Classification of Topologically Protected Gates for Local Stabilizer Codes

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Given a quantum error correcting code, an important task is to find encoded operations that can be implemented efficiently and fault tolerantly. In this Letter we focus on topological stabilizer codes and encoded unitary gates that can be implemented by a constant-depth quantum circuit. Such gates have a certain degree of protection since propagation of errors in a constant-depth circuit is limited by a constant size light cone. For the 2D geometry we show that constant-depth circuits can only implement a finite group of encoded gates known as the Clifford group. This implies that topological protection must be "turned off" for at least some steps in the computation in order to achieve universality. For the 3D geometry we show that an encoded gate U is implementable by a constant-depth circuit only if $UPU^{\dagger}$ is in the Clifford group for any Pauli operator P. This class of gates includes some non-Clifford gates such as the $\pi/8$ rotation. Our classification applies to any stabilizer code with geometrically local stabilizers and sufficiently large code distance.

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Quantum error correcting codes play a central role in all proposed schemes for fault-tolerant quantum computation. By repeatedly measuring error syndromes and applying corresponding correction operations, encoded states can be stored reliably for extended periods of time [1,2]. Furthermore, many error correcting codes support a limited set of *protected gates* that can be applied to encoded states with a high fidelity without exposing them to the environment. For example, it is often possible to apply certain encoded gates transversally, e.g., by a product of one-qubit rotations on the physical qubits. Such transversal gates are protected because an error in a single qubit location cannot spread to other qubits. Furthermore, since a transversal gate can be implemented in a single time step, it does not introduce too many errors on its own. Another example is the implementation of the CNOT gate by braiding of defects in the surface code architecture [3,4]. Here protection comes from the fact that the braiding process involves a sequence of local maps between high-distance topological codes and each local map is followed by an error correction.

Protected gates usually have a limited computational power. For example, a general no-go theorem proved by Eastin and Knill [3] asserts that for any error detecting code, the set of transversal encoded gates is a finite group and therefore cannot be computationally universal. Furthermore, in the case of the surface codes, Sarvepalli and Raussendorf showed [5] that all transversal encoded gates must belong to the Clifford group (under certain extra technical assumptions). To enhance the power of protected gates, almost all existing fault-tolerant schemes resort to preparation and distillation of certain quantum software states [6,7], which substantially increases the space and time overheads.

In this Letter we derive general limitations on the power of protected gates for a large family of codes known as *topological stabilizer codes* (TSC). The physical qubits of a TSC can be laid out on a D-dimensional regular lattice such that each syndrome bit is determined by a local Pauli stabilizer supported in a region of radius $O(1)$. Furthermore, the minimal distance of a TSC can be made arbitrarily large by increasing the lattice size.

TSCs are arguably among the most promising current approaches to the experimental realization of fault-tolerant quantum computation: the locality and Pauli structure of the stabilizers permit syndrome extraction by relatively simple measurement circuits. The placement of the qubits on a regular lattice (especially in dimension $D = 2$) appears advantageous from an engineering viewpoint. Furthermore, the dependence of the code distance on the system size suggests that robustness essentially reduces to a question of scalability. Not surprisingly, TSCs are the basis of some of the best-studied proposed schemes for fault-tolerant storage and computation. Indeed, the highest currently known fault-tolerance thresholds were established using TSCs [2,3,8].

The family of TSCs includes the 2D toric and surface codes [9,10], the 2D color codes [11], modifications of the above codes with twists [12] or punctured holes [3], as well as the punctured 3D color code developed by Bombin and Martin-Delgado [13]. It also encompasses 3D models of a self-correcting quantum memory found recently by Haah [14] and Michnicki [15]. The assumption that each stabilizer is a Pauli operator is essential for our analysis. Hence, our theorems do not apply to more general topological codes such as quantum double models [10] or the Turaev-Viro codes [16].

To formalize the intuitive notion of a protected gate, we consider encoded unitary gates that can be implemented by a constant-depth quantum circuit that maps the code space of a TSC to itself (or a code space of another TSC). Here we assume that all gates in the circuit are geometrically local; that is, each gate has support on a region of radius $O(1)$. By analogy with a transversal gate, a noisy constant-depth circuit does not introduce too many errors on its own since it can be executed in a constant time. Propagation of preexisting errors is strongly limited by the causality: any preexisting single-qubit error can only spread to an error supported on a "light cone" of width $O(1)$. In other words, constant-depth circuits preserve the set of "local" errors and are thus naturally fault tolerant. Let us say that an encoded unitary gate is *topologically protected* if it can be implemented by a constant-depth circuit as described above. It is worth pointing out that, in contrast to transversal gates, constant-depth circuits do not form a group (since the circuit's depth may grow under compositions).

