v}) = 0$ [20], where $n \equiv \mu/g$ and

$$\vec{v} \equiv -\frac{1}{m}(\vec{\nabla}\theta_{\text{reg}} + 2m\vec{\Omega}\times\vec{u} - m\Omega\epsilon_{kl}u^k\vec{\nabla}u^l). \tag{16}$$

Since we implicitly assumed that vortices are massless and hence $\vec{u}$ does not have the kinetic term $\propto \dot{\vec{u}}^2$, the equation of motion (EOM) of the displacement vector requires the balance between the Magnus force and the elastic force $\vec{F}_{\text{Magnus}} + \vec{F}_{\text{el}} = 0$, where $\vec{F}_{\text{el}} \equiv \delta E_{\text{el}}/\delta\vec{u}$ and $\vec{F}_{\text{Magnus}} = 2m\vec{\Omega}\times j^\mu\partial_\mu\vec{X} = 2mn\vec{\Omega}\times[\vec{v} - (\partial_t + \vec{v}\cdot\vec{\nabla})\vec{u}]$. All of these equations agree with those discussed in Refs. [12,13] based on the linearized hydrodynamic theory, which in turn verifies our effective Lagrangian. Note that our expressions are fully nonlinear as required by the symmetry, e.g., the third term in Eq. (16), beyond the linearized expressions in their papers.

Let us analyze the low-energy collective mode in our effective Lagrangian. If we keep only quadratic terms in the fluctuation φ and $\vec{u}$, the Lagranigan becomes

$$\mathcal{L}_{\text{eff}} \simeq \frac{n_0}{2mc_s^2}[\dot{\varphi}^2 - c_s^2(\partial_i\varphi + 2m\Omega\epsilon_{ij}u^j)^2] - n_0 m\vec{\Omega}\cdot\vec{u}\times\dot{\vec{u}} - E_{\text{el}}(\partial\vec{u}). \tag{17}$$

In order to compare our expressions to those in the literature, we have eliminated g and μ_0 in terms of the equilibrium density n_0 and the superfluid velocity c_s by $g = \mu_0/n_0$ and $\mu_0 = mc_s^2$. The remarkable feature of the effective Lagrangian is the mass term $-2mn_0\Omega^2\vec{u}^2$. Combined with the second term, which makes u^x and u^y canonically conjugate to each other, it explains the gapped mode with a gap 2Ω in the spectrum [12,13]. This mode can be identified as the collective mode with the cyclotron gap $e_{\text{eff}}B_{\text{eff}}/m = 2\Omega$ predicted by Kohn's theorem [21].

Given the gap, one can safely integrate $\vec{u}$ out by using EOM,

$$u^i = \frac{1}{2m\Omega}\epsilon^{ij}\partial_j\varphi + O(\partial_0\partial_i, \partial_i\partial_j\partial_k). \tag{18}$$

At the leading order in the derivative expansion, the remaining Lagrangian is

$$\mathcal{L}_{\text{eff}} \simeq \frac{n_0}{2mc_s^2}\left[\dot{\varphi}^2 - \frac{C_2}{2mn_0}\frac{c_s^2}{\Omega^2}(\nabla^2\varphi)^2\right], \tag{19}$$

which describes the Tkachenko mode with the dispersion relation $E(\vec{p}) = \sqrt{(C_2/2mn_0)}(c_s/\Omega)p^2 + O(p^4)$ [12,13]. The Tkachenko mode thus can be understood as the phase oscillation, and the vortex lattice simply follows transverse to the motion of the superfluid through Eq. (18).

After all, there is only one gapless mode in the vortex lattice, as expected from our general criterion. In the derivation, we introduced the redundant fields in our effective Lagrangian and observed a mass term $\propto \vec{u}^2$ for them. An effective Lagrangian of crystal phonons does not usually contain $\vec{u}$ without any derivatives, because the invariance under the shift $\vec{u}' = \vec{u} + \vec{a}$ prohibits it. This is why we usually expect gapless phonons [22]. However, in the current example, the appearance of the mass term does not contradict with the symmetry—the original magnetic translation is still exactly realized in our effective Lagrangian Eq. (14) in a nontrivial manner,

$$\vec{u}'(\vec{x} + \vec{a}, t) = \vec{u}(\vec{x}, t) + \vec{a}, \tag{20}$$

$$\theta_{\text{reg}}'(\vec{x} + \vec{a}, t) = \theta_{\text{reg}}(\vec{x}, t) - m\vec{a}\cdot\vec{\Omega}\times[\vec{u}(\vec{x}, t) - 2\vec{x}]. \tag{21}$$

This symmetry also protects the quadratic dispersion relation of the Tkachenko mode; i.e., the lower-order term $\propto (\nabla\varphi)^2$ cannot be generated by the renormalization process in Eq. (19).

It is instructive to compare the vortex lattice with a supersolid [18]. A supersolid exhibits a similar symmetry-breaking pattern; namely, it breaks both the (usual) translation and U(1) phase rotation. In contrast to

the vortex lattice case, each of the d momentum operators P^i and the number operator N independently produces a NGB, giving rise to $d+1$ NGBs in total in d-space dimensions. This is consistent with our criterion, since in the case of the supersolid, the Galilean invariance [18] (more precisely, the nonrelativistic general-coordinate invariance [23]) leads to $T^{0i} = mj^i$. Therefore, phonons originated from the translational symmetry breaking and Bogoliubov mode are not redundant.

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Note added.—Recently, the authors were informed of a related preprint [24]. Although their approach is different and limited to the example of crystals, their result is consistent with our criterion.

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[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[2] H. Watanabe and H. Murayama, Phys. Rev. Lett. **108**, 251602 (2012); (to be published); See also, Y. Hidaka, Phys. Rev. Lett. **110**, 091601 (2013); Y. Nambu, J. Stat. Phys. **115**, 7 (2004); *Proceedings of the Third Meeting on CPT and Lorentz Symmetry*, edited by V. A. Kosterlecky (World Scientific, Singapore, 2005), p. 1.

[3] Since ρ is real and antisymmetric, its rank is always even.

[4] A. Salam and J. Strathdee, Phys. Rev. **184**, 1760 (1969).

[5] D. V. Volkov, Sov. J. Part. Nucl. **4**, 1 (1973); E. A. Ivanov and V. I. Ogievetsky, Teor. Mat. Fiz. **25**, 1050 (1975).

[6] I. Low and A. V. Manohar, Phys. Rev. Lett. **88**, 101602 (2002).

[7] V. A. Miransky and I. A. Shovkovy, Phys. Rev. Lett. **88**, 111601 (2002); T. Schäfer, D. T. Son, M. A. Stephanov, D. Toublan, and J. J. M. Verbaarschot, Phys. Lett. B **522**, 67 (2001).

[8] The constant part of currents can be completely fixed by requiring the commutation relation $[Q_i, j_j^0(x)] = if_{ij}^k j_k^0(x)$.

[9] L. Radzihovsky and T. C. Lubensky, Phys. Rev. E **83**, 051701 (2011).

[10] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science **292**, 476 (2001); P. Engels, I. Coddington, P. C. Haljan, and E. A. Cornell, Phys. Rev. Lett. **89**, 100403 (2002).

[11] V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **49**, 1875 (1965) [Sov. Phys. JETP **22**, 1282 (1966)]; Zh. Eksp. Teor. Fiz. **50**, 1573 (1966) [Sov. Phys. JETP **23**, 1049 (1966)]; Zh. Eksp. Teor. Fiz. **56**, 1763 (1969) [Sov. Phys. JETP **29**, 945 (1969)].

[12] E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987); Phys. Rev. A **71**, 011603(R) (2005).

[13] G. Baym, Phys. Rev. Lett. **91**, 110402 (2003).

[14] I. Coddington, P. Engels, V. Schweikhard, and E. A. Cornell, Phys. Rev. Lett. **91**, 100402 (2003).

[15] M. Hatsuda, S. Yahikozawa, P. Ao, and D. J. Thouless, Phys. Rev. B **49**, 15 870 (1994); M. Hatsuda, M. Sato, S. Yahikozawa, and T. Hatsuda, Int. J. Mod. Phys. B **10**, 1875 (1996).

[16] C. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, England, 2008), 2nd ed.

[17] If interested in the Bogoliubov spectrum $\omega^2 = c_s^2 k^2 (1 + (k^2/4m^2c_s^2))$, one should not integrate n out and leave it as a conjugate field to the phase. In this case, the quantity μ in the text loses the meaning of the chemical potential. Instead, we define the chemical potential as $\mu(n) \equiv \partial\varepsilon(n)/\partial n$, using the internal energy $\varepsilon(n) = (1/2g)n^2$. The EOM of n describes the time evolution of the phase, $\dot{\theta}_{\rm reg} = \mu(n) + m\vec{\Omega}\cdot\vec{u}\times\dot{\vec{u}} + \frac{m}{2}\vec{v}^2 - (1/2m\sqrt{n})\nabla^2\sqrt{n}$. Other EOMs remain unchanged.

[18] D. T. Son, Phys. Rev. Lett. **94**, 175301 (2005).

[19] R. Ikeda, J. Phys. Soc. Jpn. **64**, 3925 (1995).

[20] The appearance of Eq. (16) depends on the choice of χ, while physics is independent of it. Our convention gives the expressions consistent with those in the literature, e.g., Refs. [12,13].

[21] W. Kohn, Phys. Rev. **123**, 1242 (1961).

[22] In two spatial dimensions, the exception at the quadratic order in $\vec{u}$ is $\epsilon_{ij}u^i\dot{u}^j$, which changes by a total derivative $\partial_0(\epsilon_{ij}c^iu^j)$.

[23] D. T. Son and M. Wingate, Ann. Phys. (Amsterdam) **321**, 197 (2006).

[24] A. J. Beekman, K. Wu, V. Cvetkovic, and J. Zaanen, arXiv:1301.7329.