To state our main result, let us define the *Clifford hierarchy* (CH) introduced originally by Gottesman and Chuang [17] in the context of gate teleportation. It involves a nested family of sets of k-qubit unitary operators. At the lowest level of the hierarchy is the Pauli group $\mathcal{C}_1(k)$ generated by single-qubit Pauli operators $X_1, Z_1, \ldots, X_k, Z_k$. Higher levels of the CH are defined inductively such that

$$\mathcal{C}_{j+1}(k) = \{V \in \mathcal{U}(k)\colon V\mathcal{C}_1(k)V^\dagger \subseteq \mathcal{C}_j(k)\} \qquad (1)$$

for all $j \geq 1$. Here, $\mathcal{U}(k)$ is the group of all unitary operators on k qubits. In particular, the second level of the CH coincides with the well-known Clifford group generated by the Hadamard gate, $\pi/2$ phase shift, and the CNOT gate. The third level includes some non-Clifford gates such as the Toffoli gate, the $\pi/4$ phase shift T, or the controlled $\pi/2$ phase shift. One can easily use induction to show that $\mathcal{C}_j(k) \subseteq \mathcal{C}_{j+1}(k)$ for all j. Quite surprisingly, the CH also arises in the classification of topologically protected gates. Our main theorem asserts that for any D-dimensional TSC, all topologically protected gates must belong to the level D of the CH. More precisely, we prove the following.

Theorem 1.—Let $D \geq 2$ and let $\mathcal{L}$ be the code space of a topological stabilizer code on a D-dimensional lattice. Suppose U is a constant-depth quantum circuit that maps $\mathcal{L}$ to itself. Then the restriction of U onto $\mathcal{L}$ implements an encoded gate from the level D of the Clifford hierarchy.

(Note that any stabilizer code has a basis such that all encoded Pauli operators are products of physical Pauli operators [18]. Above we implicitly assumed that such a basis is chosen for both code spaces $\mathcal{L}_i$). Let us discuss some implications of the theorem focusing first on the 2D geometry, which is arguably the most practical one. The theorem states that any encoded circuit composed of topologically protected gates must belong to the Clifford group. Since such circuits can be efficiently simulated classically [18], our result implies that certain "nontopological" methods such as magic state distillation are necessary to achieve universality in 2D. Specializing Theorem 1 to the 2D surface codes we reproduce the result of Ref. [5] which was obtained using the matroid theory.

Our proof of Theorem 1 actually covers a more general situation where one is given two *different* codes with code spaces $\mathcal{L}_1$, $\mathcal{L}_2$, and a constant-depth quantum circuit U that maps $\mathcal{L}_1$ to $\mathcal{L}_2$. In this case, one can view the code $\mathcal{L}_2$ as a "local deformation" of the code $\mathcal{L}_1$. We prove that U induces an encoded gate from the set $\mathcal{C}_D$ provided that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are D-dimensional TSCs. For the 2D geometry, this shows that any chain of local deformations $\mathcal{L}_1 \to \mathcal{L}_2 \to \ldots \to \mathcal{L}_t$ implements an encoded Clifford group operator provided that one has uniform bounds on the locality and the distance of all intermediate codes. Such chains of local deformation can be used, for instance, to describe braiding of topological defects used in the surface code architecture to implement encoded CNOT gates [3].

To the best of our knowledge, the only example of a 3D TSC with a topologically protected non-Clifford gates is the punctured 3D color code developed by Bombin and Martin-Delgado [13]. It encodes one logical qubit with a transversal $\pi/4$ phase shift which belongs to the third level of the CH.

Although the set $\mathcal{C}_D(k)$ is finite for all D, it generates a dense subgroup of $\mathcal{U}(k)$ for $D \geq 3$. Hence, Theorem 1 does not rule out a possibility that topologically protected gates can be computationally universal for $D \geq 3$. However, under certain extra assumptions Theorem 1 has the following corollary that rules out universality of topologically protected gates for any spatial dimension D.

Corollary 1. Consider any family of D-dimensional topological stabilizer codes such that the number of logical qubits k is independent of the lattice size L. Then for any fixed circuit depth $h = O(1)$ and all large enough L, the group generated by encoded gates implementable by depth-h circuits is contained in the level D of the Clifford hierarchy.

For example, Corollary 1 shows that there is no universal set of protected gates for the D-dimensional version of the toric code [2] (for any D). On the other hand, this result does not apply to Haah's 3D model [14]. Let us emphasize that our results are restricted to unitary operations. In particular, supplementing the set of available operations by measurements may yield universality; hence, our results are compatible with, e.g., the computational scheme proposed in Ref. [13].

Constant-depth circuits and, more generally, locality preserving unitary maps play an important role in the classification of different types of topological quantum order in condensed matter physics [19]. In particular, it was recently shown by Bombin *et al.* [12,20] that any translation-invariant 2D TSC on an infinite lattice is equivalent modulo constant-depth circuits to one or several copies of the surface code. However, this result does not say anything about topologically protected gates since the latter are only defined in finite settings. It is also known that constant-depth circuits by themselves are not sufficient for encoding information into a topological code [21].

In the rest of the Letter, we sketch the proof of Theorem 1 and its corollary. All technical details can be found in Ref. [22]. To illustrate the proof strategy, let us first consider the standard toric code with two logical qubits. Recall that logical Pauli operators of the toric code correspond to noncontractible closed loops on the primal and the dual lattices [10]. Let γ_1 and γ_2 be some fixed horizontal and vertical noncontractible strips of width 1, see Fig. 1. Then we can choose a complete set of 15 nontrivial logical Pauli operators supported in $\gamma \equiv \gamma_1 \cup \gamma_2$. Alternatively, we can choose noncontractible strips δ_1 and δ_2 as translations of γ_1 and γ_2, respectively, by half the lattice size, see Fig. 1. Since the toric code is translation invariant, there exists a complete set of logical Pauli operators supported on $\delta \equiv \delta_1 \cup \delta_2$.

Consider any unitary operator U implementable by a constant-depth quantum circuit with short-range gates. Let P and Q be any pair of logical Pauli operators. We can always find logical operators P_γ and Q_δ equivalent modulo stabilizers to P and Q such that P_γ is supported on γ, while Q_δ is supported on δ. The key observation is that the commutator

$$K \equiv P_\gamma (U Q_\delta U^\dagger) P_\gamma^\dagger (U Q_\delta^\dagger U^\dagger) \tag{2}$$

acts nontrivially only on $O(1)$ qubits located near the intersection of γ and δ. Indeed, the evolution $O \mapsto UOU^\dagger$ of any observable O enlarges its support at most by $\rho = hr$, where h is the depth of U and r is the maximum range of its gates. Loosely speaking, ρ is the radius of a "light cone" describing evolution of observables under U. Note that in our case $\rho = O(1)$. In particular, $V \equiv U Q_\delta U^\dagger$ is supported in $\mathfrak{B}_\rho(\delta)$—the set of all qubits within distance ρ from δ. Furthermore, the standard causality argument implies that all gates of U lying outside the light cone $\mathfrak{B}_\rho(\delta)$ can be omitted without changing V. This shows that $K = P_\gamma V P_\gamma^\dagger V^\dagger$, where V is a circuit of depth $2h+1$ composed of gates of range r. Any gate in V must overlap with the light cone $\mathfrak{B}_\rho(\delta)$. Here we used the fact that Q_δ is a product of single-qubit Pauli operators which can be regarded as a depth-1 circuit. Applying the same causality argument to the evolution $P_\gamma^\dagger \to V P_\gamma^\dagger V^\dagger$, we conclude that $K = P_\gamma W P_\gamma^\dagger W^\dagger$, where W is obtained from V by omitting all gates lying outside the light cone of γ, that is, $\mathfrak{B}_{r(2h+1)}(\gamma)$. Hence, W has support only in $\mathfrak{B}_{O(\rho)}(\gamma \cap \delta)$. The evolution $W \to P_\gamma W P_\gamma^\dagger$ does not enlarge the support of W since P_γ is a product of single-qubit Pauli operators. We conclude that K has support only in $\mathfrak{B}_{O(\rho)}(\gamma \cap \delta)$ which contains only $O(1)$ qubits.

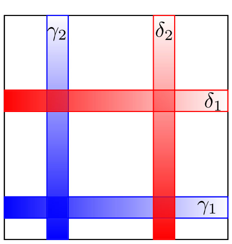

FIG. 1 (color online). Noncontractible closed strips γ_1, γ_2 and δ_1, δ_2 on the torus.

Let $\mathcal{L}$ be the four-dimensional code space of the toric code and Π be the projector onto $\mathcal{L}$. By assumption of the theorem, U preserves the code space $\mathcal{L}$, that is, $U\Pi = \Pi U$. Since the operators U, P_γ, Q_δ as well as their Hermitian conjugates preserve $\mathcal{L}$, we conclude that K preserves $\mathcal{L}$ as well. However, since K acts only on $O(1)$ qubits, the macroscopic distance property of the toric code implies that K is a trivial logical operator, that is,

$$K\Pi = c\Pi \tag{3}$$

for some complex coefficient c. We claim that in fact $c = \pm 1$. Indeed, since K is a unitary operator, one must have $|c| = 1$. Furthermore, Eq. (3) can be rewritten as $V P_\gamma V^\dagger \Pi = c P_\gamma \Pi$, where $V = U Q_\delta U^\dagger$. Since $P_\gamma^2 = e^{i\theta} I$ for some phase factor $e^{i\theta}$ this implies $e^{i\theta} = c^2 e^{i\theta}$, that is, $c = \pm 1$. To conclude, we have shown that

$$P_\gamma (U Q_\delta U^\dagger) \Pi = \pm (U Q_\delta U^\dagger) P_\gamma \Pi \tag{4}$$

for any pair of logical Pauli operators P, Q. Let $\bar{P}$, $\bar{Q}$, and $\bar{U}$ be the encoded two-qubit operators implemented by P, Q, U, respectively. Let $\bar{R} = \bar{U} \cdot \bar{Q} \cdot \bar{U}^\dagger$. From Eq. (4) one infers that $\bar{P} \cdot \bar{R} = \pm \bar{R} \cdot \bar{P}$. Since $\bar{P}$ could be an arbitrary two-qubit Pauli operator, this is possible only if $\bar{R}$ is a Pauli operator itself. However, since this is true for any Pauli $\bar{Q}$, we conclude that $\bar{U}$ belongs to the Clifford group, see Eq. (1).

The key technical result that allows one to extend the above arguments to general TSCs is the Cleaning Lemma of Ref. [6]. To illustrate the main idea, we shall now consider a general 2D TSC with a code space $\mathcal{L}$. Let P, $Q \in \mathcal{C}_1(n)$ be any Pauli operators preserving $\mathcal{L}$ and implementing encoded Pauli operators $\bar{P} = P|_\mathcal{L}$ and $\bar{Q} = Q|_\mathcal{L}$. Operators P, Q as above will be referred to as *logical Pauli operators*. Let $R = UQU^\dagger$ and $\bar{R} = R|_\mathcal{L}$. Note that $\bar{R} = \bar{U}\,\bar{Q}\,\bar{U}^\dagger$. If we can show that the commutator $K \equiv PRP^\dagger R^\dagger$ implements the encoded $\pm I$ operator, $\bar{K} = \pm I$, the same algebraic arguments as above would show that $\bar{U}$ is in the Clifford group.

We will say that a subset of physical qubits M is *correctable* iff for any logical Pauli operator P supported inside M, the encoded operator $\bar{P} = P|_\mathcal{L}$ is proportional to the identity. By definition, any subset M of size smaller than the code distance d is correctable. We will use the following facts.

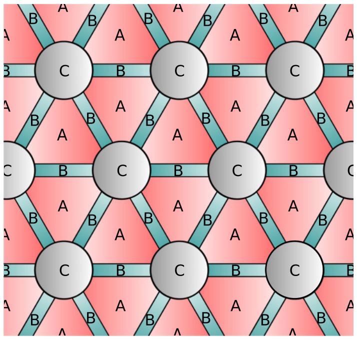

FIG. 2 (color online). Simplicial partition of the lattice $\Lambda = A \cup B \cup C$. Starting from a triangulation with regular triangles having sides of length R, let C be the union of discs of radius $R/4$ centered on the vertices of the triangulation. Let $B \subset \Lambda\backslash C$ be the union of the $R/8$ neighborhoods of each edge in the remaining surface. Finally, let $A = \Lambda\backslash(B \cup C)$ be the union of the remaining capped triangles. A similar but rectangular partition has been used in Ref. [23] to derive upper bounds on parameters of TSCs but is less suitable for generalization to $D > 2$.

Lemma 1 (*Cleaning Lemma* [6]). Suppose M is a correctable subset of qubits. Then for any logical Pauli operator P there exists a stabilizer S such that PS acts trivially on M.

Lemma 2 (*Union Lemma* [17,23]). Suppose M and K are disjoint correctable subsets of qubits such that the distance between M and K is greater than the diameter ξ of the stabilizer generators. Then the union $M \cup K$ is correctable.

Let L be the linear size of the physical lattice and ξ be the diameter of the stabilizer generators. For any integer $1 \ll R \ll L$ the lattice can be partitioned into three disjoint regions, $\Lambda = A \cup B \cup C$, such that each region $A = \cup_i A_i$, $B = \cup_j B_j$, $C = \cup_k C_k$ consists of disjoint chunks of diameter $O(R)$ separated by distance $\Omega(R)$, see Fig. 2 for an example. We assume that the lattice is large enough so we can choose ξ, $hr \ll R \ll \sqrt{d}$ (recall that r denotes the range of the gates in U, whereas h is the depth of U).

This choice guarantees for any $\rho = O(hr)$, the ρ neighborhood $\mathfrak{B}_\rho(A_j)$ of any chunk A_j contains fewer qubits than the code distance d; hence, $\mathfrak{B}_\rho(A_j)$ is correctable. Furthermore, since the separation between $\mathfrak{B}_\rho(A_i)$ and $\mathfrak{B}_\rho(A_j)$ with $i \neq j$ is larger than ξ, the Union Lemma implies that the entire region $\mathfrak{B}_\rho(A) = \cup_i \mathfrak{B}_\rho(A_i)$ is correctable. In a similar fashion, we can show that the regions $\mathfrak{B}_\rho(B)$ and C are correctable.

Applying the Cleaning Lemma to the logical Pauli operator Q and the region $\mathfrak{B}_\rho(A)$, we can find a stabilizer S_1 such that QS_1 acts trivially on $\mathfrak{B}_\rho(A)$. Applying the same arguments to the logical Pauli operator P and the region $\mathfrak{B}_\rho(B)$, we can find a stabilizer S_2 such that PS_2 acts trivially on $\mathfrak{B}_\rho(B)$. Replacing Q and P by equivalent logical operator QS_1 and PS_2 (which does not change $\bar{Q}$ and $\bar{P}$), we can now assume that

$$\mathrm{supp}(Q) \cap \mathfrak{B}_\rho(A) = \emptyset \quad \text{and} \quad \mathrm{supp}(P) \cap \mathfrak{B}_\rho(B) = \emptyset.$$

Consider the evolution $Q \to UQU^\dagger$. It enlarges the support of Q at most by $rh < \rho$, so that the light cone of Q and all gates of U overlapping with it are contained in $B \cup C$. Applying the causality argument used in the toric code example, we conclude that $UQU^\dagger$ can be implemented by a depth-$(2h+1)$ circuit V with gates of range r and all gates of V are supported in $B \cup C$. Note that $K = PVP^\dagger V^\dagger$. Applying the causality argument to the time evolution $P^\dagger \to VP^\dagger V^\dagger$ which is characterized by a light cone of radius $r(2h+1) < \rho$, we conclude that $K = PWP^\dagger W^\dagger$, where W is obtained from V by omitting all gates lying outside the light cone of (P). Our assumptions on $\mathrm{supp}(P)$ imply that any gate supported in B or overlapping with B lies outside the light cone of $\mathrm{supp}(P)$. Hence, all gates of W are supported in C. The evolution $W \to PWP^\dagger$ does not enlarge the support of W since P is a product of single-qubit Pauli operators. We conclude that K is supported in C which is a correctable region as argued above.

Let $K = \sum_\alpha c_\alpha K_\alpha$ be the expansion of K in the basis of Pauli operators, where c_α are complex coefficients and K_α are n-qubit Pauli operators. Note that all K_α are supported in C. Then

$$K\Pi = \Pi K \Pi = \sum_\alpha c_\alpha \Pi K_\alpha \Pi.$$

Since C is a correctable region, $\Pi K_\alpha \Pi = x_\alpha \Pi$ for some complex coefficient x_α. This shows that $K\Pi = c\Pi$ for some coefficient c. The same arguments as in the toric code example show that $c = \pm 1$. This proves that $\bar{K} \equiv K|_{\mathcal{L}} = \pm I$ and completes the proof of the theorem for $D = 2$ and a single TSC.

The generalization of the proof to $D > 2$ consists in a recursive application of these arguments using a partition of the lattice into $D + 1$ regions. This argument, the proof of Theorem 1 for the case of two different TSCs and the proof of Corollary 1 can be found in the long version of this Letter [22].

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